Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 1095-1106

http://www.ripublication.com

On Measurable Separable-Normal Radon Measure Manifold

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Abstract

In this paper, we show that a non-empty set of composition of C^{∞} measurable homeomorphisms and Radon measure-invariant functions induce a group structure on the set $\mathcal{M} = \{ M_1, M_2, \dots., M_n\}$ of measurable separable-normal Radon measure manifolds where the measurable separable-normal property remains invariant under measurable homeomorphisms and Radon measure-invariant function.

Keywords: Measurable separable-normal Radon measure manifold, Measurable homeomorphism and Radon measure-invariant map.

Mathematics Subject Classification: 28-XX, 54-XX, 58-XX, 57N-XX.

1. Introduction

The concept of Measure Manifold was introduced and the invariance of Radon measurable normality property on such measure manifold under C^{∞} measurable homeomorphism and Radon measure-invariant transformation was investigated and proved by the author S. C. P. Halakatti [4], [7], [8].
In this paper, the invariance of separable-normal property P, $\mu_R - a.e.$ (where μ_R is a Radon measure) on Radon measure manifold under C^{∞} measurable homeomorphism and Radon measure-invariant map is investigated by S. C. P. Halakatti. Further, it is shown that a non-empty set of such C^{∞} measurable homeomorphisms and Radon measure-invariant functions induce a group structure on $\mathcal{M}$ where the property P

remains invariant. On these developed ideas we continue to study some more properties on measurable separable-normal Radon measure manifolds.

2. Preliminaries

In this section, we consider some basic concepts:

Definition 2.1: Radon Measure on (R^n, τ, Σ, μ) [11]:

A Radon measure on a topological measure space (R^n, τ, Σ, μ) is a positive Borel measure $\mu : B \longrightarrow [0,\infty]$ which is finite on compact subsets and is inner regular in the sense that for every Borel subset E $\subset (R^n, \tau, \Sigma, \mu)$ we have

(i) μ(E) = sup{ μ(K) : K $\subseteq$ E, K $\in \mathcal{K}$} where $\mathcal{K}$ denote the family of all compact subsets. μ is outer regular on a family $\mathcal{F}$ of Borel subsets if for every E $\in \mathcal{F} \subset (R^n, \tau, \Sigma, \mu)$ we have,

(ii) μ(E) = inf{ μ(O) : O $\supseteq$ E, O $\in \mathcal{O}$} where $\mathcal{O}$ denote the family of all open subsets.

Definition 2.2: Measurable Homeomorphism [5]:

Let (M, τ) be a second countable, Hausdorff topological space and (R^n, τ, Σ, μ) be a measure space. Then the function $\varphi : U \subset M \longrightarrow (R^n, \tau, \Sigma, \mu)$ is called measurable homeomorphism if

(i) φ is bijective and bi continuous

(ii) φ and φ^{-1} are measurable

Definition 2.3: Dense subset of a topological space [10]:

A subset A of a topological space X is called dense in X if closure of A is X i.e., $\overline{\text{A}}$ = X.

Definition 2.4: Separable topological space [10]:

A topological space (R^n, τ) is said to be separable if it has countable dense subsets.

Inverse Function Theorem on Measure Manifold [5]:

Let $F : (M, \tau, \Sigma, \mu) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_1)$ be a C^∞ measurable homeomorphism and measure-invariant map of measure manifolds and suppose that $F_{*p} : T_p(M) \longrightarrow T_{f(p)}(M_1)$ is a linear isomorphism at some p of M. Then there exists a measure chart (U, φ) of p in M such that the restriction of F to (U, φ) is a diffeomorphism onto a measure chart (V, φ) of $F(p)$ in M_1. This implies for every function F which is measurable homeomorphism and measure-invariant has a C^∞ F^{-1}: $(M_1, \tau_1, \Sigma_1, \mu_1) \longrightarrow (M, \tau, \Sigma, \mu)$ which is also measurable homeomorphism and measure-invariant.

Definition 2.5: Radon measure chart [7]:
A measurable chart ((U,$\tau_{1/U}$,$\Sigma_{1/U}$), φ) of a measurable manifold (M,, τ_1, Σ_1) equipped with a Radon measure $\mu_{R_{1/U}}$ is called a Radon measure chart denoted by ((U, $\tau_{1/U}$,$\Sigma_{1/U}$, $\mu_{R_{1/U}}$), φ) satisfying the following conditions:

a) φ is measurable homeomorphism, if for every compact Borel subset $\bar{V} \in (R^n$, τ, Σ), $\varphi^{-1}(\bar{V})$ = (U, φ) ∈ (M, τ_1, Σ_1) is also Borel measurable.

b) φ is Radon measure-invariant. that is, $\mu_{R_1}(\varphi^{-1}(\bar{V})) = \mu_R(\bar{V})$ where $\varphi^{-1}(\bar{V})$ = (U, φ) a Borel measurable chart which satisfies the Radon measure conditions:

I. (i) For p ∈ V ⊂ $\bar{V}$ ∈ Σ ; $\mu_R(\bar{V}) < \infty$;

(ii) For any Borel compact subset $\bar{V} \subset (R^n$, τ, Σ), $\mu_R(\bar{V})$ = sup{$\mu_R(E_i)$; i ∈ I : $E_i \subseteq \bar{V}$: E_i compact and measurable}

II. (i) For $\varphi^{-1}(\bar{V}) \in \Sigma_1$, where Σ_1 a σ-algebra induced on second countable Hausdorff topological space, $\mu_{R_1}(\varphi^{-1}(\bar{V})) < \infty$; where $\varphi^{-1}(\bar{V})$ = (U, φ) is a measurable chart

(ii) For any Borel subset $\varphi^{-1}(\bar{V})$ ⊂ (M, τ_1 , Σ_1), $\mu_{R_1}(\varphi^{-1}(\bar{V}))$ = sup{$\mu_{R_1}(\varphi^{-1}(E_i))$; i ∈ I : $\varphi^{-1}(E_i) \subseteq \varphi^{-1}(\bar{V})$: $\varphi^{-1}(E_i)$ compact and measurable}.

Definition 2.6: Radon measure atlas [7]:
By an R^n-Radon measure atlas of class C^k (k ≥ 1) on measurable manifold (M, τ_1,Σ_1), we mean a countable collection ($\mathcal{A}$,$\tau_{1/\mathcal{A}}$,$\Sigma_{1/\mathcal{A}}$,$\mu_{R_{1/\mathcal{A}}}$) of n-dimensional Radon measure charts ((U_i,τ_{1/U_i},Σ_{1/U_i},$\mu_{R_{1/U_i}}$),φ_i) for all i ∈ N on (M, τ_1,Σ_1, μ_{R_1}) satisfying the following conditions:

(a_1) $\bigcup_{i\in I}((U_i, \tau_{1/U_i}, \Sigma_{1/U_i}, \mu_{R_{1/U_i}})$ = (M, τ_1,Σ_1, μ_{R_1}) i.e., the countable union of all Radon measure charts in ($\mathcal{A}$,$\tau_{1/\mathcal{A}}$,$\Sigma_{1/\mathcal{A}}$,$\mu_{R_{1/\mathcal{A}}}$) cover (M, τ_1,Σ_1, μ_{R_1}).

(a_2) For any pair of Radon measure charts ((U_i,τ_{1/U_i},Σ_{1/U_i},$\mu_{R_{1/U_i}}$),φ_i) and ((U_j,τ_{1/U_j},Σ_{1/U_j},$\mu_{R_{1/U_j}}$),φ_j) in ($\mathcal{A}$,$\tau_{1/\mathcal{A}}$,$\Sigma_{1/\mathcal{A}}$,$\mu_{R_{1/\mathcal{A}}}$), the transition maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are:

1. differentiable maps of class C^k (k≥1)
i.e., $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j) \subseteq (R^n, \tau_2, \Sigma_2, \mu_{R_2})$ and $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j) \subseteq (R^n, \tau_2, \Sigma_2, \mu_{R_2})$ are differentiable maps of class C^k (k ≥ 1).

2. measurable
Transition maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are measurable functions if,

a) any Borel subset K ⊆ $\varphi_i(U_i \cap U_j)$ is measurable in (R^n,τ_2,Σ_2,μ_{R_2}) then, $(\varphi_i \circ \varphi_j^{-1})^{-1}$(K) ∈ $\varphi_j(U_i \cap U_j)$ is also measurable.

b) $\varphi_j \circ \varphi_i^{-1}$ is measurable if S ⊆ $\varphi_j(U_i \cap U_j)$ is measurable in (R^n,τ_2, Σ_2,μ_{R_2}), then $(\varphi_j \circ \varphi_i^{-1})^{-1}$(S) ∈ $\varphi_i(U_i \cap U_j)$is also measurable.

c) For any two Radon measure atlases ($\mathcal{A}_1$,$\tau_{1/\mathcal{A}_1}$,$\Sigma_{1/\mathcal{A}_1}$,$\mu_{R_{1/\mathcal{A}_1}}$) and ($\mathcal{A}_2$,$\tau_{1/\mathcal{A}_2}$,$\Sigma_{1/\mathcal{A}_2}$,$\mu_{R_{1/\mathcal{A}_2}}$), we say that a mapping T: $\mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is measurable if $T^{-1}(E)$ is measurable for every Radon measurable chart A = (U,$\tau_{1/U}$,$\Sigma_{1/U}$,$\mu_{R_{1/U}}$) $\subset$ ($\mathcal{A}_2$,$\tau_{1/\mathcal{A}_2}$,$\Sigma_{1/\mathcal{A}_2}$,$\mu_{R_{1/\mathcal{A}_2}}$) and the mapping is Radon measure preserving if $\mu_{R_{1/\mathcal{A}_1}} = \mu_{R_{1/\mathcal{A}_2}}$ satisfying the Radon measure conditions:

I. (i) $\forall$ p $\in A \subset \mathcal{A}_2$, $\mu_{R_{1/\mathcal{A}_2}}(A) <\infty$; where A is a Borel compact subset of $\mathcal{A}_2$.

(ii) $\mu_{R_{1/\mathcal{A}_2}}(A)$ = sup{ $\mu_{/\mathcal{A}_2}(E_i)$; $i \in$ I : $E_i \subseteq A$: E_i compact and measurable}. T^{-1}: $\mathcal{A}_2 \longrightarrow \mathcal{A}_1$ is Radon measure preserving transformation if $\mu_{R_{1/\mathcal{A}_2}}(E)$ = $\mu_{R_{1/\mathcal{A}_1}}(T^{-1}(E))$ where $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mu_{R_{1/\mathcal{A}_2}} = \mu_{R_{1/\mathcal{A}_1}}$ satisfying the Radon measure conditions:

II. (i) $\forall\, T^{-1}$(p) $\in T^{-1}(A) \subset \mathcal{A}_1$; $\mu_{R_{1/\mathcal{A}_1}}(T^{-1}(A)) <\infty$;

(ii) $\mu_{R_{1/\mathcal{A}_1}}(T^{-1}(A))$ = sup{ $\mu_{/\mathcal{A}_2}(T^{-1}(E_i))$; $i \in$I: $T^{-1}(E_i)) \subseteq T^{-1}(A)$: $T^{-1}(E_i)$ compact and measurable}

Then, we call T a Radon-measure preserving transformation.
(a_4) If a measurable transformation T: $\mathcal{A} \longrightarrow \mathcal{A}$ preserves a Radon measure μ_{R_1}, then we say that μ_{R_1} is T-invariant.
If T is invariant and if both T and T^{-1} are measurable and Radon measure preserving then we call T an invertible Radon measure preserving transformation.
Let A_m^k(M) denotes the set of all Radon measure atlases of class C^k on (M, τ_1, Σ_1, μ_{R_1}). Two Radon measure atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ in A_m^k(M) are said to be equivalent if ($\mathcal{A}_1 \cup \mathcal{A}_2$) $\in A_m^k$(M). In order to have that $\mathcal{A}_1 \cup \mathcal{A}_2$ to be a member of A_m^k(M) we require that for every Radon measure chart $(U_i, \varphi_i) \in \mathcal{A}_1$ and $(V_j, \varphi_j) \in \mathcal{A}_2$, the set of $\varphi_i(U_i \cap V_j)$ and $\varphi_j(U_i \cap V_j)$ are Borel subsets in (R^n,τ_2, Σ_2,μ_{R_2}) and maps $\varphi_i \circ \varphi_j^{-1}$ and $\varphi_j \circ \varphi_i^{-1}$ are of class C^k and are measurable. The relation introduced is an equivalence relation in A_m^k(M) and hence partitions A_m^k(M) into disjoint equivalence classes. Each of these equivalence classes forms a differentiable structure of class C^k on (M, τ_1, Σ_1, μ_{R_1}).

Definition 2.7: Radon measure manifold [7]:
A measurable manifold (M, τ_1, Σ_1) equipped with a Radon measure μ_{R_1} is called a Radon measure manifold (M, τ_1, Σ_1, μ_{R_1}).

3. Main Results

The following results are introduced and developed by S. C. P. Halakatti and we carry our study on these results.

Definition 3.1: Measurable separable-normal Radon measure space:
A Radon measure space $(R^n, \tau, \Sigma, \mu_R)$ is said to be measurable separable-normal, if for any two disjoint dense Borel subsets $\overline{E_\iota}$ and $\overline{E_J}$ which are Radon measurable there exists two Borel subsets A_i and A_j of $(R^n, \tau, \Sigma, \mu_R)$ such that $\overline{E_\iota} \subseteq A_i$, $\overline{E_J} \subseteq A_j$, $A_i \cap A_j = \emptyset$ and $\mu_R(A_i) > 0$ and $\mu_R(A_j) > 0$.
By using theorem 3.4 [7], the measurable separable-normal property can be studied on a Radon measure manifold $(\mathrm{M}, \tau_1, \Sigma_1, \mu_{R_1})$ as follows:

Theorem 3.1: Let $(\mathrm{R}^n, \tau, \Sigma, \mu_R)$ be a measurable separable-normal Radon measure space and $(\mathrm{M}, \tau_1, \Sigma_1, \mu_{R_1})$ be a Radon measure manifold. If there exists a C^∞ measurable homeomorphism and Radon measure-invariant map $\varphi : (\mathrm{M}, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (\mathrm{R}^n, \tau, \Sigma, \mu_R)$ then $(\mathrm{M}, \tau_1, \Sigma_1, \mu_{R_1})$ is a measurable separable-normal Radon measure manifold.

Proof: Suppose $(R^n, \tau, \Sigma, \mu_R)$ is a measurable separable-normal Radon measure space and $(M, \tau_1, \Sigma_1, \mu_{R_1})$ is a Radon measure manifold. If there exists a C^∞ measurable homeomorphism and Radon measure-invariant map $\varphi : (M, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^n, \tau, \Sigma, \mu_R)$ then we show that $(M, \tau_1, \Sigma_1, \mu_{R_1})$ is a measurable separable-normal Radon measure manifold by using theorem 3.4[7].
Since $(R^n, \tau, \Sigma, \mu_R)$ is measurable separable-normal with Radon measure μ_R, then there exists two disjoint dense Borel subsets $\overline{E_\iota}$ and $\overline{E_J} \in \mathcal{K}$ and two Borel subsets A_i and $A_j \in \mathcal{G}$ which are Radon measurable in $(R^n, \tau, \Sigma, \mu_R)$ such that $\overline{E_\iota} \subseteq A_i$ and $\overline{E_J} \subseteq A_j$ and $\overline{E_\iota} = (R^n, \tau, \Sigma, \mu_R)$ or $\overline{E_J} = (R^n, \tau, \Sigma, \mu_R)$ and $A_i \cap A_j = \emptyset$, also $\mu_R(A_i) > 0$ and $\mu_R(A_j) > 0$ where $\mathcal{K}$ is countable family of countable dense/compact Borel subsets of $(R^n, \tau, \Sigma, \mu_R)$ and $\mathcal{G}$ is the family of Borel subsets of $(R^n, \tau, \Sigma, \mu_R)$. On $(R^n, \tau, \Sigma, \mu_R)$, $\overline{E_\iota}$ and $\overline{E_J}$ are Radon measurable.
Since $\overline{E_\iota}$ and $\overline{E_J}$ are compact Borel subsets of $(R^n, \tau, \Sigma, \mu_R)$, for every Borel cover $\overline{E_\iota} \subseteq \{\bigcup_{i\in I} P_i\}$ there exists a Borel sub cover $\{\bigcup_{j=1}^{n} P_{i_j}\}$ and for every Borel cover $\overline{E_J} \subseteq \{\bigcup_{i\in I} Q_i\}$ there exists a Borel sub cover $\{\bigcup_{j=1}^{n} Q_{i_j}\}$ such that $P_{i_j} \subseteq \overline{E_\iota}$ and $Q_{i_j} \subseteq \overline{E_J}$ such that:

I. For $\overline{E_\iota} \subseteq A_i \subset (R^n, \tau, \Sigma, \mu_R)$,
   (i) $\mu_R(\overline{E_\iota}) < \infty$,
   (ii) $\mu_R(\overline{E_\iota}) = \sup\{ \mu_R\left(P_{i_j}\right) : \forall\, j \in J, P_{i_j} \subseteq \overline{E_\iota}, \forall\, P_{i_j} \in \mathcal{K} \}$,
   (iii) $\mu_R(\overline{E_\iota}) = \inf\{ \mu_R(A_i) : \forall\, i \in I, A_i \supseteq \overline{E_\iota}, \forall\, A_i \in \mathcal{G} \}$

II. For $\overline{E_J} \subseteq A_j \subset (R^n, \tau, \Sigma, \mu_R)$,
   (i) $\mu_R\left(\overline{E_J}\right) < \infty$,
   (ii) $\mu_R\left(\overline{E_J}\right) = \sup\{ \mu_R\left(Q_{i_j}\right) : \forall\, j \in J, Q_{i_j} \subseteq \overline{E_J}, \forall\, Q_{i_j} \in \mathcal{K} \}$,
   (iii) $\mu_R\left(\overline{E_J}\right) = \inf\{ \mu_R\left(A_j\right) : \forall\, j \in J, A_j \supseteq \overline{E_J}, \forall\, A_j \in \mathcal{G} \}$.

If $\varphi : (M, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^n, \tau, \Sigma, \mu_R)$ is a C^∞ measurable homeomorphism and Radon measure-invariant map, then by using theorem 3.4[7], for each $\overline{E_\iota}$, $\overline{E_J} \in (R^n, \tau, \Sigma, \mu_R)$, $\exists$ $\varphi^{-1}(\overline{E_\iota}) = \overline{K_\iota}$, $\varphi^{-1}(\overline{E_J}) = \overline{K_J}$ which are dense /compact Borel subsets belonging to $(M, \tau_1, \Sigma_1, \mu_{R_1})$ and for each A_i, $A_j \in (R^n, \tau, \Sigma, \mu_R)$ $\exists$ $\varphi^{-1}(A_i) = (U_i, \varphi_i)$, $\varphi^{-1}(A_j) = (U_j, \varphi_j)$ are Radon measure charts belonging to $(M, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\overline{K_\iota} = (M, \tau_1, \Sigma_1, \mu_{R_1})$ or $\overline{K_J} = (M, \tau_1, \Sigma_1, \mu_{R_1})$ and $\overline{K_\iota} \subseteq (U_i, \varphi_i)$, $\overline{K_J} \subseteq (U_j, \varphi_j)$, $\overline{K_\iota} \cap \overline{K_J} = \emptyset$, $\mu_{R_1}(U_i) > 0$, $\mu_{R_1}(U_j) > 0$ and $\overline{K_\iota}$, $\overline{K_J} \in \varphi^{-1}(\mathcal{K}) = \mathcal{F}$ where $\mathcal{F}$ is a countable family of dense/compact Borel subsets and (U_i, φ_i), $(U_j, \varphi_j) \in \mathcal{A}$ where $\mathcal{A}$ is the family of Radon measure charts of $(M, \tau_1, \Sigma_1, \mu_{R_1})$, which is called Radon measure atlas of $(M, \tau_1, \Sigma_1, \mu_{R_1})$.

On$(M, \tau_1, \Sigma_1, \mu_{R_1})$, $\overline{K_\iota}$ and $\overline{K_J}$ are Radon measurable. Since $\overline{K_\iota}$ and $\overline{K_J}$ are compact Borel subsets of $(M, \tau_1, \Sigma_1, \mu_{R_1})$, for every Borel cover $\overline{K_\iota} \subseteq \{\bigcup_{i \in I} R_i\}$ there exists a Borel sub cover $\{\bigcup_{j=1}^{n} R_{i_j}\}$ and for every Borel cover $\overline{K_J} \subseteq \{\bigcup_{i \in I} S_i\}$ there exists a Borel sub cover $\{\bigcup_{j=1}^{n} S_{i_j}\}$ such that $R_{i_j} \subseteq \overline{K_\iota}$ and $S_{i_j} \subseteq \overline{K_J}$ such that:

III. For $\overline{K_\iota} \subseteq U_i \subset (M, \tau_1, \Sigma_1, \mu_{R_1})$,
- (i) $\mu_{R_1}(\overline{K_\iota}) < \infty$,
- (ii) $\mu_{R_1}(\overline{K_\iota}) = \sup\{ \mu_{R_1}\left(R_{i_j}\right) : \forall\, j \in J,\, R_{i_j} \subseteq \overline{K_\iota},\, \forall\, R_{i_j} \in \mathcal{F} \}$,
- (iii) $\mu_{R_1}(\overline{K_\iota}) = \inf\{ \mu_{R_1}(U_i) : \forall\, i \in I,\, U_i \supseteq \overline{K_\iota},\, \forall\, U_i \in \mathcal{A} \}$

IV. For $\overline{K_J} \subseteq U_j \subset (M, \tau_1, \Sigma_1, \mu_{R_1})$,
- (i) $\mu_{R_1}(\overline{K_J}) < \infty$,
- (ii) $\mu_{R_1}(\overline{K_J}) = \sup\{ \mu_{R_1}\left(S_{i_j}\right) : \forall\, j \in J,\, S_{i_j} \subseteq \overline{K_J},\, \forall\, S_{i_j} \in \mathcal{F} \}$,
- (iii) $\mu_{R_1}(\overline{K_J}) = \inf\{ \mu_{R_1}(U_j) : \forall\, j \in J,\, U_j \supseteq \overline{K_J},\, \forall\, U_j \in \mathcal{A} \}$.

Since φ is measurable homeomorphism and Radon measure-invariant, for each $\overline{E_\iota} \subset (R^n, \tau, \Sigma, \mu_R)$ $\exists$ $\varphi^{-1}(\overline{E_\iota}) = \overline{K_\iota} \subset (M, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\mu_{R_1}(\varphi^{-1}(\overline{E_\iota})) = \mu_R(\overline{E_\iota}) \Rightarrow \mu_{R_1}(\overline{K_\iota}) = \mu_R(\overline{E_\iota})$ and for each $\overline{E_J} \subset (R^n, \tau, \Sigma, \mu_R)$ $\exists$ $\varphi^{-1}(\overline{E_J}) = \overline{K_J} \subset (M, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\mu_{R_1}(\varphi^{-1}(\overline{E_J})) = \mu_R(\overline{E_J}) \Rightarrow \mu_{R_1}(\overline{K_J}) = \mu_R(\overline{E_J})$.

This shows that the existence of C^∞ measurable homeomorphism and Radon measure-invariant map $\varphi : (M, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (R^n, \tau, \Sigma, \mu_R)$ generates a measurable separable-normal Radon measure manifold $(M, \tau_1, \Sigma_1, \mu_{R_1})$.

■

Hence the following definition:

Definition 3.2: Measurable separable-normal Radon measure manifold:

A Radon measure manifold $(M, \tau_1, \Sigma_1, \mu_{R_1})$ is said to be measurable separable-normal if for any two disjoint dense Borel subsets $\overline{K_\iota}$ and $\overline{K_J}$ which are Radon measurable there exists two Radon measure charts (U_i, φ_i) and (U_j, φ_j) of $(M, \tau_1, \Sigma_1, \mu_{R_1})$ such that $\overline{K_\iota} \subseteq (U_i, \varphi_i)$, $\overline{K_J} \subseteq (U_j, \varphi_j)$ and $(U_i, \varphi_i) \cap (U_j, \varphi_j) = \emptyset$, $\mu_{R_1}(U_i) > 0$, $\mu_{R_1}(U_j) > 0$. Using the above theorem, The Inverse Function Theorem on Measure Manifolds [5] can be extended on Radon measure manifold as follows:

Inverse Function Theorem on Radon measure manifolds:
Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be a C^∞ measurable homeomorphism and Radon measure-invariant map of Radon measure manifolds $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where μ_{R_1} and μ_{R_2} are Radon measures on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ respectively and suppose that $F_{*p} : T_p(M_1) \longrightarrow T_{f(p)}(M_2)$ is a linear isomorphism at some point p of M_1. Then there exists a Radon measure chart (U, φ) of p in M_1 such that the restriction of F to (U, φ) is a diffeomorphism onto a Radon measure chart (V, φ) of $F(p)$ in M_2. This implies for every function F which is measurable homeomorphism and Radon measure-invariant has a C^∞ map F^{-1}: $(M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ which is also measurable homeomorphism and Radon measure-invariant.
In the following example, S. C. P. Halakatti has shown that the set D of all dyadic rationals is Radon measurable on measurable separable Radon measure manifold $(R^1, \tau, \Sigma, \mu_R)$.

Example 3.1:
The subset D of all dyadic rationals in [0,1] is Radon measurable on measurable separable Radon measure manifold $(R^1, \tau, \Sigma, \mu_R)$.

Solution:
Let $(R^1, \tau, \Sigma, \mu_R)$ be a measurable separable Radon measure manifold.
Let D = { $\frac{a}{2^b}$, where a is an integer and b is a natural number }
i.e. D = { 0,....., $\frac{1}{2^4}, \frac{1}{2^3}, \frac{3}{2^4}, \frac{1}{2^2}, \frac{5}{2^4}, \frac{3}{2^3}, \frac{7}{2^4}, \frac{1}{2}, \frac{9}{2^4}, \frac{5}{2^3}$,....., 1 }.
D is relatively small dense subset in $[0,1] \subset (R^1, \tau, \Sigma, \mu_R)$, i.e. $\overline{D} = [0,1] \subset (R^1, \tau, \Sigma, \mu_R)$. Since the subset of all dyadic rationals is dense in compact set $[0,1] \subset (R^1, \tau, \Sigma, \mu_R)$, it admits Radon measure satisfying inner and outer regularity properties. We show that D has outer regular and inner regular measure μ_R on $\overline{D}$:
(i) The outer regular measure μ_R on $\overline{D}$ is as follows:

$$\mu_R(\overline{D}) = \lim_{d\to\infty} inf_{d\geq b}\{ \frac{2^{d-b}a-c}{2^d} : d \geq b \text{ in } [0,1]\}$$
$$= 0$$

Therefore, the outer regular measure μ_R on $\overline{D}$ is 0.

(ii) The inner regular measure μ_R on $\overline{D}$ is as follows:

$$\mu_R(\overline{D}) = \lim_{d\to\infty} sup_{d\geq b}\{ \frac{2^{d-b}a-c}{2^d} : d \geq b \text{ in } [0,1]\}$$

= $\lim_{d\to\infty} sup_{d\geq b}\{ \frac{2^{d-b}a}{2^d} - \frac{c}{2^d} : d \geq b$ in [0,1]}

= $\lim_{d\to\infty} sup_{d\geq b}\{ \frac{2^d 2^{-b}a}{2^d} - \frac{c}{2^d}: d \geq b$ in [0,1]}

= $\lim_{d\to\infty} sup_{d\geq b}\{ \frac{a}{2^b} - \frac{c}{2^d}: d \geq b$ in [0,1]}

= $\frac{a}{2^b}$

If a = 0, b = 1, 2, 3 then $\frac{a}{2^b}$ = 0

If a = 1, b = 1, 2, 3 then $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}$,...

If a = 2, b = 1, 2, 3 then $\frac{2}{2}, \frac{2}{2^2}, \frac{2}{2^3}$,...

Now if a = 2, b = 1 then $\frac{a}{2^b} = \frac{2}{2^1}$ = 1

Therefore, the inner regular measure μ_R on $\overline{D}$ is 1 i.e. $\mu_R(\overline{D})$ = 1.
Hence, the Radon measure μ_R which is a positive Borel measure is outer regular having the value zero and inner regular having the value 1 on the dense subset $\mu_R(\overline{D})$ = [0,1].

Note:
In this paper, by a Radon measure manifold we mean a measurable separable-normal Radon measure manifold. Also, we denote the set of all C^∞ measurable homeomorphisms and Radon measure-invariant functions $F_1, F_2, \ldots, F_n$ by G = { $F_1, F_2, \ldots, F_n$ }.

Remark:
We observe that a measurable separable-normal property say P holds $\mu_R - a.e.$ on any dense Borel set $A \subset (M, \tau, \Sigma, \mu_R)$ if the set A = { $\forall\ \overline{K_\iota} \subset (M, \tau, \Sigma, \mu_R)$: P($\overline{K_\iota}$) is true } has positive measure i.e. $\mu_R(A) > 0$.
Suppose P does not hold $\mu_R - a.e.$ on the set $A \subset (M, \tau, \Sigma, \mu_R)$ then $\mu_R(A)$ = 0. Then we identify A as a dark region of (M, τ, Σ, μ_R).
Now by using Inverse Function Theorem on Radon measure manifold, S. C. P. Halakatti has shown the following result and we conduct study on it.

Theorem 3.2:
Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds. If $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a C^∞ measurable homeomorphism and Radon measure-invariant map and if the measurable separable-normal property say P holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then P holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Proof:
Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds of dimension n. Let P be a measurable separable-normal property holds $\mu_{R_2} - a.e.$ on the set A of dense Borel subsets $\overline{K_\iota}$ in $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$, where A = { $\forall\ \overline{K_\iota} \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$: P($\overline{K_\iota}$) is true } has positive measure i.e. $\mu_{R_2}(A) > 0$.

We show that P also holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.
Since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a C^∞ measurable homeomorphism and Radon measure-invariant map from $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ to $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ there exists F^{-1}: $(M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$. If measurable separable-normal property P holds $\mu_{R_2} - a.e.$ on the set A of dense Borel subsets $\overline{K_\iota}$ in $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where the set A = { $\forall\ \overline{K_\iota} \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$: P$(\overline{K_\iota})$ is true } has positive measure i.e. $\mu_{R_2}(A) > 0$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then the property P also holds $\mu_{R_1} - a.e.$ on the set $F^{-1}(A)$ of dense Borel subsets $F^{-1}(\overline{K_\iota})$ in $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $F^{-1}(A)$ = {$F^{-1}(\overline{K_\iota})$ $\subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$: P($F^{-1}(\overline{K_\iota})$) is true } has positive measure i.e. $\mu_{R_1}(F^{-1}(A)) > 0$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.
Also since F is C^∞ measurable homeomorphism and Radon measure-invariant map, if P holds $\mu_{R_1} - a.e.$ on any dense set $A \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where the set A = { $\forall\ \overline{K_\iota} \subset$ $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$: P$(\overline{K_\iota})$ is true } has positive measure i.e. $\mu_{R_1}(A) > 0$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, then the property P also holds $\mu_{R_2} - a.e.$ on any dense set $F(A) \subset$ $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $F(A)$ = { $\forall\ F(\overline{K_\iota}) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$: P$(F(\overline{K_\iota}))$ is true} has positive measure i.e. $\mu_{R_2}(F(A)) > 0$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$.
Therefore, if measurable separable-normal property P holds $\mu_{R_2} - a.e.$ $\forall\ A \subset$ $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then the property P also holds $\mu_{R_1} - a.e.$ $\forall\ F^{-1}(A) \subset$ $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ under C^∞ measurable homeomorphism and Radon measure-invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

■

Theorem 3.3: Let $(\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1})$, $(\mathrm{M}_2, \tau_2, \Sigma_2, \mu_{\mathrm{R}_2})$ and $(\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$ be Radon measure manifolds. Let $\mathrm{F}_1 : (\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1}) \longrightarrow (\mathrm{M}_2, \tau_2, \Sigma_2, \mu_{\mathrm{R}_2})$ and $\mathrm{F}_2 : (\mathrm{M}_2, \tau_2, \Sigma_2, \mu_{\mathrm{R}_2}) \longrightarrow (\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$ be C^∞ measurable homeomorphisms and Radon measure-invariant maps. Then if measurable separable-normal property say P holds $\mu_{\mathrm{R}_1} - \mathrm{a.e.}$ on A $\subset (\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1})$ then P also holds $\mu_{\mathrm{R}_3} - \mathrm{a.e.}$ on $(\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$ under the composition mapping $\mathrm{F}_2 \circ \mathrm{F}_1 : (\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1}) \longrightarrow (\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$.

Proof: Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Radon measure manifolds. Let $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow$ $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be C^∞ measurable homeomorphisms and Radon measure-invariant maps.
We show that, if measurable separable-normal property P holds $\mu_{\mathrm{R}_1} - \mathrm{a.e.}$ on $(\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1})$ then P holds $\mu_{\mathrm{R}_3} - \mathrm{a.e.}$ on $(\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$ under the composition mapping $\mathrm{F}_2 \circ \mathrm{F}_1 : (\mathrm{M}_1, \tau_1, \Sigma_1, \mu_{\mathrm{R}_1}) \longrightarrow (\mathrm{M}_3, \tau_3, \Sigma_3, \mu_{\mathrm{R}_3})$.
From theorem 3.2, P holds $\mu_{R_1} - a.e.$ on any dense set $A \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where A = { $\forall\ \overline{K_\iota} \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$: P$(\overline{K_\iota})$ is true } has positive measure i.e. $\mu_{R_1}(A) > 0$, then P also holds $\mu_{R_2} - a.e.$ on any dense set $F_1(A) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $F_1(A)$ = { $\forall$

$F_1(\overline{K_\iota}) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$: P($F_1(\overline{K_\iota})$) is true } has positive measure i.e. $\mu_{R_2}(F(A)) > 0$ under the C^∞ measurable homeomorphism and Radon measure-invariant map $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$.

Similarly, if the property P holds $\mu_{R_2} - a.e.$ on any dense set $F_1(A) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $F_1(A)$ = { $\forall\ F_1(\overline{K_\iota}) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$: P($F_1(\overline{K_\iota})$) is true } has positive measure i.e. $\mu_{R_2}(F_1(A)) > 0$, then the property P also holds $\mu_{R_3} - a.e.$ on any dense set $F_2(F_1(A)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $F_2(F_1(A))$ = { $\forall\ F_2(F_1(\overline{K_\iota})) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$: $F_2(F_1(\overline{K_\iota}))$ is true } has positive measure i.e. $\mu_{R_3}(F_2(F_1(A))) > 0$ under C^∞ measurable homeomorphism and Radon measure-invariant map $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Since $F_2(F_1(A)) = F_2 \circ F_1(A)$, we have $F_2 \circ F_1(A)$ = { $\forall\, F_2 \circ F_1(\overline{K_\iota}) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$: P($F_2 \circ F_1(\overline{K_\iota})$) is true} has positive measure i.e. $\mu_{R_3}(F_2 \circ F_1(A)) > 0$ and also since F_1 and F_2 are C^∞ measurable homeomorphisms and Radon measure-invariant maps, $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ is also C^∞ measurable homeomorphism and Radon measure-invariant map. Also, for every $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ there exists an inverse map $(F_2 \circ F_1)^{-1} : (M_3, \tau_3, \Sigma_3, \mu_{R_3}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ such that if P holds $\mu_{R_3} - a.e.$ on any dense Borel set $F_2 \circ F_1(A) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $F_2 \circ F_1(A)$ = { $\forall\, F_2 \circ F_1(\overline{K_\iota}) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$: P($F_2 \circ F_1(\overline{K_\iota})$) is true} has positive measure i.e. $\mu_{R_3}(F_2 \circ F_1(A)) > 0$, then the property P also holds $\mu_{R_1} - a.e.$ on any dense Borel set $(F_2 \circ F_1)^{-1}(A) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $(F_2 \circ F_1)^{-1}(A)$ = { $\forall\ (F_2 \circ F_1)^{-1} (\overline{K_\iota}) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$: P $(F_2 \circ F_1)^{-1} (\overline{K_\iota})$ is true } has positive measure i.e. $\mu_{R_1}\ (F_2 \circ F_1)^{-1}\ (A) > 0$.

Therefore, if P holds $\mu_{R_1} - a.e.$ $\forall\, A \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P also holds $\mu_{R_3} - a.e.$ $\forall\, F_2 \circ F_1(A) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under C^∞ measurable homeomorphism and Radon measure-invariant map $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Hence the composition map $F_2 \circ F_1$ preserves measurable separable-normal property $\mu - a.e.$.

■

In these two results, we have shown that, if measurable separable-normal property P holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ under C^∞ measurable homeomorphism and Radon measure-invariant function F_1 and if P holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under C^∞ measurable homeomorphism and Radon measure-invariant function F_2 and also if measurable separable-normal property P holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition $F_2 \circ F_1$.

By continuing this process, one can show that any $(M_i, \tau_i, \Sigma_i, \mu_{R_i})$ can be related to any $(M_j, \tau_j, \Sigma_j, \mu_{R_j})$ by composition of two C^∞ measurable homeomorphisms and Radon measure-invariant functions $F_i \circ F_j$ denoted as $M_i\ R\ M_j$.

Thus, if there exists $F_1 : M_1 \longrightarrow M_2$, $F_2 : M_2 \longrightarrow M_3$,....., $F_n : M_{n-1} \longrightarrow M_n$, then the composition of functions $F_1, F_2, \ldots, F_n$ induces a group structure (G,∘) on the non-empty set of Radon measure manifolds $(M_1, M_2, \ldots, M_n) = \mathcal{M}$.
That is, if (G,∘) = { $F_1, F_2, \ldots, F_n$ } is a non-empty set of composition of $F_1, F_2, \ldots, F_n$ and if $\mathcal{M}$ = { $M_1, M_2, \ldots, M_n$} is a set of Radon measure manifolds then, S. C. P. Halakatti has shown that (G,∘) induces a group structure on $\mathcal{M}$. The following result is introduced and proved by S. C. P. Halakatti and we conduct a study on it.

Theorem 3.4: Let $\mathcal{M}$ = { $\mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_\mathrm{n}$} be a non-empty set of Radon measure manifolds and G = { $\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_\mathrm{n}$ } be the set of C^∞ measurable homeomorphisms and Radon measure-invariant functions on $\mathcal{M}$. Then (G,∘) induces a group structure on $\mathcal{M}$ under the composition of C^∞ measurable homeomorphism and Radon measure-invariant functions $\mathrm{F_i}$ and $\mathrm{F_j}$, $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{n}$.

Proof: Let $\mathcal{M}$ = { $M_1, M_2, \ldots, M_n$} be the set of Radon measure manifolds and let G = { $F_1, F_2, \ldots, F_n$ } be the set of C^∞ measurable homeomorphisms and Radon measure-invariant functions on $\mathcal{M}$.

In theorem 3.3, it is shown that if F_1: $M_1 \longrightarrow M_2$ and F_2: $M_2 \longrightarrow M_3$ are C^∞ measurable homeomorphisms and Radon measure-invariant functions, then $F_2 \circ F_1$: $M_1 \longrightarrow M_3$ and $F_1 \circ F_2$: $M_3 \longrightarrow M_1$ are also C^∞ measurable homeomorphisms and Radon measure-invariant functions.

We show that (G,∘) induces a group structure on $\mathcal{M}$.

(i) If $F_i, F_j \in$ (G,∘) then $F_i \circ F_j \in$ (G,∘) [By theorem 3.3]

(ii) If $F_i, F_j, F_l \in$ (G,∘) then $(F_i \circ F_j) \circ F_l = F_i \circ (F_j \circ F_l)$

Let $F_i \circ F_j = F_R$ and $F_j \circ F_l = F_S$
Then, $F_R \circ F_l = F_i \circ F_S$ [By theorem 3.3]

(iii) For every $F_i \in$ (G,∘) there exists an inverse map $F_i^{-1} \in$ (G,∘) such that $F_i \circ F_i^{-1} = F_i^{-1} \circ F_i = id$. [By theorem 3.2, since both F_i and F_i^{-1} are C^∞ measurable homeomorphism and Radon measure-invariant functions]

(iv) For any $F_i \in$ (G,∘) there exists an identity map *id* : $F_i \longrightarrow F_i \in$ (G,∘) such that *id* $\circ F_i = F_i \circ$ id $= F_i$ holds, where *id* ∈ (G,∘). [By theorem 3.3]

Therefore, (G,∘) = { $F_1, F_2, \ldots . F_i, F_i^{-1}, id, \ldots, F_n$ } induces a group structure on $\mathcal{M}$ under the composition of C^∞ measurable homeomorphisms and Radon measure-invariant functions. ■

If $\mathcal{M}$ is a non-empty set of maximally connected measure manifolds, S. C. P. Halakatti has observed that the group structure (G,∘) on $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$ generates a Network structure ($\mathcal{M}$, G,∘) of order 1.

Conclusion
The measure manifold (M, τ, Σ, μ_R) satisfying the separable-normal property P is invariant under the composition of finite number of measurable homeomorphisms and Radon measure-invariant functions induces a group structure (G,∘) on $\mathcal{M}$. If it is maximally connected measure manifold then ($\mathcal{M}$, G, ∘) generates a Network structure of order 1.

References

[1] Bogachev V. I., 1998, "Measures on Topological Spaces," Journal of Mathematical Sciences, Volume 91, No. 4.

[2] Bogachev V. I., 2006, "Measure Theory", Volume II, Springer.

[3] Dorlas T. C., 2010, "Remainder Notes for the Course on Measure on Topological Spaces", Dublin Institute for Advanced Studies, School of theoretical Physics, 10, Dublin 4, Ireland.

[4] S. C. P. Halakatti and H. G. Haloli,, 2014,"Introducing the Concept of Measure Manifold (M,τ_1, Σ_1, μ_1)", International Journal of Scientific Research, Journal of Mathematics (IOSR-JM),10, Issue 3, Ver. II, 01-11.

[5] S. C. P. Halakatti, Akshata Kengangutti and Soubhagya Baddi, 2015,"Generating A Measure Manifold", International Journal of Mathematical Archieve, (IJMA), Volume 6, Issue 3, 164-172.

[6] S. C. P. Halakatti and H. G. Haloli, 2014,"Extended Topological Properties on Measure Manifold", International Journal of Mathematical Archieve, (IJMA), Volume 05, Issue 3, 01-18.

[7] S. C. P. Halakatti and Soubhagya Baddi,, 2015,"Radon Measure on Compact Measurable Manifold with Some Measurable properties", International Journal of Mathematical Archieve, (IJMA), Volume 6, Issue 7, 01-12.

[8] S. C. P. Halakatti and Soubhagya Baddi, 2015, "Radon Measure on Compact Topological Measurable Space", International Journal of Scientific Research, Journal of Mathematics (IOSR-JM), Volume 11, Issue 2, Ver. III, 10-14.

[9] Hunter John K, 2011, "Measure Theory, Lecture notes", University of California at Davis, 2-11, 20, 33.

[10] Patty C. W., "Foundations of Topology", Jones and Bartlett Publishers, Inc., First Indian Edition.

[11] William Arveson, 1996, "Notes on Measure and Integration in Locally Compact Spaces", Department of Mathematics, University of California, Berkeley.

[12] Fremlin D. H., (2007), "Measure Theory", Volume 4, University of Essex.