Symmetric Skew 4-Derivations on Semi Prime Rings

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Abstract

In this paper we introduce the notation of symmetric skew 4-derivation of Semiprime ring and we consider $R$ be a non-commutative 2, 3-torsion free semi prime ring, $I$ be a non zero two sided ideal of $R$, $\alpha$ be an automorphism of $R$, and $D: R^4 \rightarrow R$ be a symmetric skew 4-derivation associated with the automorphism $\alpha$. If $f$ is trace of $D$ such that $[f(x), \alpha(x)] \in Z$ for all $x \in I$, then $[f(x), \alpha(x)] = 0$, for all $x \in I$.

Keywords: Semiprime ring, Derivation, Bi derivation, Symmetric Skew 3-derivation, Symmetric Skew 4-derivation and Auto orphism.

Introduction

In 1957, the study of centralizing and commuting mappings on aprime rings was initiated by the result of E. C. Posner [2] which states that the existence of a non-zero centralizing derivation on a prime ring implies that the ring has to be commutative. Further Vukman [4, 5] extended above result for bi derivations. Recently jung and park[6]considered permuting 3-derivations on prime and semi prime rings and obtained the following:Let $R$ be a non-commutative3-torsion free semi prime ring and let $I$ be a non-zero two sided ideal of $R$. Suppose that there exists a permuting 3-derivation $D: R^3 \rightarrow R$ such that $f$ is centralizing on $I$then $f$ is commuting on $I$. A. Fosner [1] extended the above results in symmetric skew 3-derivations with prime rings and semi prime rings. Recently Faiza Shujat, Abuzaid Ansari[3] Studied some results in symmetric skew 4-derivations in prime rings. In this Paper we proved that Symmetric skew 4-derivations in semi prime rings.
Preliminaries
Throughout this paper, $R$ will be represent a ring with a center $Z$ and $\alpha$ be an automorphism of $R$. Let $n \geq 2$ be an integer. A ring $R$ is said to be $n$-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. For all $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. We make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is semi prime if $xRx = 0$ implies that $x = 0$. An additive map $d: R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$, and it is called a skew derivation ($\alpha$-derivation) of $R$ associated with automorphism $\alpha$ if $d(xy) = \alpha(x)d(y)$ for all $x, y \in R$, associated with automorphism $\alpha$ if $d(xy) = x\alpha(y)d(x)$ for all $x, y \in R$.

Before starting our main theorem, let us give some basic definitions and well-known results which we will need in our further investigation.

Let $D$ be a symmetric 4-additive map of $R$, then obviously

$$D(-(p, q, r, s)) = -D(p, q, r, s), \quad \text{for all } p, q, r, s \in R \tag{1}$$

Namely, for all $y, z \in R$, the map $D(\ldots, y, z): R \to R$ is an endomorphism of the additive group of $R$.

The map $f: R \to R$ defined by $f(x) = D(x, x, x, x)$, $x \in R$ is called trace of $D$.

Note that $f$ is not additive on $R$. But for all $x, y \in R$, we have

$$f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)]$$

Recall that by equation (1), $f$ is even function.

More precisely, for all $p, q, r, s, u, v, w, x \in R$, we have

$$D(pu, q, r, s) = D(p, q, r, s)u + \alpha(p)D(u, q, r, s),$$
$$D(p, qv, r, s) = D(p, q, r, s)v + \alpha(q)D(p, v, r, s),$$
$$D(p, qr, w, s) = D(p, q, r, s)w + \alpha(r)D(p, q, w, s),$$
$$D(p, q, rs, x) = D(p, q, r, s)x + \alpha(s)D(p, q, r, x).$$

Of course, if $D$ is symmetric, then the above four relations are equivalent to each other.

**Lemma 1:**
Let $R$ be a prime ring and $a, b \in R$. If $a[x, b] = 0$, for all $x \in R$, then either $a = 0$ or $b \in Z$.

**Proof:**
Note that

$$0 = a[x, b] = ax[y, b] + a[x, b]y = ax[y, b], \quad \text{for all } x, y \in R.$$ 

Thus $aR[y, b] = 0$, $y \in R$, and, since $R$ is prime, either $a = 0$ or $b \in Z$.

**Theorem 1:**
Let $R$ be a $2, 3$ -torsion free non commutative semiprime ring and $I$ be a nonzero ideal of $R$. Suppose $\alpha$ is an automorphism of $R$ and $D: R^4 \to R$ is a symmetric skew 4-derivation associated with $\alpha$. If $f$ is trace of $D$ such that $[f(x), \alpha(x)] \in Z$ for all $x \in R$. Then...
Let \( [f(x), \alpha(x)] \in Z \), for all \( x \in I \).

Linearization of (2) yields that, we have

\[
[f(x + y), \alpha(x + y)] \in Z
\]

\[
[ f(x + y), \alpha(x) ] + [ f(x + y), \alpha(y) ] \in Z
\]

By skew 4-derivation, we have

\[
f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)]
\]

\[
[f(x), \alpha(x)] + 4[D(x, x, x, y), \alpha(x)] + 6[D(x, x, y, y), \alpha(x)] +
\]

\[
4[D(x, y, y, y), \alpha(x)] + [f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x, x, x, y), \alpha(y)] +
\]

\[
6[D(x, x, x, y), \alpha(y)] + [f(x), \alpha(y)] + 4[D(x, x, y, y), \alpha(y)] + [f(y), \alpha(y)] \in Z,
\]

for all \( x \in I \).

From (2) & (3), we get

\[
4[D(x, x, x, x, y), \alpha(x)] + 6[D(x, x, x, y, y), \alpha(x)] +
\]

\[
4[D(x, x, x, y, y), \alpha(x)] + [f(x), \alpha(x)] + 4[D(x, x, x, y), \alpha(y)] +
\]

\[
6[D(x, x, x, y), \alpha(y)] + 4[D(x, y, y, y), \alpha(y)] \in Z,
\]

for all \( x \in I \).

Comparing (4) and (5) and using 2-torsion freeness of \( R \), we have

\[
4[D(x, x, x, y, y), \alpha(x)] + 4[D(x, x, y, y, y), \alpha(x)] + [f(x), \alpha(y)] + 6[D(x, x, y, y), \alpha(y)]
\]

\[
\in Z,
\]

for all \( x \in I \).

Substitute \( y + z \) for \( y \) in (6) and use (6), we get

\[
4[D(x, x, x, y + z), \alpha(x)] + 4[D(x, y + z, y + z, y + z), \alpha(x)] + [f(x), \alpha(y + z)]
\]

\[
+ 6[D(x, y + z, y + z, y + z), \alpha(y + z)] \in Z
\]

\[
+ 4[D(x, y, y, y), \alpha(y)] + 4[D(x, z, z, y), \alpha(x)] + 4[D(x, z, z, z), \alpha(x)]
\]

\[
+ 4[D(x, z, z, z), \alpha(x)] + [f(x), \alpha(y)] + [f(x), \alpha(z)]
\]

\[
+ 6[D(x, x, x, y), \alpha(y)] + 6[D(x, x, x, z), \alpha(y)] +
\]

\[
6[D(x, x, x, z), \alpha(y)] + 6[D(x, x, z, z), \alpha(y)] +
\]

\[
6[D(x, x, x, z), \alpha(z)] + 6[D(x, x, z, z), \alpha(z)] +
\]

\[
6[D(x, x, x, z), \alpha(z)] + 6[D(x, x, z, z), \alpha(z)] \in Z
\]
\[
4[D(x, y, y, z), \alpha(x)] + 4[D(x, y, z, y), \alpha(x)] + 4[D(x, y, z, z), \alpha(x)] \\
+ 4[D(x, z, y, y), \alpha(x)] + 4[D(x, z, y, z), \alpha(x)] \\
+ 4[D(x, z, z, y), \alpha(x)] + 6[D(x, x, y, y), \alpha(y)] \\
+ 6[D(x, x, y, z), \alpha(y)] + 6[D(x, x, z, y), \alpha(y)] \\
+ 6[D(x, x, z, z), \alpha(y)] + 6[D(x, x, y, z), \alpha(z)] \\
+ 6[D(x, x, y, y), \alpha(z)] + 6[D(x, x, z, y), \alpha(z)] + 6[D(x, x, z, z), \alpha(z)] \in Z
\]

\[
12[D(x, y, y, z), \alpha(x)] + 12[D(x, y, z, z), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] \\
+ 6[D(x, x, z, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)] + 6[D(x, x, y, y), \alpha(z)] \in Z,
\]

for all \(x, y, z \in I\).  

(7)

Replacing \(z\) in \(-z\) in (7) and compare with (7), we obtain

\[
-12[D(x, y, y, z), \alpha(x)] + 12[D(x, y, z, z), \alpha(x)] - 12[D(x, x, y, z), \alpha(y)] \\
+ 6[D(x, x, z, z), \alpha(y)] - 6[D(x, x, y, y), \alpha(z)] + 6[D(x, x, y, y), \alpha(z)] \in Z
\]

\[
2(12[D(x, z, y, y), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)]) \in Z
\]

Using of two torsion free ring, we have

\[
12[D(x, z, y, y), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, y), \alpha(z)] \in Z,
\]

for all \(x, y, z \in I\).  

(8)

Substitute \(y + u\) for \(y\) in (8) and use (8) we get

\[
12[D(x, z, y + u, y + u), \alpha(x)] + 12[D(x, x, y + u, z), \alpha(y + u)] \\
+ 6[D(x, x, y + u, y + u), \alpha(z)] \in Z
\]

\[
12[D(x, z, y, y), \alpha(x)] + 12[D(x, z, y, u), \alpha(x)] + 12[D(x, z, u, y), \alpha(x)] \\
+ 12[D(x, x, z, u), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] \\
+ 12[D(x, x, u, z), \alpha(y)] + 12[D(x, x, y, z), \alpha(u)] \\
+ 12[D(x, x, u, z), \alpha(u)] + 6[D(x, x, y, y), \alpha(z)] \\
+ 6[D(x, x, y, u), \alpha(z)] + 6[D(x, x, u, y), \alpha(z)] \\
+ 6[D(x, x, u, u), \alpha(z)] \in Z
\]

\[
24[D(x, z, y, u), \alpha(x)] + 12[D(x, x, y, z), \alpha(u)] + 12[D(x, x, u, z), \alpha(y)] + 12[D(x, x, y, u), \alpha(z)] \in Z
\]

for all \(x, y, z \in I\).  

(9)

Since \(R\) is 2 and 3-torsion free and replacing \(y, u\) by \(x\) in (9), we have

\[
24[D(x, z, x, x), \alpha(x)] + 12[D(x, x, x, z), \alpha(x)] + 12[D(x, x, z, x), \alpha(x)] + 12[D(x, x, x, x), \alpha(z)] \in Z
\]

\[
48[D(x, x, x, z), \alpha(x)] + 12[D(x, x, x, x), \alpha(z)] \in Z
\]

\[
4[D(x, x, x, z), \alpha(x)] + [f(x), \alpha(z)] \in Z, \text{ for all } x, z \in I.
\]

(10)

Again replaced \(z\) by \(xz\) in (10) and using (10) we obtain

\[
4[D(x, x, x, x), \alpha(x)] + [f(x), \alpha(x)] \in Z, \text{ for all } x, z \in I.
\]

\[
4[D(x, x, x, x), \alpha(x)] + [f(x), \alpha(x)] \in Z, \text{ for all } x, z \in I.
\]

\[
4[D(x, x, x, x)z + \alpha(x)D(x, x, x, z), \alpha(x)] + [f(x), \alpha(x)]\alpha(z) + \alpha(x)[f(x), \alpha(z)] \in Z,
\]

for all \(x, z \in I\).

\[
4[f(x)[z, \alpha(x)] + 4[f(x), \alpha(x)]z + 4\alpha(x)[D(x, x, x, z), \alpha(x)] + [f(x), \alpha(x)]\alpha(z) + \alpha(x)[f(x), \alpha(z)] \in Z,
\]

for all \(x, z \in I\).

\[
\alpha(x)((f(x), \alpha(z)] + 4[D(x, x, x, z), \alpha(x)]) + (\alpha(z) + 4\alpha(f(x), \alpha(x)]) +
\]


\[4f(x)[z, \alpha(x)] \in Z, \text{ for all } x, z \in I.\] (11)

Therefore, from (11), we get
\[\alpha(x)([f(x), \alpha(z)] + 4D(x, x, x, z), \alpha(x)), \alpha(x)] + [(\alpha(z) + 4z)f(x), \alpha(x)] + 4f(x)[z, \alpha(x)], \alpha(x)] + 4[f(x), \alpha(x)][z, \alpha(x)] = 0, \text{ for all } x, z \in I.\] (12)

\[\alpha(x)([f(x), \alpha(z)], \alpha(x)] + [\alpha(z) + 4z, \alpha(x)][f(x), \alpha(x)] + 4f(x)[z, \alpha(x)], \alpha(x)] + 4[f(x), \alpha(x)][z, \alpha(x)] = 0, \text{ for all } x, z \in I.\]

Replacing \(z\) by \(f(x)[f(x), \alpha(x)]\) in (13), we get
\[\alpha(x)([f(x), \alpha(z)], \alpha(x)] + [\alpha(z) + 8z, \alpha(x)][f(x), \alpha(x)] + 4f(x)[z, \alpha(x)], \alpha(x)] = 0, \text{ for all } x, z \in I.\]

Since \(f\) is commuties on \(I\) and we have 2, 3-torsion freeness,
\[2[f(x), \alpha(x)]^2 = 0.\] (14)

On the other hand, taking \(z = x^2\) in equation (10), we get
\[4[D(x, x, x, x^2), \alpha(x)] + [f(x), \alpha(x^2)] \in Z, \text{ for all } x \in I.\]

\[4[D(x, x, x, x)x + \alpha(x)D(x, x, x, x), \alpha(x)] + [f(x), \alpha(x^2)] \in Z, \text{ for all } x \in I.\]
4[f(x)x + α(x)f(x), α(x)] + α(x)[f(x), α(x)] + [f(x), α(x)]α(x) ∈ Z, for all x ∈ I.
4[f(x)x, α(x)] + 4[α(x)f(x), α(x)] + 2α(x)[f(x), α(x)] ∈ Z, for all x ∈ I.
4f(x)[x, α(x)] + 4[f(x), α(x)]x + 4α(x)[f(x), α(x)] + 4[α(x), α(x)]f(x) + 2α(x)[f(x), α(x)] ∈ Z, for all x ∈ I.
6α(x)[f(x), α(x)] + 4x[f(x), α(x)] + 4f(x)[x, α(x)] ∈ Z, for all x ∈ I. \[15\]

Therefore, from equation \((15)\), we get
\[\begin{align*}
[f(x), 6α(x)[f(x), α(x)] + 4x[f(x), α(x)] + 4f(x)[x, α(x)] = 0, & \text{ for all } x ∈ I. \\
[f(x), 6α(x)[f(x), α(x)] + [f(x), 4x[f(x), α(x)] + [f(x), 4f(x)[x, α(x)] = 0. \\
6α(x)[f(x), [f(x), α(x)] + 6[f(x), α(x)][f(x), α(x)] + 4x[f(x), [f(x), α(x)]] \\
+ 4[f(x), x][f(x), α(x)] + 4f(x)[f(x), [x, α(x)]] \\
+ 4[f(x), f(x)][x, α(x)] = 0. \\
6[f(x), α(x)]^2 + 4f(x)[f(x), [x, α(x)]] = 0, & \text{ for all } x ∈ I. \\
6[f(x), α(x)]^2 + 4f(x)[f(x), x, α(x) = 0, & \text{ for all } x ∈ I. \[16\]
\end{align*}\]

Since \(f\) is commutative and using equation \((16)\), we get
\[6[f(x), α(x)]^2 = 0, & \text{ for all } x ∈ I. \[16\]

We have 2-torsion freeness, we get
\[3[f(x), α(x)]^2 = 0, & \text{ for all } x ∈ I. \[17\]

Comparing \((14)\) and \((17)\) and we have 2-torsion freeness, we get
\[f(x), α(x)]^2 = 0, & \text{ for all } x ∈ I. \[17\]

Note that zero is the only nilpotent element in the center of semiprime ring. Thus, \([f(x), α(x)] = 0, & \text{ for all } x ∈ I. \[17\]

This completes the proof.

References