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Inverse Secure Domination in the Join and Corona of Graphs

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Abstract

In [6], Enriquez and Kiunisala showed that every integers k, m, and n with $1 \leq k \leq m < n$ is realizable as inverse domination number, inverse secure domination number, and order of G respectively and gave the characterization of the inverse secure dominating set with inverse secure domination number of one and two. In this paper, we characterize the inverse secure dominating sets in the join and corona of two graphs and give some important results.

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1. Introduction

In [10], Claude Berge and Oystein Ore intrduced the domination in graph. Due to Cockayne and Hedetniemi in [2], domination in graphs became an area of study by many researchers. Secure domination in graphs was studied and introduced by E.J. Cockayne et al. [3, 4, 1]. In [7] Enriquez and Canoy, introduced a variant of domination in graphs, the concept of secure convex domination in graphs. The inverse domination in graph was first found in the paper of Kulli [11] and further read in [8, 12]. In [6], Enriquez and Kiunisala introduced the inverse secure domination in graphs. In this paper, we characterize the inverse secure dominating sets in the join and corona of two graphs and give some important results. For the general concepts, the reader may refer to [9].

Let $G = (V(G), E(G))$ be a connected simple graph and $v \in V(G)$. The neighborhood of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \bigcup_{v \in S} N_G(v)$. The *closed neighborhood* of S is $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in (V(G)\backslash S)$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A dominating set S in G is called a *secure dominating set* in G if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set. The minimum cardinality of secure dominating set is called the *secure domination number* of G and is denoted by $\gamma_s(G)$. A secure dominating set of cardinality $\gamma_s(G)$ is called γ_s-*set* of G. Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-*set* of G. Motivated by the definition of inverse domination in graph, we define a new domination parameter. Let C be a minimum secure dominating set in G. The secure dominating set $S \subseteq V(G) \setminus C$ is called an *inverse secure dominating set* with respect to C. The minimum cardinality of inverse secure dominating set is called an *inverse secure domination number* of G and is denoted by $\gamma_s^{-1}(G)$. An inverse secure dominating set of cardinality $\gamma_s^{-1}(G)$ is called γ_s^{-1}-*set* of G.

2. Results

Since $\gamma_s^{-1}(G)$ does not always exists in a connected nontrivial graph G, we denote by $\mathcal{G}_s^{-1}$ be a family of all graphs with inverse secure dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered (including $G + H$ and $G \circ H$) belong to the family $\mathcal{G}_s^{-1}$.

Remark 2.1. Let $G \in \mathcal{G}_s^{-1}$. If S is a secure dominating set in G, then there exists $C \subseteq V(G) \setminus S$ such that C is a secure dominating set of G.

The *join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge-set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

A nonempty subset S of $V(G)$, where G is any graph, is a *clique* in G if the graph $\langle S \rangle$ induced by S is complete.

The following result characterized the inverse secure dominating sets in the join of two graphs.

Theorem 2.2. Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$, where $S \subseteq V(G + H) \setminus C$, is an inverse secure dominating set in $G + H$ if and only if one of the following statements holds:

(i) S is a secure dominating set of G and $|S| \geq |C| \geq 2$.

(ii) S is a secure dominating set of H and $|S| \geq |C| \geq 2$.

(iii) $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$ and

- (a) S_G is a dominating set of G and S_H is a dominating set of H
- (b) S_G is dominating set of G and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H; or
- (c) S_H is dominating set of H and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G; or
- (d) $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H.

(iv) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ ($|S_G| \geq 2$) and $S_H = \{w\} \subset V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G.

(v) $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ ($|S_H| \geq 2$) and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H.

(vi) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ ($|S_G| \geq 2$) and $S_H \subseteq V(H)$ ($|S_H| \geq 2$).

Proof. Suppose that S is an inverse secure dominating set of $G + H$. Consider the following cases:

Case 1. Suppose that $S \subseteq V(G)$ or $S \subseteq V(H)$.
If $S \subseteq V(G)$, then S is a secure dominating set of G. Now suppose that $|S| = 1$, say $S = \{a\}$. Since S is a secure dominating set of $G + H$, $\{z\}$ is a dominating set in $G + H$ (and hence in H) for every $z \in V(H)$. This implies that H is a complete graph, contrary to our assumption. Thus, $|S| \geq 2$. Similarly, $|C| \geq 2$. Since C is a minimum secure dominating set of G, it follows that $|S| \geq |C|$. This shows that statement (i) holds. Similarly, statement (ii) holds if $S \subseteq V(H)$.

Case 2. Suppose that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Then $S = S_G \cup S_H$. Consider the following subcases.

Subcase 1. Suppose that $S_G = \{v\}$ is a dominating set of G and $S_H = \{w\}$ is a dominating set of H. Then we are done with $(iiia)$. Suppose that S_G is a dominating set of G and S_H is not a dominating set of H. Let $x \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since S is a secure dominating set of $G + H$, $\{w, x\}$ is a dominating set in $G + H$ (and hence in H). Since $wx \notin E(H)$, $xy \in E(H)$ for every $y \notin N_H(w)$. This implies that $y \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(H) \setminus S_H) \setminus N_H(S_H) \rangle$ induced by $(V(H) \setminus S_H) \setminus N_H(S_H)$ is complete. Hence, $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H. This proves $(iiib)$. Similarly, if S_H is dominating set of H and S_G is not a dominating set of G, then $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This proves $(iiic)$. If S_G is not a dominating set of G and S_H is not a dominating set of H, then $(iiid)$ holds by following similar arguments in $(iiib)$ and $(iiic)$.

Subcase 2. Suppose that $S_G \subseteq V(G)$ $(|S_G| \geq 2)$ and $S_H = \{w\} \subset V(H)$. If S_G is a dominating set of G, then (i) holds. Suppose that S_G is not a dominating set of G. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since S is a secure dominating set of $G + H$, $S_x = (S \setminus \{w\}) \cup \{x\}$ is a dominating set in $G + H$ (and hence in G). Since $vx \notin E(G)$ for every $v \in S_G$, $xy \in E(G)$ for every $y \notin N_G(S_G)$ (otherwise, S_x is not dominating set in $G + H$). This implies that $y \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(G) \setminus S_G) \setminus N_G(S_G) \rangle$ induced by $(V(G) \setminus S_G) \setminus N_G(S_G)$ is complete. Hence $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This proves (iv). Similarly, (v) holds, if $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ $(|S_H| \geq 2)$.

*Subcase 3.*Suppose that $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Let $|S_G| \geq 2$. If S_G is a dominating set of G, then (i) holds. Suppose that S_G is not a dominating set of G. If $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G, then (iv) holds. Suppose that $(V(G) \setminus S_G) \setminus N_G(S_G)$ is not a clique in G. If $|S_H| = 1$, say $S_H = \{w\}$, then there exists $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$ such that $S_x = (S \setminus \{w\}) \cup \{x\}$ is not a dominating set of G(and hence in $G + H$). This contradict to our assumption that S is a secure dominating set of $G + H$. Thus, $|S_H| \geq 2$. Similarly, if $|S_H| \geq 2$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is not a clique in H, then $|S_G| \geq 2$. This proves (vi).

For the converse, suppose first that statement (i) holds. Let $u \in V(G + H)$. If $u \in V(G)$, then there exists $v \in S \cap N_G(u)$ such that $S_u = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of G (and hence S_u is a dominating set of $G + H$). Suppose that $u \in V(H)$. Since $|S| \geq 2$, $N_{G+H}[S_u] = N_{G+H}[S \setminus \{v\}] \cup N_{G+H}[\{u\}] = V(G+H)$. Thus S_u a dominating set of G and hence of $G + H$. Accordingly, S is a secure dominating set of $G + H$. Since G and H are connected non-complete graphs, there exists $C \in V(G + H)$ such that $C \cap S = \varnothing$ and C is a secure dominating set of $G + H$. If $|C| = 1$, then $G + H$ is complete contrary to our assumption. Thus, $|C| \geq 2$. Since $|C| \leq |S|$, if follows that C is a γ_s-*set* of $G + H$. Thus, $S \subseteq V(G + H) \setminus C$ is an inverse secure dominating set of $G + H$.

Similarly, if statement (ii) holds, $S \subseteq V(G + H) \setminus C$ is an inverse secure dominating set of $G + H$.

Suppose that statement $(iiia)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. Then $vx \in E(G + H)$ and $S_x = (S \setminus \{v\}) \cup \{x\} = \{w, x\}$ is a dominating set of $G + H$, that is, S is a secure dominating set of $G + H$. Since G and H are connected non-complete graphs, there exists $C \in V(G + H)$ such that $C \cap S = \varnothing$ and C is a secure dominating set of $G + H$. If $|C| = 1$, then $G + H$ is complete contrary to our assumption. Thus, $|C| \geq 2$. Let $x, y \in C$. Then for every $u \in V(G + H) \setminus C$, there exists $z \in C$, say $z = x$, such that $xu \in E(G + H)$ and $C_u = (C \setminus \{x\}) \cup \{u\}$ is a dominating set of $G + H$. Since $|C| \geq 2$, it follows that $C = \{x, y\}$ is the γ_s-set of $G + H$. Accordingly, $S \subseteq V(G + H) \setminus C$ is an inverse secure dominating set of $G + H$ if statement $(iiia)$ holds.

Suppose that statement $(iiib)$ holds. Since $S_G = \{v\}$ is a dominating set of G (and hence of $G + H$), $S = S_G \cup S_H$ is a dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Then $uv \in E(G + H)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$. If $u \in V(G)$, then S_u is a dominating set of $G + H$. Suppose that $u \in V(H)$. Then $u \notin N_H(w)$, and $u \in (V(H) \setminus S_H \setminus N_H(S_H))$. Since $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H, it follows that $N_H[S_u] = N_H[w] \cup N_H[u] = V(H)$. Thus, S_u is a dominating set of H and hence of $G + H$. Accordingly, S is a secure dominating set of G. Since G and H are connected non-complete graphs, there exists $C \in V(G + H)$ such that $C \cap S = \varnothing$ and C is a secure dominating set of $G + H$. By similar arguments used above, $S \subseteq V(G + H) \setminus C$ is an inverse secure dominating set of $G + H$ if statement $(iiib)$ holds.

Similarly, S is an inverse secure dominating set of $G + H$ if $(iiic)$ holds.

Suppose that statement $(iiid)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Consider the following cases:

Case 1. Let $u \in V(G)$. If $u \in N_G(S_G)$, the $uv \in E(G)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$ is a dominating set of $G + H$. If $u \notin N_G(S_G)$, then $u \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G, it follows that $uw \in E(G + H)$ and $S_u = (S \setminus \{w\}) \cup \{u\} = \{v, u\}$ is a dominating set of G and hence of $G + H$.

Case 2. Let $u \in V(H)$. If $u \in N_H(S_H)$, the $uw \in E(H)$ and $S_u = (S \setminus \{w\}) \cup \{u\} = \{v, u\}$ is a dominating set of $G + H$. If $u \notin N_H(S_H)$, then $u \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H, it follows that $uv \in E(G + H)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$ is a dominating set of H and hence of $G + H$.

Accordingly, $S \subseteq V(G + H) \setminus C$ is an inverse secure dominating set of $G + H$ if statement $(iiid)$ holds. Similarly, S is a inverse secure dominating set of $G + H$ if any of the following (iv), (v), or (vi) holds. ■

The following result is a quick consequence of Theorem 2.2.

Corollary 2.3. Let G and H be connected non-complete graphs and let $S_G \subset V(G)$

and $S_H \subset V(H)$. Then

$$\gamma_s^{-1}(G+H) = \begin{cases} 2, \text{ if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma_s(G) = 2 \text{ and } \gamma_s(H) = 2 \\ 3, \text{ if } |S_G| = 2 \text{ and } (V(G) \setminus S_G) \setminus N_G(S_G) \text{ is a clique in G and} \\ \quad |S_H| = 2 \text{ and } (V(H) \setminus S_H) \setminus N_H(S_H) \text{ is a clique in H} \\ 4, \text{ if otherwise.} \end{cases}$$

The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

Remark 2.4. Let G and H be nontrivial connected graphs. A nonempty subset S of $V(G \circ H)$ is a dominating set of $G \circ H$ if and only if $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v.

The following result characterize the inverse secure dominating sets in the corona of two connected graphs.

Theorem 2.5. Let G and H be nontrivial connected graphs. A nonempty subset S of $V(G \circ H)$ is an inverse secure dominating set of $G \circ H$ if and only if for each $v \in V(G)$, one of the following is satisfied.

(i) $S = V(G)$ and H is complete.

(ii) $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where for each $v \in V(G)$ $S_v = V(H^v) \setminus \{u\}$ and $\{u\}$ is a dominating set of H^v, (H^v is non-complete).

(iii) $S = (\bigcup_{v \in V(G)} S_v)$, where S_v is a secure dominating set of H^v, (H^v is non-complete).

Proof. Suppose that a nonempty subset S of $V(G \circ H)$ is an inverse secure dominating set of $G \circ H$. Since S is a dominating set, in view of Remark 2.4, $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v. Consider the following cases.

Case 1. Suppose that $V(G) \subseteq S$. If $S = V(G)$ and suppose that H is non-complete, then there exist distinct vertices $x, y \in V(H)$ such that $xy \notin E(H)$. This implies that for each $v \in V(G)$, $(S \setminus \{v\}) \cup \{x\}$ is not a dominating set of $G \circ H$ contrary to our assumption that S is a secure restrained dominating set of $G \circ H$. Thus, H is complete. This proves statement (i). If $S \neq V(G)$, then $V(G) \subset S$. Let $x \in S \setminus V(G) = \bigcup_{v \in V(G)} S_v$, where for each $v \in V(G)$, $S_v = V(H^v) \setminus \{u\}$ and $\{u\}$ is a dominating set of H^v. This implies that $\bigcup_{v \in V(G)} S_v \subset S$, that is, $V(G) \cup (\bigcup_{v \in V(G)} S_v) \subseteq S$. Since $x \in S$ implies that $x \in \bigcup_{v \in V(G)} S_v$, it follows that $x \in V(G) \cup (\bigcup_{v \in V(G)} S_v)$. Thus, $S \subseteq V(G) \cup (\bigcup_{v \in V(G)} S_v)$, that is, $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$, where for each $v \in V(G)$, $S_v = V(H^v) \setminus \{u\}$ and $\{u\}$ is a dominating set of H^v. This proves statement (ii).

Case 2. Suppose that $\bigcup_{v \in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v. In *Case1*, $V(G) \subseteq S$, suppose that $V(G) \nsubseteq S$. Then $S \subseteq \bigcup_{v \in V(G)} V(H^v)$ with S_v is a dominating set of H^v. If for each $v \in V(G)$, S_v is not a secure dominating set of H^v then there exists $u \in V(H^v) \setminus S_v$ such that for every $x \in S_v$, $xu \notin E(H^v)$ and $S'_v = (S_v \setminus \{x\}) \cup \{u\}$ is not a dominating set in H^v. Thus, $S_u = \bigcup_{v \in V(G)} S'_v$ is not a dominating set of $G \circ H$ contrary to our assumption that S is an inverse secure dominating set of $G \circ H$. This implies that for each $v \in V(G)$, S_v must be a a secure dominating set of H^v. Now, let $x \in S$. Since $x \notin V(G)$, it follows that $x \in \bigcup_{v \in V(G)} V(H^v)$. Suppose that for each $v \in V(G)$, $x \notin \bigcup_{v \in V(G)} S_v$. Then for each $v \in V(G)$, $x \notin S_v$. Since for each $v \in V(G)$, S_v is a secure dominating set of H^v, there exists $y \in S_v \cap N_{H^v}(x)$ such that $S'_v = (S_v \setminus \{y\}) \cup \{x\}$ is a dominating set in H^v for all $v \in V(G)$. This implies that $x \in S_x = \bigcup_{v \in V(G)} S'_v$, that is, $x \notin S$, a contradiction. Thus $x \in \bigcup_{v \in V(G)} S_v$ and hence $S \subseteq \bigcup_{v \in V(G)} S_v$. Accordingly, $S = \bigcup_{v \in V(G)} S_v$, where S_v is a secure dominating set of H^v. This proves statement (iii).

For the converse, suppose that for each $v \in V(G)$, statement (i) or (ii) or (iii) holds. First, if statement (i) holds, then $S = V(G)$ is a dominating set of $G \circ H$ by Remark 2.4. Now, let $u \in V(G \circ H) \setminus S$. Since H is complete, for each $v \in V(G)$, $v \in S \cap N_{G \circ H}(u)$ and $S_u = S \setminus \{v\} \cup \{u\}$ is a dominating set of $G \circ H$. This implies that S is a secure dominating set of $G \circ H$. Further, H is complete implies that for each $v \in V(G)$ and for each $u \in V(H^v)$, $S_v = \{u\}$ is a dominating set of $V(H^v)$. Thus, $C = \bigcup_{v \in V(G)} S_v$ is a dominating set in $G \circ H$ where $|C| = |V(G)|$. Let $z \in V(G \circ H) \setminus C$. Then there exists $w \in C$ such that $wz \in E(G \circ H)$ and $C_z = (C \setminus \{w\}) \cup \{z\}$. Since $N_{G \circ H}[C_z] = N_{G \circ H}[C \setminus \{w\}] \cup N_{G \circ H}[\{z\}] = V(G \circ H)$, it follows that C_z is a dominating set in $G \circ H$. Thus, C is a secure dominating set of $G \circ H$. Suppose that $C' = C \setminus \{u\}$

is a dominating set of $G \circ H$. Then for each $y \in V(G \circ H) \setminus C'$, there exists $x \in C'$ such that $xy \in E(G \circ H)$. But there exists $v \in V(G) \subset V(G \circ H) \setminus C'$ such that $vu' \notin E(G \circ H)$ for each $u' \in C'$ contrary to our assumption that C' is a dominating. Thus, C is a minimum dominating set of $G \circ H$, that is, C is a γ_s-*set* of $G \circ H$. Since $C \cap S = \varnothing$, it follows that $S \subseteq V(G \circ H) \setminus C$ is an inverse secure dominating set of $G \circ H$.

Suppose that statement (ii) holds. Then $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$ is a dominating set of $G \circ H$ where for each $v \in V(G)$ $S_v = V(H^v) \setminus \{u\}$ and $\{u\}$ is a dominating set of H^v by Remark 2.4. Thus, it is clear that S is a secure dominating set of $G \circ H$. Now, let $C = \bigcup_{v \in V(G)} (V(H^v) \setminus S_v)$. By similar arguments above, C ia a γ_s-*set* of $G \circ H$. Since $C \cap S = \varnothing$, it follows that $S \subseteq V(G \circ H) \setminus C$ is an inverse secure dominating set of $G \circ H$.

Suppose that statement (iii) holds, then $S = (\bigcup_{v \in V(G)} S_v)$ is a dominating set of $G \circ H$ by Remark 2.4. Let $u \in V(G \circ H) \setminus S$. Then there exists $x \in S$ such that $xu \in E(G \circ H)$ and $S_u = (S \setminus \{x\}) \cup \{u\} = \bigcup_{v' \in (V(G) \setminus \{v\})} S_{v'} \cup [(S_v \setminus \{x\}) \cup \{u\}]$. If $u \notin V(G)$, then for each $v \in V(G)$, $(S_v \setminus \{x\}) \cup \{u\}$ is a dominating set of H^v (note that S_v is a secure dominating set). Since for each $v' \in V(G \setminus \{v\})$, S'_v is a dominating set in $H^{v'}$, it follows that S_u is a dominating set of $V(G \circ H)$. If $u \in V(G)$, then for each $v \in V(G)$, $(S_v \setminus \{x\}) \cup \{u\}$ is a dominating set in $v + H^v$. Again S_u is a dominating set of $G \circ H$. Thus, S is a secure dominating set of $G \circ H$. This implies that there exists $C \in G \circ H$ such that $C \cap S = \varnothing$ and C is a minimum secure dominating set of $G \circ H$. Hence, S is an inverse dominating set of $G \circ H$. ■

Corollary 2.6. Let G be connected graph and $H = K_n$ with $n \geq 2$. Then $\gamma_s^{-1}(G \circ H) = |V(G)|$.

References

[1] E.C. Castillano, R.L. Ugbinada, and S.R. Canoy, Jr. Secure domination in the join of graphs, *Applied Mathematical Sciences*, 8(105), 5203–5211. http://dx.doi.org/10.12988/ams.2014.47519

[2] E.J. Cockayne, and S.T. Hedetniemi *Towards a theory of domination in graphs*, Networks, (1977) 247–261.

[3] E.J. Cockayne, O. Favaron and C.M. Mynhardt *Secure domination, weak Roman domination and forbidden subgraphs*, Bull. Inst. Combin. Appl. 39(2003) 87–100.

[4] E.J. Cockayne, *Irredundance, secure domination and maximum degree in trees*, Discrete Math, 307(2007) 12–17.

[5] E.L. Enriquez *Secure convex dominating sets in corona of graphs*, Applied Mathematical Sciences, Vol. 9, 2015, no. 120, 5961–5967. http://dx.doi.org/10.12988/ams.2015.58510

[6] E.L. Enriquez, and E.M. Kiunisala *Inverse secure domination in graphs*, Global Journal of Pure and Applied Mathematics, Vol. 12, 2016, no. 1, 147–155.

[7] E.L. Enriquez, and S.R. Canoy, Jr. Secure convex comination in a graph. *International Journal of Mathematical Analysis*, 9(7), 317–325. http://dx.doi.org/10.12988/ajma.2015.412412

[8] E.M. Kiunisala and F.P. Jamil, *Inverse domination Numbers and disjoint domination numbers of graphs under some binary operations,* Applied Mathematical Sciences, Vol. 8, 2014, no. 107, 5303–5315.

[9] G. Chartrand and P. Zhang. *A First Course in Graph Theory*, Dover Publication, Inc., New York, 2012.

[10] O. Ore. *Theory of Graphs*. American Mathematical Society, Provedence, R.I., 1962.

[11] V.R. Kulli and S.C. Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sci. Letters, 14(1991) 473–475.

[12] T. Tamizh Chelvan, T. Asir and G.S. Grace Prema, *Inverse domination in graphs*, Lambert Academic Publishing, 2013.