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Operators Generating New Structures on Banach Manifold

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Abstract

In this paper, we study some properties of bounded linear operators on Banach manifolds. We show that the relation induced by bounded linear operator forms an equivalence relation on Banach manifold. Further, we show that the set of all bijective bounded linear operators, the set of all bijective closed linear operators and the set of all bijective continuous linear operators on Banach manifold forms a group structure.

Subject classification: 46XX, 20XX, 58BXX.

Keyword: Banach manifold, bounded linear operators, closed linear operators, continuous linear operators.

1. Introduction

We know that, if M is a Banach manifold [2], [3], [4], [8] modeled on Banach space E then any subset of M is locally homeomorphic to E. If M is considered to be a Banach manifold with characteristics of Banach space then one can define linear operators between any two Banach manifolds and induce some of the basic properties of Banach spaces on to Banach manifolds.

In this paper, S. C. P. Halakatti has introduced analytical and algebraic structures on Banach manifolds, that is, the bounded linear operator on M defines a new relation $' \leq '$ as an equivalence relation that generates new structure $(M, \leq)$ on M. Further, the set of all bijective bounded linear operators on M forms a group structure under composition generating one more new structure $\left((M,\leq),(G,\circ)\right)$ on M.

2. Preliminaries

Some basic definitions and results are referred from[1],[5],[6],[7]and are as follows.

Definition 2.1: Let E_1 and E_2 be normed linear spaces, a linear operator $A: E_1 \to E_2$ is said to be continuous at $x \in E_1$ if for every $\epsilon > 0 \,\exists\, \delta > 0$ such that:
$\|u-x\| < \delta \Longrightarrow \|A(u) - A(x)\| < \epsilon$ for $u \in E_1$.

Definition 2.2: Let E_1 and E_2 be normed linear spaces, a linear operator $A: E_1 \to E_2$ is said to be bounded if it satisfies the following condition:
$\|A(x)\| \leq C\|x\| \,\forall\, x \in E_1, C > 0$.

Definition 2.3: Let E_1 and E_2 be normed linear spaces and E_0 be a subspace of E_1, a linear operator $A: E_0 \subset E_1 \to E_2$ is said to be a closed operator if for every (x_n) in E_0 $x_n \to x$ and $Ax_n \to Ax$ for some $x \in E_1$ *and* $y \in E_2$ such that $x \in E_0$ and $Ax = y$.
i.e., A is closed iff $G(A) = \{(x, Ax)/\, x \in E_0\}$ is closed.

Theorem 2.1: Let $A: E_0 \subset E_1 \to E_2$ be bounded operator. If E_0 is closed in E, then A is closed operator.

Theorem 2.2: Let E_1 and E_2 be Banach spaces, and $A \in \mathcal{B}(E_1, E_2)$ be bijective. Then $A^{-1} \in \mathcal{B}(E_1, E_2)$.

Theorem 2.3: If E_1 and E_2 are Banach spaces, then every bounded linear operator $A: E_1 \to E_2$ is continuous.

3. Main Results

In this section, two different structures are introduced on Banach manifold. The following results are introduced and proved by S. C. P. Halakatti.

Theorem 3.1: *Let M_1, M_2 and M_3 be Banach manifolds and $A: M_1 \to M_2$, $B: M_2 \to M_3$ be bounded linear operators then $B \circ A: M_1 \to M_3$ is bounded linear operator.*

Proof: Let M_1, M_2 and M_3 be Banach manifolds and $A: M_1 \to M_2$, $B: M_2 \to M_3$ be bounded linear operators then we have,
$\|A(x)\| \leq C_1\|x\| \,\forall\, x \in M_1, C_1 > 0$ and $\|B(y)\| \leq C_2\|y\| \,\forall\, y \in M_2,\, C_2 > 0$.
Consider a composition of bounded linear operators $B \circ A: M_1 \to M_3$.
Now we shall show that $B \circ A$ is also bounded linear operator.
Let $A(x) = y \in M_2$ and $B(y) = z \in M_3$,
i.e., $\|y\| \leq C_1\|x\|$ and $\|z\| \leq C_2\|y\|$ …………………(i)

Also, we know that $B \circ A(x) = B(A(x)) = B(y) = z$,
$\Rightarrow \|B \circ A(x)\| = \|z\|$,
$\Rightarrow \|z\| \leq C_2\|y\|$, ($\because$from (i))
$\leq C_1C_2\|x\|$,
$\Rightarrow \|B \circ A(x)\| \leq C\|x\|$, where $C = C_1C_2$.
Therefore we can say that the composition of two bounded linear operators is also a bounded linear operator. ■

Definition 3.1: Let M be a Banach manifold. If $A: M \to M$ is a bounded linear operator on M such that $\|A(x)\| \leq C\|x\| \; \forall \; x \in M, C > 0$ then, we say that $A(x), x \in M$ are related by the relation $' \leq '$ for all $x \in M$.

Theorem 3.2: *Let M be a Banach manifold and $A: M \to M$ is a bounded linear operator then the relation $' \leq '$ is an equivalence relation.*

Proof: Let $A: M \to M$ be bounded linear operator on M such that:
$\|A(x)\| \leq C\|x\| \; \forall \; x \in M, C > 0$.
Now we shall show that the relation induced by bounded linear operators is an equivalence relation.

i)Reflexive: Reflexive relation is trivial by considering a identity bounded linear operator $A: M \to M$ defined as $A(x) = x \; \forall \; x \in M$,
then we get $\|A(x)\| \leq C\|x\|$
i.e., $\|x\| \leq C\|x\| \; \forall \; x \in M$ and $C > 0$.
Therefore the relation is reflexive.

ii)Symmetry: Let $A: M \to M$ be a linear operator such that:
$\|A(x)\| \leq C\|x\| \; \forall \; x \in M, C > 0$, then it can be easily seen that
$C'\|x\| \leq \|A(x)\| \; \forall \; x \in M$ and $C' = \frac{1}{c}$, where $C, C' > 0$.
Therefore the relation is symmetric.

iii)Transitivity: Let $A: M \to M$ be a linear operator such that:
$\|A(x)\| \leq C\|x\| \; \forall \; x \in M, C > 0$.
Let $\|A(x)\| \leq C_1\|x\|$ and $\|A(y)\| \leq C_2\|y\| \; \forall \; x, y \in M$ & $C_1, C_2 > 0$,
Let $A(x) = y$ and $A(y) = z$
then we get, $\|y\| \leq C_1\|x\|$ and $\|z\| \leq C_2\|y\|$ …….(i)
Consider $\|A(x)\| \leq C_1\|x\|$,
i.e., $\|y\| \leq C_1\|x\| \; \forall \; x, y \in M$,
$\Rightarrow C_2\|y\| \leq C_1 C_2\|x\| \; \forall \; x, y \in M$,
$\Rightarrow \|z\| \leq C_2\|y\|$,
$\leq C_1 C_2\|x\|$,
$\Rightarrow \|z\| \leq C\|x\|$ where $C = C_1 C_2 > 0$.
i.e., $\|A(y)\| \leq C\|x\| \; \forall \; x, y \in M$ and $C > 0$.
Therefore the relation is transitive.

Hence the relation $' \leq '$ between any two members of M is an equivalence relation. ■

Therefore by the above theorem we say that the Banach manifold along with differentiable structure has one more new structure induced by bounded linear operators denoted by $(M, \leq)$.

Further, we know that the set of all bounded linear operators from M to N is denoted by $\mathcal{B}(M, N)$. If $N = M$ then we denote it by $\mathcal{B}(M)$. Now we show that the set of all bijective bounded linear operators $\mathcal{B}(M)$ on M forms a group under composition of bounded linear operator.

Theorem 3.3: *Let M be Banach manifold. Then set of all bijective bounded linear operators $\mathcal{B}(M)$ on M forms a group under composition of bounded linear operators.*

Proof: Let M be Banach manifold and $\mathcal{B}(M)$ be the set of all bijective bounded linear operators defined on M. Now we shall show that $\mathcal{B}(M)$ is group under composition of bijective bounded linear operators.

i) Let A, $B \in \mathcal{B}(M)$, by Theorem 3.1 we know that the composition of bounded linear operators is also a bounded linear operator. So, we can say that composition of bijective bounded linear operator is also a bijective bounded linear operator.

Therefore for every $A, B \in \mathcal{B}(M)$, $A \circ B \in \mathcal{B}(M)$ (∵ by theorem 3.1).

ii) Let $A, B, C \in \mathcal{B}(M)$ then we show that $A \circ (B \circ C) = (A \circ B) \circ C$.

Consider $A \circ (B \circ C) \in \mathcal{B}(M)$,

Then we get $A \circ (B \circ C) = A \circ B \circ C$ (∵ by theorem 3.1),

$= (A \circ B) \circ C \in \mathcal{B}(M)$,

Therefore $A \circ (B \circ C) = (A \circ B) \circ C$.

iii) Let $A \in \mathcal{B}(M)$ then $\exists\, A^{-1} \in \mathcal{B}(M)$,

such that $A \circ A^{-1} = I \in \mathcal{B}(M)$ (∵ by Bounded inverse theorem [7]).

Hence $\forall\, A \in \mathcal{B}(M)\ \exists\, A^{-1} \in \mathcal{B}(M)$ and $I \in \mathcal{B}(M)$.

iv) Let $A \in \mathcal{B}(M)$ then $I \in \mathcal{B}(M)$, such that $A \circ I = A \in \mathcal{B}(M), \forall\, A \in \mathcal{B}(M)$.

Therefore $I \in \mathcal{B}(M)$.

Hence $\mathcal{B}(M)$ forms a group structure on M under a composition of bijective bounded linear operators. ■

Corollary 3.1: *Let M be Banach manifold and $\mathcal{B}(M)$ be set of all bijective closed linear operators on M. If M is closed then $\mathcal{B}(M)$ forms a group under composition of closed linear operators.*

Proof: Let M be a closed Banach manifold and $\mathcal{B}(M)$ be the set of all bijective closed linear operators defined on M. We know that a bounded linear operator with its domain closed is a closed linear operator (∵ by theorem 3.17 [7]). Therefore from

Theorem 3.3 it is clear that the set of all bijective closed linear operators $\mathcal{B}(M)$ forms a group structure on M under composition of closed linear operators. ■

Corollary 3.2: *Let M be Banach manifold. Then the set of all bijective continuous linear operators $\mathcal{B}(M)$ on M forms a group under composition of continuous linear operators.*

Proof: Let M be Banach manifold and $\mathcal{B}(M)$ be the set of all bijective continuous linear operators defined on M. The Closed graph theorem[7] suggest that every closed operator defined between Banach spaces are continuous. Therefore by Theorem 3.3 and Corollary 3.1, it is clear that the set of all continuous linear operators $\mathcal{B}(M)$ forms a group structure on M under composition of continuous linear operators. ■

Conclusion

The present paper is an attempt to enrich the Banach manifold by inducing analytical and algebraic structures on it. Such an approach paves a way for a deeper understanding of intrinsic structure of Banach manifolds and its application to the Network structure.

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