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On bicontinuous functions in strongly pairwise normal spaces

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Abstract

In this paper we extend the Urysohn and Katětov theorems in a topological space about existence of a continuous and semi-continuous function, respectively, to a strongly pairwise normal bitopological space.

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1. Introduction

A set X with two topologies τ_1 and τ_2 is called a bitopological space, and it is denoted by (X, τ_1, τ_2). The elements of τ_i, $i = 1, 2$ are called τ_i-open. A set F is τ_i-closed if this is the complement of a τ_i-open set. If $A \subset X$, the interior and the closure of the set A is denoted by $int_{\tau_i} A$ and $cl_{\tau_i} A$ with respect to τ_i, respectively. In 1963, Kelly in [5] defines for the first time the separation axioms in bitopological spaces. These are knows as pairwise separation axioms: pairwise T_0, pairwise T_1, pairwise Hausdorff, pairwise regular and pairwise normal, such concepts generalize to those in a topological space, that is, in the case $\tau_1 = \tau_2$. In 2013 Ajoy Mukharjee [7] mentions that it is interesting to analyze what happens with the τ_i interior of a τ_j-open set since it is possible that these are empty. In his work Mukharjee defines the strongly pairwise separations axioms, based on interiors of open sets with respect to contrary topologies and shows the relation between them and the corresponding pairwise axioms. In this paper, in principle, some characterizations are given on the concept of pairwise normality, and are the basis for the proofs of theorems on the existence of bicontinuous functions which are established subsequently.

We use the abbreviation lsc for lower semicontinuous and usc for upper semicontinuous, always $i, j \in \{1, 2\}$ and $i \neq j$ whenever i, j appear together. Let $\Re$ be the set of all rational numbers r of the form $k/2^n$, $0 \leq k/2^n \leq 1$.

2. Preliminaries

In the spirit of making the article self-contained, we give the following known definitions and results.

Definition 2.1. A topological space (X, τ) is said to be normal, if for each pair of closed sets F and G with $F \cap G = \emptyset$, there exists open sets U and V such that $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$.

Theorem 2.2. Let (X, τ) be a topological space. The following statements are equivalent:

1) (X, τ) is normal.

2) For each closed set F and open set U, with $F \subset U$, there exists an open set W and closed set D such that $F \subset W \subset D \subset U$.

3) For each closed set F and open set U, with $F \subset U$, there exists an open set W with $F \subset W \subset clW \subset U$.

Definition 2.3. (Kelly) A bitopological space (X, τ_1, τ_2) is said to be pairwise normal if for any pair of a τ_i-closed set F and τ_j-closed set G with $F \cap G = \emptyset$, there exist a τ_j-open set U and a τ_i-open set V such that $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$.

Theorem 2.4. Let (X, τ_1, τ_2) be a bitopological space. The following statements are equivalent:

1) (X, τ_1, τ_2) is pairwise normal.

2) For any pair of a τ_i-closed set F and a τ_j-open set U with $F \subset U$, there exists a τ_j-open set W and a τ_i-closed set D such that
$$F \subset W \subset D \subset U. \tag{2.1}$$

3) For each τ_i-closed set F and τ_j-open U with $F \subset U$ there exists a τ_j-open W such that
$$F \subset W \subset cl_{\tau_i} W \subset U. \tag{2.2}$$

Definition 2.5. (Ajoy Mukharjee) A bitopological space (X, τ_1, τ_2), is said to be strongly pairwise normal if for each τ_i-closed set F and τ_j-closed set G, with $F \cap G = \emptyset$, there exists a τ_j-open set U and a τ_i-open set V such that $F \subset int_{\tau_i} U$, $G \subset int_{\tau_j} V$ and $U \cap V = \emptyset$.

Example 2.6. ([7], example 2.5, p. 168) Si $X = \mathbb{R}$, with the topologies $\tau_1 = \{X, \phi, (-\infty, a], (a, \infty)\}$ and $\tau_2 = \{X, \phi, \mathbb{R}-\{a\}, (-\infty, a), (-\infty, a], (a, \infty)\}$, (X, τ_1, τ_2) is strongly pairwise normal.

Definition 2.7. A function $f : (X, \tau_1, \tau_2) \to \mathbb{R}$ is called bicontinuous if $f : (X, \tau_i) \to \mathbb{R}$ is continuous for $i = 1$ and $i = 2$.

3. Results

Now let's see some other formulations on strongly pairwise normality, which generalize the corresponding pairwise normality (2.1) and (2.2).

Theorem 3.1. Let (X, τ_1, τ_2) be a bitopological space. The following statements are equivalent:

1) (X, τ_1, τ_2) is strongly pairwise normal.

2) For any pair of a τ_i-closed set F and a τ_j-open set U with $F \subset U$, there exist a τ_j-open set W and a τ_i-closed set D such that

$$F \subset int_{\tau_i} W \subset W \subset D \subset cl_{\tau_j} D \subset U. \tag{3.1}$$

3) For each τ_i-closed set F and τ_j-open U with $F \subset U$ there exists a τ_j-open W with

$$F \subset int_{\tau_i} W \subset cl_{\tau_i} W \subset cl_{\tau_j} \left(cl_{\tau_i} W\right) \subset U. \tag{3.2}$$

Proof. 1) $\to$ 2) The set $X - U$ is τ_j-closed and $F \cap (X - U) = \emptyset$, then there is a τ_j-open set W and a τ_i-open set V with $F \subset int_{\tau_i} W$, $X - U \subset int_{\tau_j} V$ and $W \cap V = \emptyset$. So $cl_{\tau_j} (X - V) = X - int_{\tau_j} V \subset U$ obtaining $int_{\tau_j} V \subset V \subset X - W \subset X - int_{\tau_i} W \subset X - F$ and thus $F \subset int_{\tau_i} W \subset W \subset X - V \subset X - int_{\tau_j} V = cl_{\tau_j} (X - V) \subset U$. If D is equal to the τ_i-closed set $X - V$, the result is obtained.

2) $\to$ 3) There exists a τ_j-open set $W$and a τ_i-closed set D with $F \subset int_{\tau_i} W \subset W \subset D \subset cl_{\tau_j} D \subset U$. Because D is τ_i-closed $cl_{\tau_i} W \subset D \subset cl_{\tau_j} D$ and so $cl_{\tau_j} \left(cl_{\tau_i} W\right) \subset cl_{\tau_j} D \subset U$, then (3.2) is satisfied. 3) $\to$ 1) Let F be a τ_i-closed set and G a τ_j-closed set with $F \cap G = \emptyset$, thus F is a subset of the τ_j-open $X - G$. By 3), there exists a τ_j-open set W such that $F \subset int_{\tau_i} W \subset cl_{\tau_i} W \subset cl_{\tau_j} \left(cl_{\tau_i} W\right) \subset X - G$, then $G \subset X - cl_{\tau_j} \left(cl_{\tau_i} W\right) = int_{\tau_j} [\left(cl_{\tau_i} W\right)^c]$ and $F \subset int_{\tau_i} W$. Clearly $W \cap \left(cl_{\tau_i} W\right)^c = \emptyset$. ■

Kelly in [5] generalizes the Urysohn Theorem to pairwise normal bitopological spaces, this generalization establishes: *If* (X, τ_1, τ_2) *is a pairwise normal bitopological space, then for each pair of a* τ_i*-closed set* A *and* a τ_j*-closed set* B*, with* $A \cap B = \emptyset$*, there exists a function* $f : X \to [0, 1]$ τ_i*-lsc* and τ_j*-usc such that* $f_{|A} \equiv 0$ *and* $f_{|B} \equiv 1$.

Now we give the following result.

Theorem 3.2. Let (X, τ_1, τ_2) be a strongly pairwise normal space. If $A, B \subset X$ are sets with A τ_i-closed, B τ_j-closed and $A \cap B = \phi$, then there is a bicontinuous function $f : X \longrightarrow [0, 1]$ such that

$$f_{|A} \equiv 0 \text{ and } f_{|B} \equiv 1.$$

Proof. We will follow the schema of the classical proof of the Urysohn Theorem (see, for example [2]). By doing $D_0 = A$ and $V_1 = X - B$. D_0 is a τ_i-closed set, V_1 is a τ_j-open and $D_0 \subset V_1$. By (3.1) there are $D_{\frac{1}{2}}$ τ_i-closed and $V_{\frac{1}{2}}$ τ_j-open, such that

$D_0 \subset int_{\tau_i} V_{\frac{1}{2}} \subset V_{\frac{1}{2}} \subset D_{\frac{1}{2}} \subset cl_{\tau_j} D_{\frac{1}{2}} \subset V_1$. Applying the same property to the pairs $D_0 \subset V_{\frac{1}{2}}$ and $D_{\frac{1}{2}} \subset V_1$, affirm the existence of $D_{\frac{1}{4}}$, $D_{\frac{3}{4}}$ τ_i-closed and $V_{\frac{1}{4}}$, $V_{\frac{3}{4}}$ τ_j-open sets, with

$$\begin{aligned} D_0 &\subset int_{\tau_i} V_{\frac{1}{4}} \subset V_{\frac{1}{4}} \subset D_{\frac{1}{4}} \subset cl_{\tau_j} D_{\frac{1}{4}} \subset V_{\frac{1}{2}} \subset D_{\frac{1}{2}} \\ &\subset int_{\tau_i} V_{\frac{3}{4}} \subset V_{\frac{3}{4}} \subset D_{\frac{3}{4}} \subset cl_{\tau_j} D_{\frac{3}{4}} \subset V_1. \end{aligned}$$

Continuing in this way we obtain two families of the sets $\{D_{\frac{k}{2^n}}\}$ $k = 0, 1, \ldots, 2^n$-1, $n = 1, 2, \ldots$ and $\{V_{\frac{k}{2^n}}\}$ $k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$ that satisfy

$$D_{\frac{k-1}{2^n}} \subset int_{\tau_i} V_{\frac{k}{2^n}} \subset V_{\frac{k}{2^n}} \subset D_{\frac{k}{2^n}} \subset cl_{\tau_j} D_{\frac{k}{2^n}} \subset V_{\frac{k+1}{2^n}} \subset D_{\frac{k+1}{2^n}} \tag{3.3}$$

for $k = 1, \ldots, 2^n$-1. For convenience let $D_1 = X$ and $V_0 = \emptyset$. Thus, in a general way if $q, p, r \in \Re$ with $q < p < r$

$$D_q \subset int_{\tau_i} V_p \subset V_p \subset D_p \subset cl_{\tau_j} D_p \subset V_r \subset D_r. \tag{3.4}$$

Now, by defining the function $f : X \longrightarrow [0, 1]$ by

$$f(x) = \inf\{p \in \Re : x \in V_p\}, \ x \in X. \tag{3.5}$$

It is clear to see, as mentioned in [5] and in [1], also $f(x) = \inf\{p \in \Re : x \in D_p\}$ for each $x \in X$. In the same study it is shown that f is τ_i-lsc y τ_j-usc, now we prove that this is τ_i-usc y τ_j-lsc. To perform the test let's assert that $f(x)$ is also expressible as:

$$\inf\{p \in \Re : x \in int_{\tau_i} V_p\}, \quad \text{and} \tag{3.6}$$

$$\inf\{p \in \Re : x \in cl_{\tau_j} D_p\}. \tag{3.7}$$

The equality of (3.6) with $f(x)$ follows from $\inf\{p : x \in D_p\} \leq \inf\{p : x \in int_{\tau_i} V_p\}$, because $\{p : x \in int_{\tau_i} V_p\} \subset \{p : x \in D_p\}$. Now if $p_1 = \inf\{p : x \in D_p\} < \inf\{p : x \in int_{\tau_i} V_p\} = p_2$, then there exist $q, r \in (p_1, p_2) \cap \Re$ with $q < r$ for which they (by (3.4)):

$$x \in D_q \subset int_{\tau_i} V_r \subset V_r \subset D_r \subset cl_{\tau_j} D_r \text{ contradicting the definition of } p_2.$$

Similarly, it is verified that (3.7) is equal to $f(x)$. Clearly $0 \leq f(x) \leq 1$, $x \in X$; $f(x) = 0$, $x \in A$ and $f(x) = 1$, $x \in B$.

To test that f is τ_i-usc, we see that for α with $0 < \alpha \leq 1$, the set $f^{-1}([0, \alpha))$ is τ_i-open, because by (3.6)

$$f(x) < \alpha \iff \exists p, \ p < \alpha, \ x \in int_{\tau_i} V_p \iff x \in \cup_{p<\alpha} int_{\tau_i} V_p,$$

So $f^{-1}([0, \alpha)) = \cup_{p<\alpha} int_{\tau_i} V_p$ is a τ_i-open set. The τ_j-lsc of f follows from the expression (3.7) of $f(x)$ and that

$$f(x) > \alpha \iff \exists p, \ p > \alpha, \ x \in X\text{-}cl_{\tau_j} D_p \iff x \in \cup_{p>\alpha}(X\text{-}cl_{\tau_j} D_p).$$

Showing that the set $f^{-1}((\alpha, 1])$ is a τ_j-open set. ■

In 1952 Katětov in [3], [4] establishes the result that in a normal topological space (X, τ), for each pair of functions $f, g : X \longrightarrow \mathbb{R}$, with f usc, g lsc and $f \leq g$, there exists a continuous function $h : X \longrightarrow \mathbb{R}$ such that $f \leq h \leq g$. Later in 1967, Lane in [6] generalizes the Theorem of Katětov for bitopological spaces: *a bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if for each pair of functions* $f, g : X \longrightarrow \mathbb{R}$, *with* f τ_i*-usc,* g τ_j*-lsc and* $f \leq g$ *there exists a function* $h : X \longrightarrow \mathbb{R}$ τ_i*-usc and* τ_j*-lsc, such that* $f \leq h \leq g$. Now we establish the result to strongly pairwise normal spaces.

Theorem 3.3. A bitopological space (X, τ_1, τ_2) is strongly pairwise normal if and only if for each pair of functions $f, g : X \longrightarrow \mathbb{R}$ with f τ_i-usc, g τ_j-lsc and $f \leq g$ there exists a bicontinuous function $h : X \longrightarrow \mathbb{R}$ such that $f \leq h \leq g$.

For the proof of this theorem first we see the following result, whose proof follows the idea used in the corresponding theorem for pairwise normal bitopological spaces [1].

Lemma 3.4. Let (X, τ_1, τ_2) be a strongly pairwise normal space. If $A, B \subset X$ are sets with A a τ_i-F_σ, B a τ_j-F_σ and

$$cl_{\tau_i} A \cap B = \emptyset = A \cap cl_{\tau_j} B. \tag{3.8}$$

Then there exists U τ_j-open and V τ_i-open sets, with $A \subset int_{\tau_i} U$, $B \subset int_{\tau_j} V$ and $U \cap V = \emptyset$; furthermore these sets satisfy

$$A \subset int_{\tau_i} U \subset U \subset cl_{\tau_i} U \subset cl_{\tau_j}\left(cl_{\tau_i} U\right) \subset X - B, \text{ and}$$

$$B \subset int_{\tau_j} V \subset V \subset cl_{\tau_j} V \subset cl_{\tau_i}\left(cl_{\tau_j} V\right) \subset X\text{-}A.$$

Proof. If $A = \cup_{n=1}^{\infty} A_n$ and $B = \cup_{n=1}^{\infty} B_n$, with A_n τ_i-closed and B_n τ_j-closed. Applying (3.2) to A_1 and $X - cl_{\tau_j} B$, there exists a τ_j-open set U_1 satisfying

$$A_1 \subset int_{\tau_i} U_1 \subset U_1 \subset cl_{\tau_i} U_1 \subset cl_{\tau_j}\left(cl_{\tau_i} U_1\right) \subset X\text{-}cl_{\tau_j} B.$$

As $B_1 \cap \left(cl_{\tau_i} A \cup cl_{\tau_i} U_1\right) \subset \left(B \cap cl_{\tau_i} A\right) \cup \left(B \cap cl_{\tau_i} U_1\right) = \emptyset$, by the same (3.2) applied to B_1 and $X - \left(cl_{\tau_i} A \cup cl_{\tau_i} U_1\right)$, there exists a τ_i-open set V_1 such that

$$B_1 \subset int_{\tau_j} V_1 \subset V_1 \subset cl_{\tau_j} V_1 \subset cl_{\tau_i}(cl_{\tau_j} V_1) \subset X\text{-}\left(cl_{\tau_i} A \cup cl_{\tau_i} U_1\right).$$

Now, given that $\left(A_2 \cup cl_{\tau_i} U_1\right) \cap \left(cl_{\tau_j} B \cup cl_{\tau_j} V_1\right) = \emptyset$ by the same argument applied to $A_2 \cup cl_{\tau_i} U_1 \subset X - \left(cl_{\tau_j} B \cup cl_{\tau_j} V_1\right)$, there is a τ_j -open set U_2 with

$$A_2 \cup cl_{\tau_i} U_1 \subset int_{\tau_i} U_2 \subset cl_{\tau_i} U_2 \subset cl_{\tau_j}\left(cl_{\tau_i} U_2\right) \subset X\text{-}\left(cl_{\tau_j} B \cup cl_{\tau_j} V_1\right).$$

Again applying (3.2) to $B_2 \cup cl_{\tau_j} V_1$ and $X\text{-}\left(cl_{\tau_i} A \cup cl_{\tau_i} U_1 \cup cl_{\tau_i} U_2\right)$, since $(B_2 \cup cl_{\tau_j} V_1) \cap \left(cl_{\tau_i} A \cup cl_{\tau_i} U_1 \cup cl_{\tau_i} U_2\right) = \emptyset$, we obtain a τ_i-open set V_2 such that:

$$B_2 \cup cl_{\tau_j} V_1 \subset int_{\tau_j} V_2 \subset cl_{\tau_j} V_2 \subset cl_{\tau_i}(cl_{\tau_j} V_2) \subset X\text{-}\left(cl_{\tau_i} A \cup cl_{\tau_i} U_1 \cup cl_{\tau_i} U_2\right).$$

Continuing in this way, the U_n τ_j-open and V_n, τ_i-open sets are defined inductively, satisfying

$$\begin{aligned} A_n \cup cl_{\tau_i} U_1 \cup \cdots \cup cl_{\tau_i} U_{n-1} &\subset int_{\tau_i} U_n \subset U_n \subset cl_{\tau_i} U_n \subset cl_{\tau_j}\left(cl_{\tau_i} U_n\right) \\ &\subset X\text{-}\left(cl_{\tau_j} B \cup cl_{\tau_j} V_1 \cup \cdots \cup cl_{\tau_j} V_{n-1}\right) \end{aligned}$$

and

$$\begin{aligned} B_n \cup cl_{\tau j} V_1 \cup \cdots \cup cl_{\tau_j} V_{n-1} &\subset int_{\tau_j} V_n \subset V_n \subset cl_{\tau_j} V_n \subset cl_{\tau_i}(cl_{\tau_j} V_n) \\ &\subset X - \left(cl_{\tau_i} A \cup cl_{\tau_i} U_1 \cup \cdots \cup cl_{\tau_i} U_n\right) \end{aligned}$$

for all natural numbers n, where $U_0 = V_0 = \emptyset$.

By doing $U = \cup_{n=0}^{\infty} U_n$, U is a τ_j-open, whereas $V = \cup_{n=0}^{\infty} V_n$ is τ_i-open and clearly

$$A \subset \cup_n int_{\tau_i} U_n \subset U, \;\; B \subset \cup_n int_{\tau_j} V_n \subset V \text{and} \;\; U \cap V = \emptyset.$$

Given that V is τ_i-open and $U \cap V = \emptyset$, it follows that $\left(cl_{\tau_i} U\right) \cap V = \emptyset$, and then $\left(cl_{\tau_i} U\right) \cap B = \emptyset$. Furthermore, of $\cup_n int_{\tau_i} U_n \subset int_{\tau_i}(\cup_n U_n)$, we obtain

$$A \subset int_{\tau_i} U \subset cl_{\tau_i} U \subset X - B.$$

Applying (3.2) to the sets $cl_{\tau_i} U$ and $X - B$ a τ_j-open set W is obtained, such that

$$cl_{\tau_i} U \subset int_{\tau_i} W \subset cl_{\tau_i} W \subset cl_{\tau_j}\left(cl_{\tau_i} W\right) \subset X - B.$$

Then of $cl_{\tau_j}(cl_{\tau_i} U) \subset cl_{\tau_j}\left(cl_{\tau_i} W\right)$,

$$A \subset int_{\tau_i} U \subset cl_{\tau_i} U \subset cl_{\tau_j}(cl_{\tau_i} U) \subset X - B.$$

Similarly, $B \subset int_{\tau_j} V \subset cl_{\tau_j} V \subset cl_{\tau_i}(cl_{\tau_j} V) \subset X - A$ is true. ■

Proof (of the theorem). Suppose first that $f, g : X \longrightarrow [0, 1]$. By doing $D_0 = \emptyset$ and for $r \in \mathbb{Q} \cap [0, 1]$, $D_r = g^{-1}([0, r))$ and $O_r = f^{-1}((r, 1])$. The set $g^{-1}([0, \acute{r}])$ is τ_j-closed because g is a τ_j-lsc function, and $D_r = \cup_{0<\acute{r}<r} g^{-1}([0, \acute{r}])$ shows D_r to be a τ_j-F_σ set. Now, given that f is τ_i-usc, $f^{-1}([\acute{r}, 1])$ is a τ_j-closed set and then $O_r = \cup_{r<\acute{r}<1} f^{-1}([\acute{r}, 1])$ shows O_r to be a τ_j-F_σ. It is easy to verify

$$cl_{\tau_i} D_r \cap O_r = \phi = D_r \cap cl_{\tau_j} O_r. \tag{3.9}$$

By (3.9) the sets D_1 and O_1 satisfy the hypothesis of (3.8), then there exists a τ_j-open set U_1 such that

$$D_1 \subset int_{\tau_i} U_1 \subset U_1 \subset cl_{\tau_i} U_1 \subset cl_{\tau_j}(cl_{\tau_i} U_1) \subset X\text{-}O_1.$$

In the same way, there exists a τ_j-open set $U_{\frac{1}{2}}$ with

$$D_{\frac{1}{2}} \subset int_{\tau_i} U_{\frac{1}{2}} \subset U_{\frac{1}{2}} \subset cl_{\tau_i} U_{\frac{1}{2}} \subset cl_{\tau_j}(cl_{\tau_i} U_{\frac{1}{2}}) \subset [X\text{-}O_{\frac{1}{2}}] \cap U_1.$$

Afterwards, there are two τ_j-open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$$D_{\frac{1}{4}} \subset int_{\tau_i} U_{\frac{1}{4}} \subset U_{\frac{1}{4}} \subset cl_{\tau_i} U_{\frac{1}{4}} \subset cl_{\tau_j}(cl_{\tau_i} U_{\frac{1}{4}}) \subset [X\text{-}O_{\frac{1}{4}}] \cap U_{\frac{1}{2}} \text{ and}$$

$$(D_{\frac{3}{4}} \cup cl_{\tau_i} U_{\frac{1}{2}}) \subset int_{\tau_i} U_{\frac{3}{4}} \subset U_{\frac{3}{4}} \subset cl_{\tau_i} U_{\frac{3}{4}} \subset cl_{\tau_j}(cl_{\tau_i} U_{\frac{3}{4}}) \subset [X\text{-}O_{\frac{3}{4}}] \cap U_1$$

Continuing in this manner, one obtains by induction, the τ_j-open sets U_r satisfying

$$(D_{\frac{2i-1}{2^m}} \cup cl_{\tau_i} U_{\frac{i-1}{2^{m-1}}}) \subset int_{\tau_i} U_{\frac{2i-1}{2^m}} \subset cl_{\tau_j}(cl_{\tau_i} U_{\frac{2i-1}{2^m}}) \subset [X\text{-}O_{\frac{2i-1}{2^m}}] \cap U_{\frac{i}{2^{m-1}}}$$

for $i = 1, 2, \ldots, 2^{m-1}$, $m \in \mathbb{N}$, where $U_0 = \emptyset$. Thus, in general

$$D_r \subset int_{\tau_i} U_r \subset U_r \subset cl_{\tau_i} U_r \subset cl_{\tau_j}(cl_{\tau_i} U_r) \subset X\text{-}O_r \text{ for all } r \in \Re \text{ and}$$

$$U_r \subset cl_{\tau_j}(cl_{\tau_i} U_r) \subset int_{\tau_i} U_{r'} \subset U_{r'} \text{ for all } r, r' \in \Re \text{ with } r < r'.$$

Now, if the function h is defined by $h(x) = \inf\{r \in \Re : x \in U_r\}$, $x \in X$, then as in the definition of f in the first theorem of this paper, there are other expressions for h

$$\begin{aligned} h(x) &= \inf\{r \in \Re : x \in int_{\tau_i} U_r\} = \inf\{r \in \Re : x \in cl_{\tau_i} U_r\} \\ &= \inf\{r \in \Re : x \in cl_{\tau_j}(cl_{\tau_i} U_r)\}, \ x \in X \end{aligned}$$

The bicontinuity of h can be deduced in the same way as for the f mentioned.

To prove that $f \leq h \leq g$, if $x \in X$ and $r \in \Re$ are such that $r < f(x)$, then for all $r' \in \Re$ with $0 \leq r' < r$, it must be $x \notin X\text{-}O_{r'}$ and therefore $x \notin U_{r'}$ implying that $h(x) \geq r$. But if $h(x) \geq r$ for all $r \in \mathcal{R}$ with $r < f(x)$, then $f(x) \leq h(x)$. Now, if $r \in \Re$ is such that $g(x) < r$, then $x \in D_r \subset U_r$ and then $h(x) \leq r$ for all $r \in \mathcal{R}$ with $g(x) < r$, which implies that $h(x) \leq g(x)$.

The general case $f : X \longrightarrow \mathbb{R}$ can be deduced as is done in [1] page 17, using the homeomorphism $\varphi : \mathbb{R} \longrightarrow (0, 1)$ given by

$$\varphi(x) = \frac{|x| + x + 1}{2(|x| + 1)}, \ x \in \mathbb{R}.$$

Inversely, suppose that A and B are two disjoint sets, τ_i-closed and τ_j-closed, respectively. The characteristic functions χ_A and χ_{X-B} are τ_i-usc, τ_j-lsc and $\chi_A \leq \chi_{X-B}$. By hypothesis there exists a bicontinuous function $h : X \longrightarrow [0, 1]$ such that $\chi_A \leq h \leq \chi_{X-B}$, and then $U = h^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ is a τ_i-open and τ_j-open set, whereas $D = h^{-1}\left(\left[\frac{1}{4}, 1\right]\right)$ is a τ_i-closed and τ_j-closed set. Clearly

$$A \subset int_{\tau_i} U \subset U \subset D \subset cl_{\tau_j} D \subset X - B.$$

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