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Multivariate fermionic p-adic integral on $\mathbb{Z}_p$ associated with Frobenius-Euler polynomials and numbers

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Abstract

In this paper, we investigate some integral equations which are related to the fermionic p-adic integral on $\mathbb{Z}_p$. From our investigation, we derive some identities of Frobenius-Euler polynomials.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denoted by the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$.

Let $f(x)$ be continuous function on $\mathbb{Z}_p$. Then the *fermionic p-adic integral on* $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_i(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [9, 5]).}\end{aligned} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f_n(x)d\mu_{-1}(x) + (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2\sum_{l=0}^{n-1} f(l)(-1)^{n-1-l}, \tag{1.2}$$

where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$, (see [5-10]). In particular, if we take $n = 1$, then we have

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{1.3}$$

For $u \in \mathbb{C}_p$ with $|1-u|_p < 1$ and $u \neq 1$, the *Frobenius-Euler polynomials* are defined by the generating function to be

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x|u)\frac{t^n}{n!}, \text{ (see [1-4]).} \tag{1.4}$$

When $x = 0$, $H_n(0|u) = H_n(u)$ are called the *Frobenius-Euler numbers*.

For $r \in \mathbb{N}$, the *higher-order Frobenius-Euler polynomials* are given by the generating function to be

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\frac{t^n}{n!}, \text{ (see [6-14]).} \tag{1.5}$$

When $x = 0$, $H_n^{(r)}(0|u) = H_n^{(r)}(u)$ are called the *higher-order Frobenius-Euler numbers*.

The *multiple Frobenius-Euler polynomials* are defined by Kim to be

$$\begin{aligned} &\left(\frac{1-u_1}{e^t-u_1}\right) \times \left(\frac{1-u_2}{e^t-u_2}\right) \times \cdots \times \left(\frac{1-u_r}{e^t-u_r}\right) e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|u_1, u_2, \ldots, u_r)\frac{t^n}{n!}, \text{ (see [8, 10]).} \end{aligned} \tag{1.6}$$

When $x = 0$, $H_n^{(r)}(0|u_1, u_2, \ldots, u_r)$ are called *multiple Frobenius-Euler numbers*.

From (1.3), we can derive the following equation:

$$\frac{1}{2}\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{1}{qe^t+1} = \sum_{n=0}^{\infty} H_n(-q^{-1})\frac{t^n}{n!}\frac{1}{(1+q^{-1})q} \tag{1.7}$$

and

$$\frac{1}{2}\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{1}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} H_n(x|-q^{-1})\frac{t^n}{n!}\frac{1}{(1+q)}, \tag{1.8}$$

where $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $q \neq 1$.

By (1.8), we get

$$\frac{1}{2}\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = \frac{1}{1+q} H_n(x|-q^{-1}),\ (n \geq 0). \tag{1.9}$$

In this paper, we consider the multivariate fermionic p-adic integral on $\mathbb{Z}_p$ and investigate some equations of those integrals. From our investigation, we derive new and interesting identities for the Frobenius-Euler numbers and polynomials.

2. Fermionic p-adic integral on $\mathbb{Z}_p$ associated with Frobenius-Euler polynomials

In this section, we assume that $q_i (i = 0, 1, 2, \ldots) \in \mathbb{C}_p$ with $|1 - q_i|_p < 1$. Now we consider the following multivariate fermionic p-adic integral on $\mathbb{Z}_p$: for $r \in \mathbb{N}$

$$I = \frac{1}{2^r}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q_1^{x_1} q_2^{x_2} \cdots q_r^{x_r} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.1}$$

From (1.3), we note that

$$\begin{aligned} I &= \frac{1}{(q_1e^t+1)(q_2e^t+1)\cdots(q_re^t+1)} e^{xt} \\ &= \left(\frac{1+q_1^{-1}}{e^t+q_1^{-1}}\right)\left(\frac{1+q_2^{-1}}{e^t+q_2^{-1}}\right)\cdots\left(\frac{1+q_r^{-1}}{e^t+q_r^{-1}}\right)\frac{1}{(q_1+1)(q_2+1)\cdots(q_r+1)} e^{xt} \\ &= \left(\prod_{l=1}^{r}\frac{1}{q_l+1}\right)\sum_{n=0}^{\infty} H_n^{(r)}(x|-q_1^{-1}, -q_2^{-1}, \ldots, -q_r^{-1})\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

Thus, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$ and $r \in \mathbb{N}$, we have

$$\begin{aligned} &\frac{1}{2^r}\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\prod_{l=1}^{r} q_l^{x_l})(x_1+x_2+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) \\ &\quad = \left(\prod_{l=1}^{r}\frac{1}{q_l+1}\right) H_n^{(r)}(x|-q_1^{-1}, -q_2^{-1}, \ldots, -q_r^{-1}). \end{aligned}$$

Let $x = 0$ in Theorem 2.1. Then, we have

$$H_n^{(r)}(0|-q_1^{-1}, \ldots -q_r^{-1}) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \ldots, l_r} \prod_{i=1}^{r} H_{l_i}(0|-q_i^{-1}).$$

It is easy to show that

$$\begin{aligned}&\frac{1}{2^{r-1}}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\prod_{l=1}^{r}q_l^{x_l}\right)e^{(x_1+\cdots+x_r+x)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\&=\sum_{k=1}^{r}\int_{\mathbb{Z}_p}q_k^{x_k}e^{x_kt}d\mu_{-1}(x_k)\left(\prod_{j=1,j\neq k}^{r}\left(1-\frac{q_j}{q_k}\right)^{-1}\right)e^{xt}.\end{aligned}\tag{2.3}$$

Now, we observe that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{Z}_p}q_k^{x_k}e^{(x_k+x)t}d\mu_{-1}(x_k)&=\frac{1+q_k^{-1}}{q_ke^t+1}e^{xt}\frac{1}{1+q_k^{-1}}\\&=\left(\frac{1}{q_k+1}\right)\left(\frac{1+q_k^{-1}}{e^t+q_k^{-1}}e^{xt}\right).\end{aligned}\tag{2.4}$$

From (1.4), (2.2), (2.3) and (2.4), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}H_n^{(r)}(x|-q_1^{-1},-q_2^{-1},\ldots,-q_r^{-1})\frac{t^n}{n!}\left(\prod_{l=1}^{r}\frac{1}{q_l+1}\right)\\&=\sum_{n=0}^{\infty}\left\{\sum_{k=1}^{r}\left(\frac{1}{q_k+1}\right)H_n(x|-q_k^{-1})\left(\prod_{j=1,j\neq k}^{r}\left(1-\frac{q_j}{q_k}\right)^{-1}\right)\right\}\frac{t^n}{n!}.\end{aligned}\tag{2.5}$$

By comparing the coefficients on the both sides of (2.5), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{1}{(q_1+1)(q_2+1)\cdots(q_r+1)}H_n^{(r)}(x|-q_1^{-1},-q_2^{-1},\ldots,-q_r^{-1})\\&=\sum_{k=1}^{r}\frac{H_n(x|-q_k^{-1})}{q_{k+1}}\prod_{j=1,j\neq k}^{r}\left(1-\frac{q_j}{q_k}\right)^{-1}.\end{aligned}$$

It is easy to show that

$$\begin{aligned}&\frac{1}{(q_0e^t-1)(q_1e^t-1)\cdots(q_{r-1}e^t-1)}\\&=\frac{1}{q_0q_1\cdots q_{r-1}e^{rt}-1}\sum_{i=0}^{r-1}\left(\prod_{l=0}^{i-1}\frac{q_le^t}{q_le^t-1}\right)\left(\prod_{l=i+1}^{r-1}\frac{1}{q_le^t-1}\right).\end{aligned}\tag{2.6}$$

Thus, by (2.6), we get

$$
\begin{aligned}
&H_n^{(r)}(0|q_0^{-1}, q_1^{-1}, \dots, q_{r-1}^{-1}) \\
&\quad = \sum_{i=0}^{r-1} \frac{q_0q_1\cdots q_{i-1}(q_i-1)}{q_0q_1\cdots q_{r-1}-1} \sum_{l_0+l_1+\cdots+l_{r-1}=n} \binom{n}{l_0, l_1, \dots, l_{r-1}} r^{l_i} H_{l_0}(1|q_0^{-1})\cdots \\
&\quad \times H_{l_{i-1}}(1|q_{i-1}^{-1}) H_{l_i}(0|q_0^{-1}q_1^{-1}\cdots q_{r-1}^{-1}) H_{l_{i+1}}(0|q_{i+1}^{-1})\cdots H_{l_{r-1}}(0|q_{r-1}^{-1}),
\end{aligned}
\tag{2.7}
$$

where $r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$.

We observe that

$$
\begin{aligned}
&q_{r-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\prod_{l=0}^{r-1} q_l^{x_l}\right) e^{(x_0+x_1+\cdots+x_{r-1}+x+1)t}\, d\mu_{-1}(x_0)\cdots d\mu_{-1}(x_r) \\
&\quad + \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\prod_{l=0}^{r-1} q_l^{x_l}\right) e^{(x_0+x_1+\cdots+x_{r-1}+x)t}\, d\mu_{-1}(x_0)\cdots d\mu_{-1}(x_{r-1}) \\
&\quad = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\prod_{l=0}^{r-2} q_l^{x_l}\right) e^{(x_0+x_1+\cdots+x_{r-2}+x)t}\, d\mu_{-1}(x_0)\cdots d\mu_{-1}(x_{r-2}).
\end{aligned}
\tag{2.8}
$$

From (2.8), we have

$$
\begin{aligned}
&q_{r-1}H_n^{(r)}(x+1|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-1}^{-1}) + H_n^{(r)}(x|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-1}^{-1}) \\
&\quad = \frac{2}{q_{r-1}+1} H_n^{(r-1)}(x|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-2}^{-1}).
\end{aligned}
\tag{2.9}
$$

where $r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have

$$
\begin{aligned}
&q_{r-1}H_n^{(r)}(x+1|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-1}^{-1}) + H_n^{(r)}(x|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-1}^{-1}) \\
&\quad = \frac{2}{q_{r-1}+1} H_n^{(r-1)}(x|-q_0^{-1}, -q_1^{-1}, \dots, -q_{r-2}^{-1}).
\end{aligned}
$$

It is not difficult to show that

$$
q_0q_1\cdots q_{r-1}e^{rt} - 1 = \sum_{i=0}^{r-1}(q_ie^t - 1)\left(\prod_{j=0}^{i-1} q_je^t\right), \tag{2.10}
$$

where $\left.\left(\prod_{j=0}^{i-1} q_je^t\right)\right|_{i=0} = 1$.

We observe that

$$q_ie^t - 1 = \left(\prod_{l=0}^{i-1} \frac{1}{q_le^t - 1}\right)\left(\prod_{l=i+1}^{r-1} \frac{1}{q_le^t - 1}\right)\left(\prod_{l=0}^{r-1} \frac{1}{q_le^t - 1}\right), \tag{2.11}$$

where $i \in \mathbb{N}$.

Thus, by (2.10) and (2.11), we get

$$\prod_{l=0}^{r-1} \frac{1}{q_le^t - 1} = \frac{1}{q_0q_1\cdots q_{r-1}e^{rt} - 1} \sum_{i=0}^{r-1} \left(\prod_{l=0}^{i-1} \frac{q_le^t}{q_le^t - 1}\right)\left(\prod_{l=i+1}^{r-1} \frac{1}{q_le^t - 1}\right). \tag{2.12}$$

From (2.12), we can derive the following equations:

$$\begin{aligned} &H_n^{(r)}(0|q_0^{-1}, q_1^{-1}, \ldots, q_{r-1}^{-1}) \\ &= \frac{q_0q_1\cdots q_{i-1}(q_i - 1)}{q_0q_1\cdots q_{r-1} - 1} \sum_{m=0}^{n}\sum_{n_1=0}^{m} \binom{m}{n_1}\binom{n}{m} H_{n_1}^{(i)}(r|q_0^{-1}, \ldots, q_{i-1}^{-1}) \\ &\quad \times H_{m-n_1}^{(r-i-1)}(0|q_{i+1}, \ldots, q_{r-1}) H_{n-m}(0|q_0^{-1}q_1^{-1}\cdots q_{r-1}^{-1})r^{n-m}. \end{aligned} \tag{2.13}$$

where $n \geq 0$.

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