On Complete Network Measure Manifold

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Abstract

In this paper, we show that the maximal path connectedness property induces a new equivalence relation on a complete measure manifold, generating a network structure on M namely a complete network measure manifold.

Keywords: complete measure manifold, locally path connected, interconnected, maximally path connected, complete network measure manifold.

Mathematics Subject Classification: 28XX, 54XX, 58XX.

1. Introduction

The complete measure manifolds are special class of differentiable manifolds with additional structures introduced by S. C. P. Halakatti[1]. Such manifolds have advantage over differentiable manifolds if we introduce the maximally path connectedness property. In this connection [3] S.C.P. Halakatti has introduced different modes of connectedness on a complete measure manifold and introduced a new relation between the measure atlases $A_i$ and $A_j \in A^k(M)$ that is $A_i$ is related to $A_j$ if $A_i$ is maximally path connected to $A_j$ in $A^k(M)$. Further we show that this new relation on complete measure manifold is an equivalence relation and paves a way for a new manifold called complete network measure manifold.
2. Preliminaries

We use the basic concepts of connectedness like locally path connectedness $\mu_1$-a.e., interconnected $\mu_1$-a.e. and maximally path connected $\mu_1$-a.e. on complete measure manifold introduced by S.C.P. Halakatti[3].

Definition 2.1: Measurable Homeomorphism[2]

Let $(M, \tau)$ be a second countable, Hausdorff topological space and $(R^n, \tau, \Sigma, \mu)$ be a measure space. Then the function $\phi: U \subset M \to (R^n, \tau, \Sigma, \mu)$ is called measurable homeomorphism if

(i) $\phi$ is bijective and bi continuous

(ii) $\phi$ and $\phi^{-1}$ are measurable.

Definition 2.2: Measure Invariant[4][8][9]

Let $(R^n, \tau_1, \Sigma_1, \mu_1)$ and $(R^m, \tau_2, \Sigma_2, \mu_2)$ be measure spaces and $T: (R^n, \tau_1, \Sigma_1, \mu_1) \to (R^m, \tau_2, \Sigma_2, \mu_2)$ be a measurable function. Then, $T$ is said to be measure preserving if for all $A \in \Sigma_2$ we have that, $\mu_1(T^{-1}(A)) = \mu_2(A)$

Definition 2.3: Complete measure space $(R^n, \tau, \Sigma, \mu)$[3]

Let $(R^n, \tau, \Sigma, \mu)$ be a measure space of dimension $n$. Suppose that for every Borel subset $U \subset (R^n, \tau, \Sigma, \mu)$, $\mu(U)=0$ and every $V \subseteq U$, $\mu(V)=0$ then $(R^n, \tau, \Sigma, \mu)$ is a complete measure space.

Let $(M, \tau_1, \Sigma_1, \mu_1)$ be a complete measure manifold of dimension $n$ which is measurable homeomorphic to a measure space $(R^n, \tau, \Sigma, \mu)$. Let $\{f_n\}, \{g_n\}$ be measurable real valued functions converging to $f$ and $g$ respectively in $(R^n, \tau, \Sigma, \mu)$. The ordered pair $\{(f_n \circ \phi), f \circ \phi\}$ induces a Borel subset $S \in (U, \phi) \in (M, \tau_1, \Sigma_1, \mu_1)$ satisfying the following condition:

$S = \{ p \in (M, \tau_1, \Sigma_1, \mu_1) : |(f_n \circ \phi)(p) - (f \circ \phi)(p)| < \varepsilon, \forall n \in \mathbb{N} \}$ on the chart $(U, \phi)$ for which $\mu_1(S) > 0$.

Thus by a complete measure manifold we mean to say that every subset of a measure zero is zero and every measurable function on $M$ is convergent.

Definition 2.4: Locally path connected $\mu_1$-a.e. on complete measure manifold[3]

The Borel subset $S$ is locally path connected $\mu_1$-a.e. if there exists a $C^\infty$-map $\gamma: [0,1] \to S \in (U, \phi)$ such that $\gamma(0)=p \in S \gamma(1)=q \in S$, such that $\mu_1(S) > 0$.

That is, $p$ is locally path connected $\mu_1$-a.e. to $q$ in $S \in (U, \phi) \in \mathbb{A} \in A^k(M)$.

Definition 2.5: Interconnected $\mu_1$-a.e. on complete measure manifold[3]

The Borel subset $S \in (U, \phi) \in \mathbb{A} \in (M, \tau_1, \Sigma_1, \mu_1)$ is interconnected to the Borel subset $R \in (V, \psi) \in \mathbb{A} \in (M, \tau_1, \Sigma_1, \mu_1)$ $\mu_1$-a.e. if there exists a $C^\infty$-map $\gamma: [0,1] \to S \cup R \in \mathbb{A} \in A^k(M)$ such that $\gamma(0)=p \in S \in \mathbb{A} \gamma(1)=q \in R \in \mathbb{A}$, such that $\mu_1(S) > 0$ and $\mu_1(R) > 0$.

That is, $p$ is interconnected $\mu_1$-a.e. to $q$ in $S \cup R \in \mathbb{A} \in A^k(M)$.
Definition 2.6: Maximal path connected \( \mu_1 \)-a.e. on complete measure manifold[3]
Let \((M, \tau_1, \Sigma_1, \mu_1)\) be a complete measure manifold and let \( \mathcal{A}_i, \mathcal{A}_j \) and \( \mathcal{A}_l \) be atlases on \((M, \tau_1, \Sigma_1, \mu_1)\). Let \( S, R \) and \( Q \) be Borel subsets of \( \mathcal{A}_i, \mathcal{A}_j \) and \( \mathcal{A}_l \). Then, we say that \( A^k(M) \in (M, \tau_1, \Sigma_1, \mu_1) \) is maximally connected if \( \exists \) a map \( \gamma : [0,1] \rightarrow \mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l \in A^k(M) \) such that, \( \gamma(0)= p \in S \in (U, \phi) \in \mathcal{A}_i \in A^k(M) \) for which \( \mu_1(S) > 0 \) \( \gamma(1/2) = q \in R \in (V, \psi) \in \mathcal{A}_j \in A^k(M) \) for which \( \mu_1(R) > 0 \) and \( \gamma(1)= r \in Q \in (W, \chi) \in \mathcal{A}_l \in A^k(M) \) for which \( \mu_1(Q) > 0 \).
That is, for each \( p \in (U, \phi) \in \mathcal{A}_i \) is path connected to each \( q \in (V, \psi) \in \mathcal{A}_j \), for \( \mathcal{A}_i \cup \mathcal{A}_j \in A^k(M) \), \( \mu_1(\mathcal{A}_i \cup \mathcal{A}_j) > 0 \) for each \( q \in \mathcal{A}_j \) is path connected to each \( r \in (W, \chi) \in \mathcal{A}_l \in A^k(M) \) and for, \( \mathcal{A}_j \cup \mathcal{A}_l \in A^k(M) \), \( \mu_1(\mathcal{A}_j \cup \mathcal{A}_l) > 0 \). Then, if for each \( p \in (U, \phi) \in \mathcal{A}_i \) is path connected to each \( r \in (W, \chi) \in \mathcal{A}_l \in A^k(M) \) and for \( \mathcal{A}_i \cup \mathcal{A}_l \in A^k(M) \), \( \mu_1(\mathcal{A}_i \cup \mathcal{A}_l) > 0 \) then \( \mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l \in A^k(M) \in (M, \tau_1, \Sigma_1, \mu_1) \) is maximally path connected if \( \mu_1(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l) > 0 \) on complete measure manifold.

3. Main Results
In [1][2][3] S. C. P. Halakatti has introduced the concept of different types of connectedness like locally path connected \( \mu_1 \)-a.e., interconnected \( \mu_1 \)-a.e. and maximal path connected \( \mu_1 \)-a.e. on complete measure manifold. In this section it is shown that these properties remain invariant under the composition of measurable homeomorphism and measure invariant map. Also, we show that maximal path connectedness is an equivalence relation on a complete measure manifold and generates a network structure on \( M \). The following results are introduced by S.C.P.Halakatti.

Theorem 3.1: Let \((M_1, \tau_1, \Sigma_1, \mu_1)\), \((M_2, \tau_2, \Sigma_2, \mu_2)\) and \((M_3, \tau_3, \Sigma_3, \mu_3)\) be complete measure manifolds of dimension \( n \) and let \( F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2) \), \( G : (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3) \) be measurable homeomorphism and measure invariant maps and if \((M_1, \tau_1, \Sigma_1, \mu_1)\) and \((M_2, \tau_2, \Sigma_2, \mu_2)\) are locally path connected then \( \exists \) a measurable homeomorphism and measure invariant map \( G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3) \) such that if \((M_1, \tau_1, \Sigma_1, \mu_1)\) is locally path connected then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also locally path connected.

Proof: Let \((M_1, \tau_1, \Sigma_1, \mu_1)\), \((M_2, \tau_2, \Sigma_2, \mu_2)\) and \((M_3, \tau_3, \Sigma_3, \mu_3)\) be complete measure manifolds of dimension \( n \) and \( F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2) \), \( G : (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3) \) be measurable homeomorphism and measure invariant maps and in Theorem 4.7 of [3] it is shown that if \((M_1, \tau_1, \Sigma_1, \mu_1)\) is locally path connected \( \mu_2 \)-a.e. in \( S \subset M_2 \), \( \mu_1(S) > 0 \) then \((M_2, \tau_2, \Sigma_2, \mu_2)\) is also locally path connected \( \mu_2 \)-a.e. in \( F(S) = R \subset M_2 \) with \( \mu_2(F(S)) > 0 \). If \((M_2, \tau_2, \Sigma_2, \mu_2)\) is locally path connected \( \mu_3 \)-a.e. in \( R \subset M_2 \), \( \mu_2(R) > 0 \) then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also locally path connected \( \mu_3 \)-a.e. in \( G(R) = Q \subset M_3 \) with \( \mu_3(G(R)) > 0 \).
Now we shall show that the composition map $G \circ F$ is also a measurable homeomorphism and measure invariant, $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \to (M_3, \tau_3, \Sigma_3, \mu_3)$ such that $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also locally path connected $\mu_3$-a.e. in $G \circ F(S) \subset M_3$, $\mu_3(G \circ F(S)) > 0$.

Let us consider, $S = \{ p \in (M_1, \tau_1, \Sigma_1, \mu_1) : |(f_n \circ \phi)(p) - (f \circ \phi)(p)| < \varepsilon, \forall \ n \in \mathbb{N} \}$ on the chart $(U, \phi)$ for which $\mu_1(S) > 0$.

Now for every $S \in (M_1, \tau_1, \Sigma_1, \mu_1)$, $G \circ F(S) = \{ q \in (M_3, \tau_3, \Sigma_3, \mu_3) : |G \circ F(f_n \circ \phi) - G \circ F(f \circ \phi)(q)| < \varepsilon, \forall \ n \in \mathbb{N} \}$ for which $\mu_3(G \circ F(S)) > 0$.

Let $(U, \phi)$, $(W, \chi)$ be charts in $M_1$ and $M_3$ respectively and $G \circ F(p_1)$, $G \circ F(p_2) \in G \circ F(S) \in (M_3, \tau_3, \Sigma_3, \mu_3)$.

The map $G \circ F$ is measurable homeomorphism and measure invariant, [ since F, G are measurable homeomorphism and measure invariant] there exists $(G \circ F)^{-1}$ such that for every measure chart $(W, \chi) \in (M_3, \tau_3, \Sigma_3, \mu_3)$ with $\mu_3(G \circ F(S)) > 0$, there exists $(G \circ F)^{-1}(G \circ F(p_1)) = p_1 \in (U, \phi) \in (M_1, \tau_1, \Sigma_1, \mu_1)$, $(G \circ F)^{-1}(G \circ F(p_2)) = p_2 \in (U, \phi) \in (M_1, \tau_1, \Sigma_1, \mu_1)$ since $(M_1, \tau_1, \Sigma_1, \mu_1)$ is locally path connected $\exists$ a $C^\infty$-map $\gamma : [0,1] \to (M_1, \tau_1, \Sigma_1, \mu_1)$ such that $\gamma(0) = p_1 \in S \in (U, \phi)$, $\mu_1(S) > 0$ and $\gamma(1) = p_2 \in S \in (U, \phi)$, $\mu_1(S) > 0$.

If $\mu_1(S) = 0$ then $p_1$ is not locally path connected to $p_2$.

Now, since $G \circ F$ is homeomorphism $\exists$ a map $G \circ F \gamma : [0,1] \to (M_3, \tau_3, \Sigma_3, \mu_3)$ such that, $G \circ F \gamma(0) = G \circ F(p_1) = q_1 \in G \circ F(S) \in (W, \chi)$, $\mu_3(G \circ F(S)) > 0$, $G \circ F \gamma(1) = G \circ F(p_2) = q_2 \in G \circ F(S) \in (W, \chi)$, $\mu_3(G \circ F(S)) > 0$.

If $\mu_3(G \circ F(S)) = 0$ then $q_1$ is not locally path connected to $q_2$.

Therefore, $q_1$ is locally path connected $\mu_1$-a.e. to $q_2$ by $G \circ F \gamma$ in $G \circ F(S) \in (W, \chi)$ $\in (M_3, \tau_3, \Sigma_3, \mu_3)$.

If $q_1$ is locally path connected $\mu_1$-a.e. to $q_2$ by $G \circ F \gamma$ in $G \circ F(S) \in (W, \chi)$ $\in (M_3, \tau_3, \Sigma_3, \mu_3)$ then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is locally path connected.

Therefore, if $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ are locally path connected $\mu_1$-a.e. and $\mu_2$-a.e. respectively then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also locally path connected $\mu_3$-a.e.

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**Theorem 3.2:** Let $(M_1, \tau_1, \Sigma_1, \mu_1)$, $(M_2, \tau_2, \Sigma_2, \mu_2)$ and $(M_3, \tau_3, \Sigma_3, \mu_3)$ be complete measure manifolds of dimension $n$ and let $F : (M_1, \tau_1, \Sigma_1, \mu_1) \to (M_2, \tau_2, \Sigma_2, \mu_2)$, $G : (M_2, \tau_2, \Sigma_2, \mu_2) \to (M_3, \tau_3, \Sigma_3, \mu_3)$ be measurable homeomorphism and measure invariant maps and if $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ are interconnected then $\exists$ a measurable homeomorphism and measure invariant map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \to (M_3, \tau_3, \Sigma_3, \mu_3)$ such that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is interconnected then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also interconnected.

**Proof:** Let $(M_1, \tau_1, \Sigma_1, \mu_1)$, $(M_2, \tau_2, \Sigma_2, \mu_2)$ and $(M_3, \tau_3, \Sigma_3, \mu_3)$ be complete measure manifolds of dimension $n$ and $F : (M_1, \tau_1, \Sigma_1, \mu_1) \to (M_2, \tau_2, \Sigma_2, \mu_2)$, $G : (M_2, \tau_2, \Sigma_2, \mu_2) \to (M_3, \tau_3, \Sigma_3, \mu_3)$ be measurable homeomorphism and measure invariant maps and in Theorem 4.8 of [3] it is shown that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is interconnected $\mu_1$-a.e. in $\mathbb{A} \subset (M_1, \tau_1, \Sigma_1, \mu_1)$ with $\mu_1(S_1) > 0, \mu_1(R_1) > 0$ then $(M_2, \tau_2, \Sigma_2, \mu_2)$ is also interconnected $\mu_2$-a.e. in $\mathbb{B} \subset (M_2, \tau_2, \Sigma_2, \mu_2)$ with
In theorem 4.9 of [3] it is shown that if invariant manifolds of dimension \(n\) and

\[
\mu_2(F(S_1)) > 0, \mu_2(F(R_1)) > 0.
\]

Similarly, if \((M_2, \tau_2, \Sigma_2, \mu_2)\) is interconnected \(\mu_2\)-a.e. in \(\mathcal{B} \subset (M_2, \tau_2, \Sigma_2, \mu_2)\) with \(\mu_2(S_2) > 0, \mu_2(R_2)) > 0\) then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also interconnected \(\mu_3\)-a.e. in \(\mathcal{C} \subset (M_3, \tau_3, \Sigma_3, \mu_3)\) with \(\mu_3(G(S_2)) > 0, \mu_3(G(R_2)) > 0\).

Now, let us now show that there exists a measurable homeomorphism and measure invariant map \(G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)\) such that if \((M_1, \tau_1, \Sigma_1, \mu_1)\) is interconnected \(\mu_1\)-a.e. in \(\mathcal{A} \subset (M_1, \tau_1, \Sigma_1, \mu_1)\) with \(\mu_1(S_1) > 0, \mu_1(R_1) > 0\) then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also interconnected \(\mu_3\)-a.e. in \(\mathcal{C} \subset (M_3, \tau_3, \Sigma_3, \mu_3)\) with \(\mu_3(G \circ F(S_1)) > 0, \mu_3(G \circ F(R_1)) > 0\).

Let, \(S_1 = \{ p_1 \in (M_1, \tau_1, \Sigma_1, \mu_1) : |(f_n \circ \phi)(p_1)-(f \circ \phi)(p_1)| < \varepsilon, \forall \ n \in \mathbb{N} \} \) on the chart \((U_1, \phi_1)\) for which \(\mu_1(S_1) > 0\). and \(R_1 = \{ p_2 \in (M_1, \tau_1, \Sigma_1, \mu_1) : |(f_n \circ \phi)(p_2)-(f \circ \phi)(p_2)| < \varepsilon, \forall \ n \in \mathbb{N} \} \) on the chart \((U_2, \phi_2)\) for which \(\mu_1(R_1) > 0\).

Now for every \(S_1 \in (M_1, \tau_1, \Sigma_1, \mu_1)\), \(G \circ \Sigma(S_1) = \{ q_1 \in (M_2, \tau_2, \Sigma_2, \mu_2) : |G \circ \Sigma(f_n \circ \phi)(q_1)| < \varepsilon, \forall \ n \in \mathbb{N} \} \) on the chart \((W_1, \chi_1)\) for which \(\mu_2(G \circ \Sigma(S_1)) > 0\). and \(G \circ \Sigma (R_1) = \{ q_2 \in (M_2, \tau_2, \Sigma_2, \mu_2) : |G \circ \Sigma(f_n \circ \phi)(q_2)| < \varepsilon, \forall \ n \in \mathbb{N} \} \) on the chart \((W_2, \chi_2)\) for which \(\mu_2(G \circ \Sigma(R_1)) > 0\).

The map \(G \circ F\) is measurable homeomorphism and measure invariant [ since \(G, F\) are measurable homeomorphism and measure invariant] there exists a \(C^\infty\)-map \(G \circ F \circ \gamma : [0,1] \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)\) such that \(G \circ F \circ \gamma (0) = r_1 \in (W_1, \chi_1) \in \mathcal{C} \subset (M_3, \tau_3, \Sigma_3, \mu_3)\). \(\mu_3(G \circ F(S_1)) > 0\) and \(G \circ F \circ \gamma (1) = r_2 \in (W_2, \chi_2) \in \mathcal{C} \subset (M_3, \tau_3, \Sigma_3, \mu_3)\). \(\mu_3(G \circ F(R_1)) > 0\). \(\Rightarrow r_1\) is interconnected to \(r_2\) in \((M_3, \tau_3, \Sigma_3, \mu_3)\).

Therefore, if \((M_1, \tau_1, \Sigma_1, \mu_1)\) and \((M_2, \tau_2, \Sigma_2, \mu_2)\) are interconnected \(\mu_1\)-a.e. and \(\mu_2\)-a.e. respectively then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also interconnected \(\mu_3\)-a.e.

Hence, interconnectedness is invariant under the measurable homeomorphism and measure invariant map if \(\mu_1(S_1) > 0, \mu_1(R_1) > 0\) and \(\mu_3(G \circ F(S_1)) > 0\) and \(\mu_3(G \circ F(R_1)) > 0\). ***

**Theorem 3.3:** Let \((M_1, \tau_1, \Sigma_1, \mu_1)\), \((M_2, \tau_2, \Sigma_2, \mu_2)\) and \((M_3, \tau_3, \Sigma_3, \mu_3)\) be complete measure manifolds of dimension \(n\) and let \(F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2)\), \(G : (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)\) be measurable homeomorphism and measure invariant maps and if \((M_1, \tau_1, \Sigma_1, \mu_1)\) and \((M_2, \tau_2, \Sigma_2, \mu_2)\) are maximally path connected then \(\exists\) a measurable homeomorphism and measure invariant map \(G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)\) such that if \((M_1, \tau_1, \Sigma_1, \mu_1)\) is maximally path connected then \((M_3, \tau_3, \Sigma_3, \mu_3)\) is also maximally path connected.

**Proof:** Let \((M_1, \tau_1, \Sigma_1, \mu_1)\), \((M_2, \tau_2, \Sigma_2, \mu_2)\) and \((M_3, \tau_3, \Sigma_3, \mu_3)\) be complete measure manifolds of dimension \(n\) and \(F : (M_1, \tau_1, \Sigma_1, \mu_1) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_2)\), \(G : (M_2, \tau_2, \Sigma_2, \mu_2) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_3)\) be measurable homeomorphism and measure invariant maps.

In theorem 4.9 of [3] it is shown that if \((M_1, \tau_1, \Sigma_1, \mu_1)\) is maximally path connected \(\mu_1\)-a.e. in \(\mathcal{A}_l \cup \mathcal{A}_j \cup \mathcal{A}_i \in A^K(M_1)\) with \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_l \in A^K(M_1)\) such that \(\mu_1(S_1) > 0, \mu_1(R_1) > 0\) and \(\mu_1(Q_1) > 0\) then \((M_2, \tau_2, \Sigma_2, \mu_2)\) is also maximally path
connected $\mu_2$-a.e. in $\mathbb{B}_j \cup \mathbb{B}_j \cup \mathbb{B}_l \in A^K(M_2)$ such that $\mu_2(F(S_1)) > 0 , \mu_2(F(R_1)) > 0$ and $\mu_3(F(Q_1)) > 0$. Similarly, if $(M_2, \tau_2, \Sigma_2, \mu_2)$ is maximally path connected $\mu_2$-a.e. in $\mathbb{B}_j \cup \mathbb{B}_j \cup \mathbb{B}_l \in A^K(M_2)$ with $\mathbb{B}_j, \mathbb{B}_l \in A^K(M_2)$ with $\mu_1(S_2) > 0$, $\mu_1(R_2) > 0$ and $\mu_1(Q_2) > 0$ then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected $\mu_3$-a.e. in $\mathcal{C}_l \cup \mathcal{C}_j \cup \mathcal{C}_l \in A^K(M_3)$ such that $\mu_3(G(S_2)) > 0$, $\mu_3(G(R_2)) > 0$ and $\mu_3(G(Q_2)) > 0$.

Now, let us show that there exists a measurable homeomorphism and measure invariant map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_1) \to (M_3, \tau_3, \Sigma_3, \mu_3)$ such that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected $\mu_1$-a.e. then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected $\mu_3$-a.e.

Since $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected $\mu_1$-a.e. in $\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l \in A^K(M_1)$, there exist three Borel subsets $S_1, R_1$ and $Q_1$ such that, $S_1 = \{ p_1 \in (M_1, \tau_1, \Sigma_1, \mu_1) : \left| \left( f_n \circ \phi \right)(p_1) - \left( f \circ \phi \right)(p_1) \right| < \varepsilon, \forall \ n \in \mathbb{N} \}$ on the chart $(U_1, \phi_1) \in \mathcal{A}_i \in A^K(M_1)$, for which $\mu_1(S_1) > 0$ and $R_1 = \{ p_2 \in (M_1, \tau_1, \Sigma_1, \mu_1) : \left| \left( g_n \circ \phi \right)(p_2) - \left( g \circ \phi \right)(p_2) \right| < \varepsilon, \forall \ n \in \mathbb{N} \}$ on the chart $(U_2, \phi_2) \in \mathcal{A}_j \in A^K(M_1)$, for which $\mu_1(R_1) > 0$ and $Q_1 = \{ p_3 \in (M_1, \tau_1, \Sigma_1, \mu_1) : \left| \left( h_n \circ \phi \right)(p_3) - \left( h \circ \phi \right)(p_3) \right| < \varepsilon, \forall \ n \in \mathbb{N} \}$ on the chart $(U_3, \phi_3) \in \mathcal{A}_l \in A^K(M_1)$, for which $\mu_1(Q_1) > 0$.

Then, there exist a path $\gamma : [0, 1] \to S_1 \cup R_1 \cup Q_1 \subset \mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l \subset A^K(M_1)$ in a complete measure manifold $(M_1, \tau_1, \Sigma_1, \mu_1)$ such that, $\gamma(0) = p_1 \in S_1 \subset (U_1, \phi_1) \subset \mathcal{A}_i \subset A^K(M_1)$ for which $\mu_1(S_1) > 0$ and $\gamma(1 - 1/2^n) = p_n \in R_1 \subset (U_2, \phi_2) \subset \mathcal{A}_j \subset A^K(M_1)$ for which $\mu_1(R_1) > 0$ and $\gamma(1) = p_3 \in Q_1 \subset (U_3, \phi_3) \subset \mathcal{A}_l \subset A^K(M_1)$ for which $\mu_1(Q_1) > 0$ where each $p_1 \in S_1 \in (U_1, \phi_1) \subset \mathcal{A}_i \subset A^K(M_1)$ is maximally path connected to each $p_2 \in R_1 \in (U_2, \phi_2) \subset \mathcal{A}_j$ and each $p_3 \in Q_1 \in (U_3, \phi_3) \subset \mathcal{A}_l$ is maximally path connected to each $p_3 \in Q_1 \subset (U_3, \phi_3) \subset \mathcal{A}_l \subset \mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_l \in A^K(M_1)$.

Since $G \circ F$ is measurable homeomorphism and measure invariant, $\left[ F, G \right]$ are measurable homeomorphism and measure invariant] for $p_1, p_2, p_3 \in S_1 \subset (U_1, \phi_1) \subset \mathcal{A}_i \subset (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $q_1 \in F(S_1) \subset (W_1, \chi_1) \subset \mathcal{C}_i \subset (M_3, \tau_3, \Sigma_3, \mu_3)$ such that $G \circ F(S_1) = \{ G \circ F(p_1) \in \tau_1 \subset (M_3, \tau_3, \Sigma_3, \mu_3) : | G \circ F(g_n \circ \phi)(p_1) - G \circ F(g \circ \phi)(p_1) | < \varepsilon, \forall n \in \mathbb{N} \}$ on the chart $(W_1, \chi_1) \subset \mathcal{C}_i \subset A^K(M_3)$, for which $\mu_3(G \circ F(S_1)) > 0$.

Similarly, for every $p_2 \in R_1 \subset (U_2, \phi_2) \subset \mathcal{A}_j \subset (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $r_2 \in G \circ F(R_1) \subset (W_2, \chi_2) \subset \mathcal{C}_j \subset (M_3, \tau_3, \Sigma_3, \mu_3)$ such that $G \circ F(R_1) = \{ G \circ F(p_2) \in \tau_2 \subset (M_3, \tau_3, \Sigma_3, \mu_3) : | G \circ F(g_n \circ \phi)(p_2) - G \circ F(g \circ \phi)(p_2) | < \varepsilon, \forall n \in \mathbb{N} \}$ on the chart $(W_2, \chi_2) \subset \mathcal{C}_j \subset A^K(M_3)$, for which $\mu_3(G \circ F(R_1)) > 0$ and also, for every $p_3 \in Q_1 \subset (U_3, \phi_3) \subset \mathcal{A}_l \subset (M_1, \tau_1, \Sigma_1, \mu_1)$ there exist $q_3 \in G \circ F(Q_1) \subset (W_3, \chi_3) \subset \mathcal{C}_l \subset (M_3, \tau_3, \Sigma_3, \mu_3)$ such that $G \circ F(Q_1) = \{ G \circ F(p_3) \in \tau_3 \subset (M_3, \tau_3, \Sigma_3, \mu_3) : | G \circ F(h_n \circ \phi)(p_3) - G \circ F(h \circ \phi)(p_3) | < \varepsilon, \forall n \in \mathbb{N} \}$ on the chart $(W_3, \chi_3) \subset \mathcal{C}_l \subset A^K(M_3)$, for which $\mu_3(G \circ F(Q_1)) > 0$.

Since $G \circ F$ is measurable homeomorphism and measure invariant $\left[ F, G \right]$ are measurable homeomorphism and measure invariant] for every, $\gamma : [0, 1] \to S_1 \cup R_1 \cup Q_1 \subset A^K(M_1)$ in $(M_1, \tau_1, \Sigma_1, \mu_1)$ which connects $p_1, p_2$ and $p_3$ maximally, there
exist a corresponding path $G \circ F \circ \gamma : [0,1] \to G \circ F(S_1) \cup G \circ F(R_1) \cup G \circ F(Q_1)$ in $(M_3, \tau_3, \Sigma_3, \mu_3)$ which connects maximally $G \circ F(p_1) = r_1$ to $G \circ F(p_2) = r_2$ and $G \circ F(p_2) = r_2$ to $G \circ F(p_2) = r_2$ in $C_i \cup C_j \cup C_k \in A^k(M_3)$ satisfying $\mu_3(G \circ F(S_1)) > 0$, $\mu_3(G \circ F(R_1)) > 0$ and $\mu_3(G \circ F(Q_1)) > 0$ then $r_1$ is maximally path connected to $r_2$ and $r_2$ is maximally path connected to $r_3$.

Therefore, we have shown that if $(M_1, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected then $(M_3, \tau_3, \Sigma_3, \mu_3)$ is also maximally path connected.

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**Theorem 3.4:** A complete measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected if and only if it is locally path connected and internally path connected.

**Proof:** Suppose $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected.

**To prove:** $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected and interconnected.

Since $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally path connected $\exists$ a map $\gamma : [0,1] \to S \cup R \cup Q \in A_i \cup A_j \cup A_l \in A^k(M_1)$ such that, for each $p \in (U, \phi) \in A_i$ is path connected to each $q \in (V, \psi) \in A_j$, for $A_i \cup A_j \in A^k(M)$, $\mu_1(A_i \cup A_j) > 0$ for each $q \in (V, \psi) \in A_j$ is path connected to each $r \in (W, \chi) \in A_l \in A^k(M)$. Similarly, each $p \in (U, \phi) \in A_i$ is path connected to each $r \in (W, \chi) \in A_l \in A^k(M)$. This is nothing but every point of a chart $(U, \phi)$ is connected to every other point of $(U, \phi)$.

$\Rightarrow \exists (M, \tau_1, \Sigma_1, \mu_1)$ is locally connected.

Also, each chart $(U, \phi)$ and $(V, \psi)$ of an atlas $A_i$ are connected.

$\Rightarrow (M, \tau_1, \Sigma_1, \mu_1)$ is internally connected.

This implies, $(M, \tau_1, \Sigma_1, \mu_1)$ is locally path connected and internally path connected.

Conversely, suppose $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected and is also interconnected.

We show that $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally connected.

Since $(M, \tau_1, \Sigma_1, \mu_1)$ is locally connected $\Rightarrow$ for every $p, q \in S \in (U, \phi), \exists \ a \ C^\infty$ map $\gamma : [0,1] \to S$ such that, $\gamma(0) = p \in S, \gamma(1) = q \in S$, such that $\mu_1(S) > 0$ .... (i) and also $(M, \tau_1, \Sigma_1, \mu_1)$ is interconnected $\Rightarrow$ for every $p \in S \in (U, \phi)$ and $q \in R \in (V, \psi) \exists \ a \ C^\infty$-map $\gamma : [0,1] \to S \cup R$ such that, $\gamma(0) = p \in S, \gamma(1) = q \in R$, such that $\mu_1(S) > 0$ and $\mu_1(R) > 0$ .... (ii)

From (i) and (ii) it implies that, each point of a chart are connected and each chart of an atlas are connected.

Therefore $(M, \tau_1, \Sigma_1, \mu_1)$ is maximally connected.

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**Definition 3.5:** Equivalence relation on complete measure manifold

Let $A_i, A_j \in A^k(M)$ be measure atlases on complete measure manifold. We say that $A_i$ is related to $A_j$ if $A_i$ is maximally path connected to $A_j$.

**Theorem 3.6:** The relation maximal path connectedness, is an equivalence relation on a complete measure manifold $(M, \tau_1, \Sigma_1, \mu_1)$. 

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Proof: Let \((M, \tau_1, \Sigma_1, \mu_1)\) be a complete measure manifold with maximal path connectedness.

Now, we shall show that the relation, the maximal path connectedness between \(A_i\) and \(A_j\) on \((M, \tau_1, \Sigma_1, \mu_1)\) is an equivalence relation.

We say \(A_i\), \(A_j\) and \(A_k\) in \((M, \tau_1, \Sigma_1, \mu_1)\) are maximally path connected if for each \(p_1 \in S_1 \in (U_1, \phi_1) \in A_i\), \(p_2 \in R_1 \in (U_2, \phi_2) \in A_j\), \(p_3 \in Q_1 \in (U_3, \phi_3) \in A_k\) there exist a path \(\gamma: [0,1] \rightarrow A^k(M)\) such that \(\gamma(0) = p_1 \in S_1 \in (U_1, \phi_1) \in A_i \in A^k(M)\) for which \(\mu_1(S_1) > 0\) \(\gamma(1/2) = p_2 \in R_1 \in (U_2, \phi_2) \in A_j \in A^k(M)\) for which \(\mu_1(R_1) > 0\) and \(\gamma(1) = p_3 \in Q_1 \in (U_3, \phi_3) \in A_k \in A^k(M)\) for which \(\mu_1(Q_1) > 0\).

(i) Reflexive:
Reflexive relation is trivial by considering a constant path \(\gamma: [0,1] \rightarrow A^k(M)\) such that \(\gamma(t) = p_1 \in (U_1, \phi_1) \in A^k(M), \forall t \in [0,1]\) with \(\mu_1(S_1) > 0\).

(ii) Symmetry:
Suppose \(\gamma_1\) is a path from \(p_1\) to \(p_3\) then let \(\gamma_2: [0,1] \rightarrow A^k(M)\) such that \(\gamma_2(t) = \gamma_1(1-t), \forall t \in [0,1]\) then \(\gamma_2\) is a path from \(p_3\) to \(p_1\) such that: \(\gamma(0) = \gamma_1(1-0) = \gamma_1(1) = p_3 \in Q_1 \in (U_3, \phi_3) \in A_k \in A^k(M)\) for which \(\mu_1(Q_1) > 0\).

\(\gamma_2(1/2) = \gamma_1(1/2) = p_2 \in R_1 \in (U_2, \phi_2) \in A_j \in A^k(M)\) for which \(\mu_1(R_1) > 0\).

\(\gamma_2(1) = \gamma_1(1-1) = \gamma_1(0) = p_1 \in S_1 \in (U_1, \phi_1) \in A_i \in A^k(M)\) for which \(\mu_1(S_1) > 0\).

(iii) Transitivity:
Suppose \(\gamma_1\) is a path from \(p_1\) to \(p_3\) and \(\gamma_2\) is a path from \(p_3\) to \(p_5\), to show that \(\exists\) a path \(\gamma_3\) from \(p_1\) to \(p_5\).

Let \(\gamma_3: [0,1] \rightarrow A^k(M)\) be defined as,
\[
\gamma_3(t) = \begin{cases} 
\gamma_1(2t) & \text{if } 0 \leq t \leq 1/2 \\
\gamma_2(2t-1) & \text{if } 1/2 \leq t \leq 1 
\end{cases}
\]

Then \(\gamma_3\) is well defined since \(\gamma_1(1) = p_3 = \gamma_2(0)\).

Also, \(\gamma_3\) is measurable since \([0,1]\) is measurable that is \(\gamma_3(0) = \gamma_1(0) = p_1 \in S_1 \in (U_1, \phi_1) \in A_i\) for which \(\mu_1(S_1) > 0\).

\(\gamma_3\) is (1/2)

\(\gamma_3(1) = \gamma_2(1) = p_5 \in R_1 \in (U_5, \phi_5) \in A_5 \in A^k(M)\), \(\mu_1(R_1) > 0\).

\(\exists\) \(\gamma_3\), a path from \(p_1\) to \(p_5\). Hence maximal path connectedness is an equivalence relation.

Observation:
Thus maximal path connectedness induces a new equivalence relation on the complete measure manifold \(M\). In other wards, any atlas \(A_i\) is equivalent to any other atlas \(A_j\) if and only if they are maximally path connected where \(\mu_1(A_i) > 0\) and \(\mu_1(A_j) > 0\). So
maximal path connectedness induces a new equivalence relation on a complete measure manifold. Also by theorems 3.4 and theorem 3.6 it is clear that the relation local and interconnectedness also induces an equivalence relation on a complete measure manifold.

In the following we exemplify that \( S^1 \) is a locally path connected complete measure manifold.

We consider \( S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \) the unit circle in \( \mathbb{R}^2 \). An \( \mathbb{R}^1 \)-atlas \( \mathbb{A} \) of class \( C^\infty \) on \( S^1 \) is given by \( \mathbb{A} = \{(U_1, \phi_1), (U_2, \phi_2)\} \), where \( U_1 = S^1 \setminus \{(0,1)\} \), \( U_2 = S^1 \setminus \{(0, -1)\} \), and \( \phi_1: U_1 \to \mathbb{R}, \phi_2: U_2 \to \mathbb{R} \) are defined by \( \phi_1(\sin \pi t, \cos \pi t) = t, 0 < t < 2 \), \( \phi_2(\sin \pi t, \cos \pi t) = t, -1 < t < 1 \) are \( C^\infty \)-maps and the transition maps \( \phi_1 \circ \phi_2^{-1}, \phi_2 \circ \phi_1^{-1} \) are also differentiable of class \( C^\infty \) making \( S^1 \) a differentiable manifold of dimension 1.

**Example 1:** A 1-dimensional differentiable manifold \( S^1 \) is locally connected complete measure manifold.

**Solution:** Let \( f^n: (R, \tau, \Sigma, \mu) \to \mathbb{R} \) be measurable real valued function converging to \( f \).

The ordered pair \( \{(f^n \circ \phi_1), f \circ \phi_2\} \) induces a Borel subset \( S \subseteq (U_1, \phi_1) \in (M, \tau_1, \Sigma_1, \mu_1) \) satisfying the following condition:

\[ S = \{p \in (M, \tau_1, \Sigma_1, \mu_1) : |(f^n \circ \phi_1)(p) - f \circ \phi_2(p)| < \varepsilon, \forall n \in \mathbb{N} \} \]

on the chart \((U_1, \phi_1)\) for which \( \mu_1(S) > 0 \).

For \( S \) of \( (U_1, \phi_1) \) \( \exists \) a \( C^\infty \)-map \( \gamma: [0,1] \to S \) defined by \( \gamma(x) = \phi_1^{-1}(x + \delta) \) where \( 0 < \delta < 1 \) such that \( \gamma(0) = \phi_1^{-1}(0 + \delta) = (\sin \pi \delta, \cos \pi \delta) = p \), \( \gamma(1) = \phi_1^{-1}(1 + \delta) = (\sin \pi (1 + \delta), \cos \pi (1 + \delta)) = q \).

That is, \( p \) is locally path connected to \( q \) in \( S \subseteq (U_1, \phi_1) \).

Thus, for different values of \( \delta \), \( \exists \) a \( C^\infty \)-map \( \gamma \) which connects every pair of points \( p, q \) \( \in S \) locally, making \( S^1 \) a locally path connected complete measure manifold.

**Conclusion**

The complete measure manifolds are special class of differentiable manifolds satisfying new equivalence relation between any two measure atlases generating a network structure on \( M \). The advantage of such work will be explored geometrically, algebraically and analytically in our further study of fibre bundles which have rich application in the field of network structure.

**References**


