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On Complete Network Measure Manifold

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Abstract

In this paper, we show that the maximal path connectedness property induces a new equivalence relation on a complete measure manifold, generating a network structure on M namely a complete network measure manifold.

Keywords: complete measure manifold, locally path connected, interconnected, maximally path connected, complete network measure manifold.

Mathematics Subject Classification: 28XX, 54XX, 58XX.

1. Introduction

The complete measure manifolds are special class of differentiable manifolds with additional structures introduced by S. C. P. Halakatti[1]. Such manifolds have advantage over differentiable manifolds if we introduce the maximally path connectedness property. In this connection [3] S.C.P. Halakatti has introduced different modes of connectedness on a complete measure manifold and introduced a new relation between the measure atlases $\mathbb{A}_i$ and $\mathbb{A}_j \in A^k(M)$ that is $\mathbb{A}_i$ is related to $\mathbb{A}_j$ if $\mathbb{A}_i$ is maximally path connected to $\mathbb{A}_j$ in $A^k(M)$**.** Further we show that this new relation on complete measure manifold is an equivalence relation and paves a way for a new manifold called complete network measure manifold.

2. Preliminaries

We use the basic concepts of connectedness like locally path connectedness μ_1-a.e., interconnected μ_1-a.e. and maximally path connected μ_1-a.e. on complete measure manifold introduced by S.C.P. Halakatti[3].

Definition 2.1:Measurable Homeomorphism[2]
Let (M, τ) be a second countable, Hausdorff topological space and (R^n, τ, Σ, μ) be a measure space. Then the function ϕ: U ⊂ M → (R^n, τ, Σ, μ) is called measurable homeomorphism if
(i) ϕ is bijective and bi continuous
(ii) ϕ and ϕ^{-1} are measurable.

Definition 2.2: Measure Invariant[4] [8][9]
Let $(R^n, \tau_1, \Sigma_1, \mu_1)$ and $(R^m, \tau_2, \Sigma_2, \mu_2)$ be measure spaces and T: $(R^n, \tau_1, \Sigma_1, \mu_1) \rightarrow (R^m, \tau_2, \Sigma_2, \mu_2)$ be a measurable function. Then, T is said to be measure preserving if for all A $\in \Sigma_2$ we have that, $\mu_1(T^{-1}(A)) = \mu_2(A)$

Definition 2.3: Complete measure space $(\boldsymbol{R^n}, \boldsymbol{\tau}, \boldsymbol{\Sigma}, \boldsymbol{\mu})$[3]
Let (R^n, τ, Σ, μ) be a measure space of dimension n. Suppose that for every Borel subset U ⊂ (R^n, τ, Σ, μ), μ(U)=0 and every V ⊆ U, μ(V) = 0 then (R^n, τ, Σ, μ) is a complete measure space.
Let $(M, \tau_1, \Sigma_1, \mu_1)$ be a complete measure manifold of dimension n which is measurable homeomorphic to a measure space (R^n, τ, Σ, μ). Let $\{f_n\}$, $\{g_n\}$ be measurable real valued functions converging to f and g respectively in (R^n, τ, Σ, μ). The ordered pair $(\{f_n \circ \phi\}, f \circ \phi)$[7] induces a Borel subset S $\in (U, \phi) \in (M, \tau_1, \Sigma_1, \mu_1)$ satisfying the following condition:
S = { p $\in (M, \tau_1, \Sigma_1, \mu_1)$: | $(f_n \circ \phi)$(p)-$(f \circ \phi)$(p)| $< \varepsilon$, ∀ n ∈ N } on the chart (U, ϕ) for which μ_1(S) > 0.
Thus by a complete measure manifold we mean to say that every subset of a measure zero is zero and every measurable function on M is convergent.

Definition 2.4: Locally path connected $\boldsymbol{\mu_1}$-a.e. on complete measure manifold[3]
The Borel subset S is locally path connected μ_1-a.e. if ∃ a C^∞-map γ: [0,1]→ S $\in (U, \phi)$ such that γ(0)=p ∈ S γ(1)=q ∈ S, such that μ_1(S) > 0
That is, p is locally path connected μ_1-a.e. to q in S $\in (U, \phi) \in \mathbb{A} \in A^k(M)$.

Definition 2.5: Interconnected $\boldsymbol{\mu_1}$-a.e. on complete measure manifold[3]
The Borel subset S $\in (U, \phi) \in \mathbb{A} \in (M, \tau_1, \Sigma_1, \mu_1)$ is interconnected to the Borel subset R $\in (V, \psi) \in \mathbb{A} \in (M, \tau_1, \Sigma_1, \mu_1)$ μ_1-a.e. if ∃ a C^∞-map γ: [0,1]→ S ∪ R $\in \mathbb{A} \in A^k(M)$ such that γ(0)=p ∈ S $\in \mathbb{A}$ γ(1)=q ∈ R $\in \mathbb{A}$,such that μ_1(S) > 0 and μ_1(R) > 0.
That is, p is interconnected μ_1-a.e. to q in S ∪ R $\in \mathbb{A} \in A^k(M)$.

Definition 2.6: Maximal path connected μ_1-a.e. on complete measure manifold[3] Let $(M, \tau_1, \Sigma_1, \mu_1)$ be a complete measure manifold and let $\mathbb{A}_i$, $\mathbb{A}_j$ and $\mathbb{A}_l \in A^k(M)$ be atlases on $(M, \tau_1, \Sigma_1, \mu_1)$. Let S, R and Q be Borel subsets of $\mathbb{A}_i$, $\mathbb{A}_j$ and $\mathbb{A}_l$. Then, we say that $A^k(M) \in (M, \tau_1, \Sigma_1, \mu_1)$ is maximally connected if $\exists$ a map γ: [0,1]→ $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$ such that, γ(0)= $p \in S \in (U, \phi) \in \mathbb{A}_i \in A^k(M)$ for which $\mu_1(S) > 0$ $\gamma(1/2)$= $q \in R \in (V, \psi) \in \mathbb{A}_j \in A^k(M)$ for which $\mu_1(R) > 0$ and γ(1)= $r \in Q \in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ for which $\mu_1(Q) > 0$.
That is, for each p $\in (U, \phi) \in \mathbb{A}_i$ is path connected to each q $\in (V, \psi) \in \mathbb{A}_j$**,** for $\mathbb{A}_i \cup \mathbb{A}_j \in A^k(M)$, $\mu_1(\mathbb{A}_i \cup \mathbb{A}_j) > 0$ for each q $\in \mathbb{A}_j$ is path connected to each r $\in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ and for, $\mathbb{A}_j \cup \mathbb{A}_l \in A^k(M)$, $\mu_1(\mathbb{A}_j \cup \mathbb{A}_l) > 0$. Then, if for each p $\in (U, \phi) \in \mathbb{A}_i$ is path connected to each r $\in (W, \chi) \in \mathbb{A}_l \in A^k(M)$ and for $\mathbb{A}_i \cup \mathbb{A}_l \in A^k(M)$, $\mu_1(\mathbb{A}_i \cup \mathbb{A}_l) > 0$ then $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M) \in (M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected if $\mu_1(\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l) > 0$ on complete measure manifold.

3. Main Results

In [1][2][3] S. C. P. Halakatti has introduced the concept of different types of connectedness like locally path connected μ_1-a.e., interconnected μ_1-a.e.and maximal path connected μ_1-a.e. on complete measure manifold. In this section it is shown that these properties remain invariant under the composition of measurable homeomorphism and measure invariant map. Also, we show that maximal path connectedness is an equivalence relation on a complete measure manifold and generates a network structure on M. The following results are introduced by S.C.P.Halakatti.

Theorem 3.1: *Let $(M_1, \tau_1, \Sigma_1, \mu_1)$, $(M_2, \tau_2, \Sigma_2, \mu_2)$ and $(M_3, \tau_3, \Sigma_3, \mu_3)$ be complete measure manifolds of dimension n and let $F: (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$, $G: (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)$ be measurable homeomorphism and measure invariant maps and if $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ are locally path connected then $\exists$ a measurable homeomorphism and measure invariant map $G \circ F: (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)$ such that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is locally path connected then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also locally path connected.*

Proof: Let $(M_1, \tau_1, \Sigma_1, \mu_1)$, $(M_2, \tau_2, \Sigma_2, \mu_2)$ and $(M_3, \tau_3, \Sigma_3, \mu_3)$ be complete measure manifolds of dimension n and $F: (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$, $G: (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)$ be measurable homeomorphism and measure invariant maps and in Theorem 4.7 of [3] it is shown that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is locally path connected μ_1-a.e. in S $\subset M_1$, $\mu_1(S) > 0$ then $(M_2, \tau_2, \Sigma_2, \mu_2)$ is also locally path connected μ_2-a.e. in $F(\mathrm{S}) = R \subset M_2$ with $\mu_2(F(S)) > 0$. If $(M_2, \tau_2, \Sigma_2, \mu_2)$ is locally path connected μ_2-a.e. in $R \subset M_2$, $\mu_2(R) > 0$ then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also locally path connected μ_3-a.e. in $G(R) = Q \subset M_3$ with $\mu_3(G(R)) > 0$.

Now we shall show that the composition map $G \circ F$ is also a measurable homeomorphism and measure invariant map, $G \circ F:(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ such that $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also locally path connected μ_3-a.e. in $G \circ F$(S) $\subset M_3$, $\mu_3(G \circ F$(S)) > 0.
Let us consider, S = { p $\in (M_1,\tau_1,\Sigma_1,\mu_1)$: $|(f_n \circ \phi)$(p)-$(f \circ \phi)$(p)$| < \varepsilon$, $\forall$ n $\in$ N } on the chart (U,ϕ) for which μ_1(S) > 0.
Now for every S $\in (M_1,\tau_1,\Sigma_1,\mu_1)$, G∘F(S) = { q $\in (M_3,\tau_3,\Sigma_3,\mu_3)$: |G∘F$(f_n \circ \phi)$(q)-G∘F$(f \circ \phi)(q)| < \varepsilon$, $\forall$ n $\in$ N} for which μ_3(G∘F(S))> 0.
Let (U,ϕ), (W,χ) be charts in M_1 and M_3 respectively and G∘F(p_1), G∘F(p_2) $\in$ G∘F(S) $\in (M_3,\tau_3,\Sigma_3,\mu_3)$.
The map G∘F is measurable homeomorphism and measure invariant, [since F, G are measurable homeomorphism and measure invariant] there exists $(G \circ F)^{-1}$ such that for every measure chart $(W,\chi) \in (M_3,\tau_3,\Sigma_3,\mu_3)$ with μ_3 (G∘F(S)) > 0, there exists $(G \circ F)^{-1}$(G∘F(p_1)) $= p_1 \in (U,\phi) \in (M_1,\tau_1,\Sigma_1,\mu_1)$, $(G \circ F)^{-1}$(G∘F(p_2)) $= p_2 \in (U,\phi) \in (M_1,\tau_1,\Sigma_1,\mu_1)$ since $(M_1,\tau_1,\Sigma_1,\mu_1)$ is locally path connected $\exists$ a C^∞-map γ: [0,1] $\rightarrow (M_1,\tau_1,\Sigma_1,\mu_1)$ such that $\gamma(0) = p_1 \in$ S $\in (U,\phi)$, μ_1(S) > 0 and $\gamma(1) = p_2 \in$ S $\in (U,\phi)$, μ_1(S) > 0
If μ_1(S) = 0 then p_1 is not locally path connected to p_2.
Now, since G∘F is homeomorphism $\exists$ a map G∘F∘γ:[0,1] $\rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ such that, G∘F∘$\gamma(0)$ = G∘F$(p_1) = q_1 \in$ G∘F(S) $\in (W,\chi)$, μ_3(G∘ F(S)) > 0, G∘F∘$\gamma(1)$ = G∘F$(p_2) = q_2 \in$ G∘F(S) $\in (W,\chi)$, μ_3(G∘ F(S)) > 0.
If μ_3(G∘F(S)) = 0 then q_1 is not locally path connected to q_2.
Therefore, q_1 is locally path connected μ_1-a.e. to q_2 by G∘F∘γ in G∘F(S) $\in (W,\chi)$ $\in (M_3,\tau_3,\Sigma_3,\mu_3)$.
If q_1 is locally path connected μ_1-a.e. to q_2 by G∘F∘γ in G∘F(S) $\in (W,\chi) \in (M_3,\tau_3,\Sigma_3,\mu_3)$ then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is locally path connected.
Therefore, if $(M_1,\tau_1,\Sigma_1,\mu_1)$ and $(M_2,\tau_2,\Sigma_2,\mu_2)$ are locally path connected μ_1-a.e. and μ_2-a.e. respectively then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also locally path connected μ_3-a.e.

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Theorem 3.2: *Let $(M_1,\tau_1,\Sigma_1,\mu_1)$, $(M_2,\tau_2,\Sigma_2,\mu_2)$ and $(M_3,\tau_3,\Sigma_3,\mu_3)$be complete measure manifolds of dimension n and let $F:(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_2,\tau_2,\Sigma_2,\mu_2)$, $G:(M_2,\tau_2,\Sigma_2,\mu_2) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ be measurable homeomorphism and measure invariant maps and if $(M_1,\tau_1,\Sigma_1,\mu_1)$ and $(M_2,\tau_2,\Sigma_2,\mu_2)$ are interconnected then $\exists$ a measurable homeomorphism and measure invariant map $G \circ F:(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ such that if $(M_1,\tau_1,\Sigma_1,\mu_1)$ is interconnected then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also interconnected.*

Proof: Let $(M_1,\tau_1,\Sigma_1,\mu_1)$, $(M_2,\tau_2,\Sigma_2,\mu_2)$ and $(M_3,\tau_3,\Sigma_3,\mu_3)$ be complete measure manifolds of dimension n and $F:(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_2,\tau_2,\Sigma_2,\mu_2)$, $G:(M_2,\tau_2,\Sigma_2,\mu_2) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ be measurable homeomorphism and measure invariant maps and in Theorem 4.8 of [3] it is shown that if $(M_1,\tau_1,\Sigma_1,\mu_1)$ is interconnected μ_1-a.e. in $\mathbb{A} \subset (M_1,\tau_1,\Sigma_1,\mu_1)$ with $\mu_1(S_1) > 0, \mu_1(R_1) > 0$ then $(M_2,\tau_2,\Sigma_2,\mu_2)$ is also interconnected μ_2-a.e. in $\mathbb{B} \subset (M_2,\tau_2,\Sigma_2,\mu_2)$ with

$\mu_2(F(S_1)) > 0$, $\mu_2(F(R_1)) > 0$.
Similarly, if $(M_2,\tau_2,\Sigma_2,\mu_2)$ is interconnected μ_2-a.e in $\mathbb{B} \subset (M_2,\tau_2,\Sigma_2,\mu_2)$ with $\mu_2(S_2) > 0$, $\mu_2(R_2)) > 0$ then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also interconnected μ_3-a.e. in $\mathbb{C} \subset (M_3,\tau_3,\Sigma_3,\mu_3)$ with $\mu_3(G(S_2)) > 0$, $\mu_3(G(R_2)) > 0$.
Now, let us show that there exists a measurable homeomorphism and measure invariant map G $\circ$ F: $(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ such that if $(M_1,\tau_1,\Sigma_1,\mu_1)$ is interconnected μ_1-a.e. in $\mathbb{A} \subset (M_1,\tau_1,\Sigma_1,\mu_1)$ with $\mu_1(S_1) > 0, \mu_1(R_1) > 0$ then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also interconnected μ_3-a.e. in $\mathbb{C} \subset (M_3,\tau_3,\Sigma_3,\mu_3)$ with $\mu_3(G \circ F(S_1)) > 0$, $\mu_3(G \circ F(R_1)) > 0$.
Let, S_1={ $p_1 \in (M_1,\tau_1,\Sigma_1,\mu_1)$: $|(f_n \circ \phi)(p_1)$-$(f \circ \phi)(p_1)| < \varepsilon$, ∀ n ∈ N } on the chart (U_1,ϕ_1) for which $\mu_1(S_1) > 0$. and R_1={ $p_2 \in (M_1,\tau_1,\Sigma_1,\mu_1)$: | $(f_n \circ \phi)(p_2)$-$(f \circ \phi)(p_2)| < \varepsilon$, ∀ n ∈ N } on the chart (U_2,ϕ_2) for which $\mu_1(R_1) > 0$.
Now for every $S_1 \in (M_1,\tau_1,\Sigma_1,\mu_1)$, G∘F($S_1$) ={ $q_1 \in (M_2,\tau_2,\Sigma_2,\mu_2)$: | G∘F($f_n \circ \phi$)(q_1)-G∘F($f \circ \phi$)(q_1)| $< \varepsilon$, ∀ n ∈ N } on the chart (W_1,χ_1) for which μ_3(G∘F(S_1)) > 0. and G∘F(R_1) ={ $q_2 \in (M_2,\tau_2,\Sigma_2,\mu_2)$: | G∘F($f_n \circ \phi$)(q_2)-G∘F($f \circ \phi$)(q_2)| $< \varepsilon$, ∀ n ∈ N } on the chart (W_2,χ_2) for which μ_3(G∘F(R_1)) > 0.
The map G $\circ$ F is measurable homeomorphism and measure invariant [since G, F are measurable homeomorphism and measure invariant] there exists a C^∞-map G∘F∘γ :[0,1] $\rightarrow$ $(M_3,\tau_3,\Sigma_3,\mu_3)$ such that G∘F∘γ(0) = $r_1 \in (W_1,\chi_1) \in \mathbb{C} \in (M_3,\tau_3,\Sigma_3,\mu_3)$, μ_3(G∘F(S_1)) > 0 and G∘F∘γ(1) = $r_2 \in (W_2,\chi_2) \in \mathbb{C} \in (M_3,\tau_3,\Sigma_3,\mu_3)$, μ_3(G∘F(R_1)) > 0. ⇒ r_1 is interconnected to r_2 in $(M_3,\tau_3,\Sigma_3,\mu_3)$.
If μ_3(G∘F(S_1)) = 0 and μ_3(G∘F(R_1)) = 0 then r_1 is not interconnected to r_2.
Therefore, if $(M_1,\tau_1,\Sigma_1,\mu_1)$ and $(M_2,\tau_2,\Sigma_2,\mu_2)$ are interconnected μ_1-a.e. and μ_2-a.e. respectively then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also interconnected μ_3-a.e.
Hence, interconnectedness is invariant under the measurable homeomorphism and measure invariant map if $\mu_1(S_1) > 0$, $\mu_1(R_1) > 0$ and μ_3(G∘F(S_1))>0 and μ_3(G∘F(R_1))>0. ∗∗∗

Theorem 3.3: *Let $(M_1,\tau_1,\Sigma_1,\mu_1)$, $(M_2,\tau_2,\Sigma_2,\mu_2)$ and $(M_3,\tau_3,\Sigma_3,\mu_3)$ be complete measure manifolds of dimension n and let F:$(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_2,\tau_2,\Sigma_2,\mu_2)$, G: $(M_2,\tau_2,\Sigma_2,\mu_2) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ be measurable homeomorphism and measure invariant maps and if $(M_1,\tau_1,\Sigma_1,\mu_1)$ and $(M_2,\tau_2,\Sigma_2,\mu_2)$ are maximally path connected then ∃ a measurable homeomorphism and measure invariant map $G \circ F$:$(M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ such that if $(M_1,\tau_1,\Sigma_1,\mu_1)$ is maximally path connected then $(M_3,\tau_3,\Sigma_3,\mu_3)$ is also maximally path connected.*

Proof: Let $(M_1,\tau_1,\Sigma_1,\mu_1)$, $(M_2,\tau_2,\Sigma_2,\mu_2)$ and $(M_3,\tau_3,\Sigma_3,\mu_3)$ be complete measure manifolds of dimension n and $F : (M_1,\tau_1,\Sigma_1,\mu_1) \rightarrow (M_2,\tau_2,\Sigma_2,\mu_2)$, G:$(M_2,\tau_2,\Sigma_2,\mu_2) \rightarrow (M_3,\tau_3,\Sigma_3,\mu_3)$ be measurable homeomorphism and measure invariant maps.
In theorem 4.9 of [3] it is shown that if $(M_1,\tau_1,\Sigma_1,\mu_1)$ is maximally path connected μ_1-a.e. in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$ with $\mathbb{A}_i$, $\mathbb{A}_j$, $\mathbb{A}_l \in A^k(M_1)$ such that $\mu_1(S_1) > 0$, $\mu_1(R_1) > 0$ and $\mu_1(Q_1) > 0$ then $(M_2,\tau_2,\Sigma_2,\mu_2)$ is also maximally path

connected μ_2-a.e. in $\mathbb{B}_i \cup \mathbb{B}_j \cup \mathbb{B}_l \in A^k(M_2)$such that $\mu_2(F(S_1)) > 0$,$\mu_2(F(R_1)) > 0$ and $\mu_2(F(Q_1)) > 0$. Similarly, if $(M_2, \tau_2, \Sigma_2, \mu_2)$ is maximally path connected μ_2-a.e. in $\mathbb{B}_i \cup \mathbb{B}_j \cup \mathbb{B}_l \in A^k(M_2)$ with $\mathbb{B}_i$, $\mathbb{B}_j$, $\mathbb{B}_l \in A^k(M_2)$ with $\mu_1(S_2) > 0$, $\mu_1(R_2) > 0$ and $\mu_1(Q_2) > 0$ then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected μ_3-a.e. in $\mathbb{C}_i \cup \mathbb{C}_j \cup \mathbb{C}_l \in A^k(M_3)$ such that $\mu_3(G(S_2)) > 0$, $\mu_3(G(R_2)) > 0$ and $\mu_3(G(Q_2)) > 0$.

Now, let us show that there exists a measurable homeomorphism and measure invariant map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)$ such that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected μ_1-a.e. then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected μ_3-a.e.

Since $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected μ_1-a.e. in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$, then $\exists$ three Borel subsets S_1, R_1 and Q_1 such that, $S_1 = \{ p_1 \in (M_1, \tau_1, \Sigma_1, \mu_1)$: $|(f_n \circ \phi)(p_1) - (f \circ \phi)(p_1)| < \varepsilon$, $\forall$ n $\in$ N } on the chart $(U_1, \phi_1) \in \mathbb{A}_i \in A^k(M)$, for which $\mu_1(S_1) > 0$ and $R_1 = \{ p_2 \in (M_1, \tau_1, \Sigma_1, \mu_1)$: $|(g_n \circ \phi)(p_2) - (g \circ \phi)(p_2)| < \varepsilon$, $\forall$ n $\in$ N } on the chart $(U_2, \phi_2) \in \mathbb{A}_j \in A^k(M)$, for which $\mu_1(R_1) > 0$ and $Q_1 = \{ p_3 \in (M_1, \tau_1, \Sigma_1, \mu_1)$: $|(h_n \circ \phi)(p_3) - (h \circ \phi)(p_3)| < \varepsilon$, $\forall$ n $\in$ N } on the chart $(U_3, \phi_3) \in \mathbb{A}_l \in A^k(M)$, for which $\mu_1(Q_1) > 0$.

Then, there exist a path γ:[0,1] $\rightarrow S_1 \cup R_1 \cup Q_1 \in \mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$ in a complete measure manifold $(M_1, \tau_1, \Sigma_1, \mu_1)$ such that, γ(0)= $p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_i \in A^k(M_1)$ for which $\mu_1(S_1) > 0$ $\gamma(1/2)$= $p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_j \in A^k(M_1)$ for which $\mu_1(R_1) > 0$ and γ(1)= $p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_l \in A^k(M_1)$for which $\mu_1(Q_1) > 0$ where each $p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_i$ is maximally path connected to each $p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_j$ and each $p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_j$ is maximally path connected to each $p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_l$ in $\mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$.

Since $G \circ$ F is measurable homeomorphism and measure invariant, [F, G are measurable homeomorphism and measure invariant] for $p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_i \in (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $q_1 \in$ F(S_1) $\in (W_1, \chi_1) \in \mathbb{C}_i \in (M_3, \tau_3, \Sigma_3, \mu_3)$ such that G $\circ$ F(S_1) = {G $\circ$ F(p_1) = $r_1 \in (M_3, \tau_3, \Sigma_3, \mu_3)$: | G $\circ$ F ($f_n \circ \phi$)(p_1)-G $\circ$ F ($f \circ \phi$)(p_1)| $< \varepsilon$, $\forall$ n $\in$ N } on the chart $(W_1, \chi_1) \in \mathbb{C}_i \in A^k(M_3)$, for which μ_3(G $\circ$ F(S_1)) > 0.

Similarly, for every $p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_j \in (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $r_2 \in$ G $\circ$ F(R_1) $\in (W_2, \chi_2) \in \mathbb{C}_j \in (M_3, \tau_3, \Sigma_3, \mu_3)$ such that, G $\circ$ F(R_1) = {G $\circ$ F(p_2) = $r_2 \in (M_3, \tau_3, \Sigma_3, \mu_3)$: | G $\circ$ F ($g_n \circ \phi$)(p_1)-G $\circ$ F ($g \circ \phi$)(p_1)| $< \varepsilon$, $\forall$ n $\in$ N } on the chart $(W_2, \chi_2) \in \mathbb{C}_j \in A^k(M_3)$, for which μ_3(G $\circ$ F(R_1)) > 0 and also, for every $p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_l \in (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $r_3 \in$ G $\circ$ F(Q_1) $\in (W_3, \chi_3) \in \mathbb{C}_l \in (M_3, \tau_3, \Sigma_3, \mu_3)$ such that, G $\circ$ F(Q_1) = {G $\circ$ F(p_3) = $r_3 \in (M_3, \tau_3, \Sigma_3, \mu_3)$: | G $\circ$ F ($h_n \circ \phi$)(p_3)-G $\circ$ F ($h \circ \phi$)(p_3)| $< \varepsilon$, $\forall$ n $\in$ N } on the chart $(W_3, \chi_3) \in \mathbb{C}_l \in A^k(M_3)$, for which μ_3(G $\circ$ F(Q_1)) > 0.

Since G $\circ$ F is measurable homeomorphism and measure invariant [F,G are measurable homeomorphism and measure invariant] for every, γ: [0,1] $\rightarrow S_1 \cup R_1 \cup Q_1 \in A^k(M_1)$in $(M_1, \tau_1, \Sigma_1, \mu_1)$ which connects p_1, p_2 and p_3 maximally, there

exist a corresponding path G ∘ F ∘ γ: [0,1] → G ∘ F(S_1) ∪ G ∘ F(R_1) ∪ G ∘ F(Q_1) in $(M_3, \tau_3, \Sigma_3, \mu_3)$ which connects maximally G ∘ F(p_1) = r_1 to G ∘ F(p_2) = r_2 and G ∘ F(p_2) = r_2 to G ∘ F(p_3) = r_3 in $\mathbb{C}_i \cup \mathbb{C}_j \cup \mathbb{C}_l \in A^k(M_3)$ satisfying μ_3(G ∘ F(S_1)) > 0, μ_3(G ∘ F(R_1)) > 0 and μ_3(G ∘ F(Q_1)) > 0 then r_1 is maximally path connected to r_2 and r_2 is maximally path connected to r_3.
Therefore, we have shown that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected.

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Theorem 3.4: *A complete measure manifold* $(M, \tau_1, \Sigma_1, \mu_1)$ *is maximally path connected if and only if it is locally path connected and internally path connected.*

Proof: Suppose $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected.

To prove: $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected and interconnected.
Since $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected ∃ a map γ : [0,1] → S ∪ R ∪ Q $\in \mathbb{A}_i \cup \mathbb{A}_j \cup \mathbb{A}_l \in A^k(M_1)$ such that, for each p $\in (U, \phi) \in \mathbb{A}_i$ is path connected to each q $\in (V, \psi) \in \mathbb{A}_j$, for $\mathbb{A}_i \cup \mathbb{A}_j \in A^k(M)$, $\mu_1(\mathbb{A}_i \cup \mathbb{A}_j) > 0$ for each q $\in (V, \psi) \in \mathbb{A}_j$ is path connected to each r $\in (W, \chi) \in \mathbb{A}_l \in A^k(M)$. Similarly, each p $\in (U, \phi) \in \mathbb{A}_i$ is path connected to each r $\in (W, \chi) \in \mathbb{A}_l \in A^k(M)$. This is nothing but every point of a chart (U, ϕ) is connected to every other point of (U, ϕ).
⇒ $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected.
Also, each chart (U, ϕ) and (V, ψ) of an atlas $\mathbb{A}_i$ are connected.
⇒ $(M, \tau_1, \Sigma_1, \mu_1)$ is internally connected.
This implies, $(M, \tau_1, \Sigma_1, \mu_1)$ is locally path connected and internally path connected.
Conversely, suppose $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected and is also interconnected.
We show that $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally connected.
Since $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected ⇒ for every p,q ∈ S $\in (U, \phi)$, ∃ a C^∞ map γ: [0,1] → S such that, $\gamma(0)$ = p ∈ S, $\gamma(1)$ = q ∈ S, such that μ_1(S)> 0 ... ... (i)
and also $(M, \tau_1, \Sigma_1, \mu_1)$ is interconnected ⇒ for every p ∈ S $\in (U, \phi)$ and q ∈ R ∈ (V, ψ) ∃ a C^∞-map γ: [0,1] → S ∪ R such that, $\gamma(0)$ = p ∈ S, $\gamma(1)$ = q ∈ R, such that μ_1(S)> 0 and μ_1(R)> 0 ... ... (ii)
From (i) and (ii) it implies that, each point of a chart are connected and each chart of an atlas are connected.
Therefore $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally connected.

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Definition 3.5: Equivalence relation on complete measure manifold
Let $\mathbb{A}_i$, $\mathbb{A}_j \in A^k(M)$ be measure atlases on complete measure manifold. We say that $\mathbb{A}_i$ is related to $\mathbb{A}_j$ if $\mathbb{A}_i$ is maximally path connected to $\mathbb{A}_j$.

Theorem 3.6: *The relation maximal path connectedness, is an equivalence relation on a complete measure manifold* $(M, \tau_1, \Sigma_1, \mu_1)$.

Proof: Let $(M, \tau_1, \Sigma_1, \mu_1)$ be a complete measure manifold with maximal path connectedness.

Now, we shall show that the relation, the maximal path connectedness between $\mathbb{A}_{\boldsymbol{i}}$ and $\mathbb{A}_{\boldsymbol{j}}$ on $(M, \tau_1, \Sigma_1, \mu_1)$ is an equivalence relation.

We say $\mathbb{A}_{\boldsymbol{i}}$, $\mathbb{A}_{\boldsymbol{j}}$ and $\mathbb{A}_{\boldsymbol{l}} \in (M, \tau_1, \Sigma_1, \mu_1)$ are maximally path connected if for each $p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_{\boldsymbol{i}}$, $p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_{\boldsymbol{j}}$, $p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_{\boldsymbol{l}}$there exist a path γ:[0,1] $\rightarrow A^k(M)$ such that $\gamma(0) = p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_{\boldsymbol{i}} \in A^k(M)$ for which $\mu_1(S_1) > 0$ $\gamma(1/2) = p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_{\boldsymbol{j}} \in A^k(M)$ for which $\mu_1(R_1) > 0$ and $\gamma(1) = p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_{\boldsymbol{l}} \in A^k(M)$ for which $\mu_1(Q_1) > 0$

(i) Reflexive:

Reflexive relation is trivial by considering a constant path γ: [0,1] $\rightarrow A^k(M)$ such that γ(t) = $p_1 \in (U_1, \phi_1) \in A^k(M)$, $\forall$ t $\in$ [0,1] with $\mu_1(S_1) > 0$.

(ii) Symmetry:

Suppose γ_1 is a path from p_1 to p_3 then let γ_2: [0,1] $\rightarrow A^k(M)$ such that γ_2(t) = γ_1(1-t), $\forall$ t $\in$ [0,1] then γ_2 is a path from p_3 to p_1 such that: $\gamma_2(0) = \gamma_1$(1-0) $= \gamma_1(1) = p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_{\boldsymbol{l}} \in A^k(M)$ for which $\mu_1(Q_1) > 0$. $\gamma_2(1/2) = \gamma_1(1\text{-}1/2) = \gamma_1(1/2) = p_2 \in R_1 \in (U_2, \phi_2) \in \mathbb{A}_{\boldsymbol{j}} \in A^k(M)$ for which $\mu_1(R_1) > 0$. $\gamma_2(1) = \gamma_1(1\text{-}1) = \gamma_1(0) = p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_{\boldsymbol{i}} \in A^k(M)$ for which $\mu_1(S_1) > 0$.

(iii) Transitivity:

Suppose γ_1 is a path from p_1 to p_3 and γ_2 is a path from p_3 to p_5, to show that $\exists$ a path γ_3 from p_1 to p_5.

Let γ_3:[0,1] $\rightarrow A^k(M)$ be defined as,

$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq \text{t} \leq 1/2 \\ \gamma_2(2t-1) & \text{if } 1/2 \leq \text{t} \leq 1 \end{cases}$$

Then γ_3 is well defined since $\gamma_1(1) = p_3 = \gamma_2(0)$.

Also, γ_3 is measurable since [0,1] is measurable that is $\gamma_3(0) = \gamma_1(0) = p_1 \in S_1 \in (U_1, \phi_1) \in \mathbb{A}_{\boldsymbol{i}}$ for which $\mu_1(S_1) > 0$. $\gamma_3\left(\frac{1}{2}\right) =$

$$\begin{cases} \gamma_1(1) = p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_{\boldsymbol{l}} \in A^k(M), \mu_1(Q_1) > 0. \\ \gamma_2(0) = p_3 \in Q_1 \in (U_3, \phi_3) \in \mathbb{A}_{\boldsymbol{l}} \in A^k(M), \mu_1(Q_1) > 0. \end{cases}$$

$\gamma_3(1) = \gamma_2(1) = p_5 \in R_1 \in (U_5, \phi_5) \in \mathbb{A}_{\boldsymbol{m}} \in A^k(M)$, $\mu_1(R_1) > 0$.

$\exists\, \gamma_3$, a path from p_1 to p_5. Hence maximal path connectedness is an equivalence relation.

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Observation:

Thus maximal path connectedness induces a new equivalence relation on the complete measure manifold M. In other wards, any atlas $\mathbb{A}_{\boldsymbol{i}}$ is equivalent to any other atlas $\mathbb{A}_{\boldsymbol{j}}$ if and only if they are maximally path connected where $\mu_1(A_i) > 0$ and $\mu_1(A_j) > 0$. So maximal path connectedness induces a new equivalence relation on a complete measure manifold. Also by theorems 3.4 and theorem 3.6 it is clear that the relation local and interconnectedness also induces an equivalence relation on a complete measure manifold.

In the following we exemplify that S^1 is a locally path connected complete measure manifold.

We consider S^1= { (x,y) $\in R^2$: $x^2 + y^2$=1} the unit circle in R^2. An R^1-atlas $\mathbb{A}$ of class C^∞ on S^1 is given by $\mathbb{A}$ ={(U_1,ϕ_1),(U_2,ϕ_2)}, where U_1= S^1 \ {(0,1)}, U_2= S^1 \ {(0,-1)}, and ϕ_1: $U_1 \to$ R, ϕ_2: $U_2 \to$ R are defined by ϕ_1(sinπt,cosπt) = t, 0 $<$ t $<$ 2 ϕ_2(sinπt,cosπt) = t,-1 $<$ t $<$ 1 are C^∞-maps and the transition maps $\phi_1 \circ \phi_2^{-1}$, $\phi_2 \circ \phi_1^{-1}$ are also differentiable of class C^∞ making S^1 a differentiable manifold of dimension 1.

Example 1: A 1-dimensional differentiable manifold S^1 is locally connected complete measure manifold.

Solution: Let f^n: $(R,\tau,\Sigma,\mu) \to$ R be measurable real valued function converging to f.

The ordered pair $(\{f^n \circ \phi\}, f \circ \phi)$ induces a Borel subset S $\subseteq$ (U_1,ϕ_1) $\in$ $(M,\tau_1,\Sigma_1,\mu_1)$ satisfying the following condition:

S={p $\in$ $(M,\tau_1,\Sigma_1,\mu_1)$: $|(f^n \circ \phi)(p) - f \circ \phi(p)| < \varepsilon$, $\forall$ n $\in$ N} on the chart (U_1,ϕ_1) for which μ_1(S) $>$0.

For S of (U_1,ϕ_1) $\exists$ a C^∞-map γ:[0,1] $\to$ S defined by γ(x)=ϕ_1^{-1}(x+δ) where 0 $<\delta<$ 1 such that γ(0)= ϕ_1^{-1}(0+δ)=(sin$\pi\delta$,cos$\pi\delta$)=p γ(1)= ϕ_1^{-1}(1+δ)=(sin$\pi(1+\delta)$,cos$\pi(1+\delta)$)=q.

That is, p is locally path connected to q in S $\subseteq$ (U_1,ϕ_1).

Thus, for different values of δ $\exists$ a C^∞-map γ which connects every pair of points p, q $\in$ S locally, making S^1 a locally path connected complete measure manifold.

Conclusion

The complete measure manifolds are special class of differentiable manifolds satisfying new equivalence relation between any two measure atlases generating a network structure on M. The advantage of such work will be explored geometrically, algebraically and analytically in our further study of fibre bundles which have rich application in the field of network structure.

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