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Numerical Solutions of Parabolic Constrained Control Problems

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Abstract

Optimal control problems governed by partial differential equations are a very active field of research in applied mathematics. The huge recent progress is remarkable for what has been achieved in the past in this area concerning the theoretical aspects as well as numerical results. These problems deal with the processes of hydro and gas dynamics, heat physics, aeronautics, mechanical engineering, the physics of plasma and other fields of life sciences. In this paper, we deal with a class of the constrained optimal control problem for quasilinear parabolic systems. The necessary optimality conditions and a numerical algorithm for solving the considering optimal control problem are studied. The questions of the computational realization are discussed and the instances of numerical experiments are given.

Keywords: Constrained optimal control problems, quasilinear parabolic systems, Adjoint system, Exterior penalty function method, Necessary optimality conditions.

1. Problem Formulation

Optimal control problems governed by partial differential equations are a very active field of research in applied mathematics. The huge recent progress is remarkable for what has been achieved in the past in this area concerning the theoretical aspects as well as numerical results [1-3]. These problems deal with the processes of hydro and gas dynamics, heat physics, aeronautics, mechanical engineering, the physics of plasma and other fields of life sciences [4-6].

Let D be a bounded domain of the N-dimensional Euclidean space in E_N, let l, T are given positive numbers, $0 \le t \le T$, $\Omega = D \times (0,T]$, $D = [0,l]$.

Let $V=\{v: v=(v_1, v_1, \cdots, v_N)\in E_N(\Omega), \|v\|_{E_N}\le R\}$ be a space of controls and $R>0$ are positive numbers. We consider the heat exchange process described by the equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\lambda(u,v)\frac{\partial u}{\partial x}\right)+B(u,v)\frac{\partial u}{\partial x}=f(x,t),\ (x,t)\in\Omega \tag{1}$$

with initial and boundary conditions

$$u(x,0)=\phi(x)\ ,\ x\in D \tag{2}$$

$$\lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=0}=Y_0(t)\ ,\quad \lambda(u,v)\left.\frac{\partial u}{\partial x}\right|_{x=l}=Y_1(t)\quad 0\le t\le T \tag{3}$$

where $\phi(x)\in L_2(D),\ Y_m(t)\in L_2(0,T)$, $m=1,2$, $f(x,t)$ are given functions, the functions $\lambda(u,v),\ B(u,v)$ are measured on $(x,t)\in\Omega$ at $\forall(x,t)\in\Omega$ is continuous and have continuous derivatives in u and $\frac{\partial\lambda(u,v)}{\partial u}, \frac{\partial B(u,v)}{\partial u}$ are bounded.

On the set V, under the conditions (1)-(3) and additional restrictions

$$\nu_0\le\lambda(u,v)\le\nu_1,\ \mu_0\le B(u,v)\le\mu_1\ ,\ r_1\le u(x,t)\le r_2 \tag{4}$$

is required to minimize the function

$$J_\alpha(v)=\beta_0\int_0^T[u(0,t)-f_0(t)]^2\,dt+\beta_1\int_0^T[u(l,t)-f_1(t)]^2\,dt+\alpha\|v-w\|_{E_N}^2 \tag{5}$$

where $f_m(t)\in L_2(0,T)$, $m=0,1$ are given functions, $\alpha\ge 0$ and $\beta_m\ge 0, m=0,1$, $\beta_0+\beta_1\ne 0$ are given numbers, are also given: $\omega=(\omega_0\ ,\omega_1,\cdots,\omega_N)\in E_N$.

Definition 1:

The problem of finding a function $u=u(x,t;v)\in V_2^{1,0}(\Omega)$ from conditions (1)-(4) at given $v\in V$ is called the reduced problem.

Definition 2:

The solution of the reduced problem (1)-(4) corresponding to the $v\in V$ is a function $u=u(x,t;v)\in V_2^{1,0}(\Omega)$ and satisfies the integral identity

$$\int_\Omega\left[u\frac{\partial\eta}{\partial t}-\lambda(u,v)\frac{\partial u}{\partial x}\frac{\partial\eta}{\partial x}-B(u,v)\frac{\partial u}{\partial x}\eta+f(x,t)\,\eta(x,t)\right]dx\,dt$$
$$=-\int_0^l\phi(x)\,\eta(x,0)\,dx-\int_0^T Y_0(t)\,\eta(0,t)\,dt-\int_0^T Y_1(t)\,\eta(l,t)\,dt \tag{6}$$

for all $\eta=\eta(x,t)\in W_2^{1,1}(\Omega)$ that is equal to zero for $t=T$.

The solution of the reduced problem (1)-(3) explicitly depends on the control v, therefore we shall also use the notation $u=u(x,t;v)$.

On the basis of adopted assumptions and the results of [7] follows that for every $v \in V$ the solution ofthe problem (1)-(4) is existed, unique and $|u_x \leq C_0|, \forall (x,t) \in \Omega$, $\forall\ v \in V$, where C_0 is a certain constant.

Optimal control problems of the coefficients of differential equations do not always have solution [8]. In [9], we proved the existenceand uniqueness of the solution of problem (1)-(5) as follows:

Lemma 1.1:

At above adopted assumptions for the solution of the reduced problem (1)-(5) the following estimation is valid

$$\|\delta u\|_{V_2^{1,0}(\Omega)} \leq C_1 \left(\left\| \delta\lambda \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega)} + \left\| \delta B \frac{\partial u}{\partial x} \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}} \tag{7}$$

where C_1 is constant not depending on δv .

Lemma 1.2:

The function $J_0(v)$ is continuous on V .

Theorem 1.1

The problem (1)-(5) at any $\alpha \geq 0$ has at least one solution.

Theorem 1.2

The problem (1)-(5) at $\alpha > 0$ at almost all $\omega \in E_N$ has a unique solution.

The inequality constrained problem (1) through (5) is converted to a problemwithout inequality constraints by adding a penalty function [10] to the objective (5), yielding the following $\Phi_{\alpha,s}(v, A_k)$ function:

$$\Phi_{\alpha,s}(v, A_k) \equiv \Phi(v) = J_\alpha(v) + P_k(v) \tag{8}$$

where

$$S^1(u,v) = [\max\{ v_0 - \lambda(u,v); 0\}]^2 + [\max\{ \lambda(u,v) - v_1; 0\}]^2$$

$$S^2(u,v) = [\max\{ \mu_0 - B(u,v); 0\}]^2 + [\max\{ B(u,v) - \mu_1; 0\}]^2$$

$$Q^1(u) = [\max\{ r_1 - u(x,t); 0\}]^2, Q^1(u) = [\max\{ u(x,t) - r_2; 0\}]^2$$

$$P_s(v) = A_k \int_0^l \int_0^T \left[S^1(u,v) + S^2(u,v) + Q^1(u) + Q^2(u) \right] dx\ dt$$

and $A_k, k = 1,2,\cdots$ are positive numbers, $\lim_{k \to \infty} A_k = +\infty$.

2. Necessary Optimality Condition

In the following theorem give the sufficient differentiability conditions of function (8) andits gradient formulae.

Theorem 2.1

It is assumed that the following conditions are fulfilled:

(i) The functions $\lambda(u,v), B(u,v)$ satisfy the Lipshitz condition for V.

(ii) The first derivatives of the functions $\lambda(u,v), B(u,v)$ with respect to V are continuous functionsand for any $v \in V$ such that $\|v\|_{E_N} \le R$, the functions $\lambda(u,v), B(u,v)$ belong to $L_\infty(\Omega)$.

(iii) Operators $\int_0^l \int_0^T \lambda_v(u,v)\, dx\, dt$, $\int_0^l \int_0^T B_v(u,v)\, dx\, dt$ are bounded in E_N.

Therefore, the function $\Phi_{\alpha,s}(v) \equiv \Phi(v)$ is differentiable and its gradient is given by the expression:

$$\frac{\partial \Phi(v)}{\partial v} = -\frac{\partial H}{\partial v} \equiv \left(-\frac{\partial H}{\partial v_1}, -\frac{\partial H}{\partial v_2}, \cdots, -\frac{\partial H}{\partial v_N} \right) \tag{9}$$

where the Hamiltonian function $H(u,\Theta,v)$ is defined as follows:

$$\begin{aligned} H(u,\Theta,v) = &-\int_0^l \int_0^T \left[\lambda(u,v)\Theta_x\, u_x - B(u,v)\Theta\, u_x\right] dx\, dt \\ &+ A_k \int_0^l \int_0^T \left\{S^1(u,v) + S^2(u,v)\right\} dx\, dt - \alpha \|v-\omega\|_{E_N}^2 \end{aligned} \tag{10}$$

and $\Theta(x,t)$ is the solution of the following conjugate boundary value problem for the problem (1)-(4), (8):

$$\begin{aligned} &\frac{\partial \Theta}{\partial t} - \frac{\partial}{\partial x}\left(\lambda(u,v)\frac{\partial \Theta}{\partial x}\right) + \left\{\lambda_v(u,v)\frac{\partial \Theta}{\partial x} - B(u,v)\,\Theta(x,t)\right\}\frac{\partial u}{\partial x} \\ &\qquad = -A_k\left[\frac{\partial S^1}{\partial u} + \frac{\partial S^2}{\partial u}\right],\ (x,t) \in \Omega \end{aligned} \tag{11}$$

$$\Theta(x,T) = 0 \ , x \in D \tag{12}$$

$$\left.\begin{aligned} \lambda(u,v)\frac{\partial \Theta}{\partial x}\Big|_{x=0} &= 2\beta_0 \ [u(0,t) - y_0(t)] \\ \lambda(u,v)\frac{\partial \Theta}{\partial x}\Big|_{x=l} &= -2\beta_1 \ [u(l,t) - y_1(t)] \end{aligned} \quad 0 \le t \le T \right\} \tag{13}$$

The proof of theorem 2.1 is then directly followed by theorem 4.2 in [11].
Using the above theorem, the necessary optimality conditions for the optimal control problem (1)-(3), (8) will be as the following :

Theorem 2.2

In order that a function $v^* \in V$ be a solution of problem (1)-(2), (5) it is necessary that

$$H(u^*, \Theta_k^*, v^*) = \max_{v \in V} H(u^*, \Theta^*, v) \tag{14}$$

where $u^*(x,t)$ and $\Theta^*(x,t)$ are, respectively, solutions of the basic problem and adjoint problem $v^* \in V$.

Proof:

Suppose that $v^* \equiv (v_0^*, v_1^*, v_2^*)$ such that $v_m^* \equiv (v_{0m}^*, v_{1m}^*, \cdots, v_{im}^*, \cdots) \in l_2, m = \overline{0,2}$ is an optimal control. Suppose the contrary, i.e. will be found such a control $\overline{v} = v^* + h\delta v \in V$ and number $\alpha > 0$ for which

$$H(u^*, \Theta_k^*, \overline{v}) - H(u^*, \Theta_k^*, v^*) \ge \alpha > 0, \tag{15}$$

where $h > 0$ is a constant and $\overline{v} = (\overline{v}_0, \overline{v}_1, \overline{v}_2) \equiv (v_0^* + h\delta v_0, v_1^* + h\delta v_1, v_2^* + h\delta v_2), \delta v_m = (\delta v_{0m}, \delta v_{1m}, \cdots, \delta v_{im}, \cdots) \in l_2, m = \overline{0,2}$.

In view of (15), this implies that

$$h \sum_{m=0}^{2} \langle \frac{\partial \Psi_{\beta,k}(\hat{v}_m)}{\partial v_m}, \delta v_m \rangle_{l_2} \le -\alpha < 0, \tag{16}$$

where $\hat{v}_0 = (h\theta_0 \delta v_0, \overline{v}_1, \overline{v}_2)$,$\hat{v}_1 = (v_0^*, h\theta_1 \delta v_1, \overline{v}_2)$,$\hat{v}_2 = (v_0^*, v_1^*, h\theta_2 \delta v_2)$ and $\theta_m \in (0,1), m = \overline{0,2}$ is positive number.

Using the formula of finite increment, we obtain that

$$\Psi_{\beta,k}(\overline{v}) - \Psi_{\beta,k}(v^*) = h \textstyle\sum_{m=0}^{2} \langle \frac{\partial \Psi_{\beta,k}(v_m^+)}{\partial v_m}, \delta v_m \rangle_{l_2}$$
$$\le -\alpha + h \textstyle\sum_{m=0}^{2} \langle \frac{\partial \Psi_{\beta,k}(v_m^+)}{\partial v_m} - \frac{\partial \Psi_{\beta,k}(\hat{v}_m)}{\partial v_m}, \delta v_m \rangle_{l_2} \le -\alpha + h \sum_{m=0}^{2} O(\|\delta v_m\|_{l_2}), \tag{17}$$

where $v_0^+ = (h\gamma_0\delta v_0, \overline{v}_1, \overline{v}_2), v_1^+ = (v_0^*, h\gamma_0\delta v_1, \overline{v}_2), v_2^+ = (v_0^*, v_1^*, h\gamma_0\delta v_2)$ and $\gamma_m \in (0,1), m = \overline{0,2}$ is positive number. Let $0 < h_1 < h$ is such a number that $-\alpha + h_1 \sum_{m=0}^{2} O(\|\delta v_m\|_{l_2}) < 0$.

Put $\overline{\overline{v}} = (\overline{\overline{v}}_0, \overline{\overline{v}}_1, \overline{\overline{v}}_2)$ such that $\overline{\overline{v}}_m = (v_{0m}^* + h_1\delta v_{0m}, v_{1m}^* + h_1\delta v_{1m}, \cdots, v_{im}^* + h_1\delta v_{im})$. Consequently, we obtain

$$\Psi_{\beta,k}(\overline{\overline{v}}) - \Psi_{\beta,k}(v^*) \le -\alpha + h_1 \sum_{m=0}^{2} O(\|\delta v_m\|_{l_2}) < 0. \tag{18}$$

This contradicts to the optimality of control v^*. Then the validity of relation (14). This ends the proof.

3. Discretization of State and Adjoint systems

In this section, we use forward finite difference approximation for discretization the state and adjint systems, (1)-(3), (11)-(13) respectively, for the above problem. Therefore, we divide [0, l] by N − 1 equal subintervals $x_0 = 0$, $x_1 = h, \ldots$, $x_i = ih, \ldots$, $x_N = l$, when $h = \frac{1}{N}$ and [0, T] by M − 1 equal subintervals, $t_\circ = 0$, $t_1 = \tau, \ldots$, $t_j = j\tau, \ldots$, $t_M = T$. For a function u(x, t), defined on [0, l] × [0, T], we set $u_i^j = u(x_i, t_j)$.

We discretize the state system (1)-(3) as follows:

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{1}{h}\left(\lambda_{i+1}^j \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h} - \lambda_i^j \frac{u_i^{j+1} - u_{i-1}^{j+1}}{h}\right) - B_i^j \frac{\left(u_{i+1}^j - u_i^j\right)}{h} = f_i^{j+1}, \; i = \overline{0, N-1}, \; j = \overline{0, M-1} \tag{19}$$

$$u_i^0 = \varphi_i, \; i = \overline{1, N} \tag{20}$$

$$\lambda_0^j = \frac{g_0^j}{(u_0^j)_x}, \quad \lambda_N^j = \frac{g_1^j}{(u_1^j)_x}, \; j = \overline{0, M} \tag{21}$$

The system (19)-(21) may be written in the following form

$$A\left((u_i^j)^{(\nu)}\right)\left(u_{i-1}^{j+1}\right)^{(\nu+1)} - C\left((u_i^j)^{(\nu)}\right)\left(u_i^{j+1}\right)^{(\nu+1)} + S\left((u_i^j)^{(\nu)}\right)\left(u_{i+1}^{j+1}\right)^{(\nu+1)} = -D(u_i^j)^{(\nu)}, \; i = \overline{1, N-1}, \; j = \overline{0, M-1} \tag{22}$$

Where

$$A\left((u_i^j)^{(\nu)}\right) = \frac{\tau\lambda_i}{h^2} \tag{23}$$

$$S\left(\left(u_i^j\right)^{(\nu)}\right)=\frac{\tau\lambda_{i+1}^j}{h^2}+\frac{\tau B_i^j}{h^2} \tag{24}$$

$$C\left(\left(u_i^j\right)^{(\nu)}\right)=A\left(\left(u_i^j\right)^{(\nu)}\right)+B\left(\left(u_i^j\right)^{(\nu)}\right)+1 \tag{25}$$

$$D\left(\left(u_i^j\right)^{(\nu)}\right)=\tau\ f_i^{j+1}+\left(u_i^j\right)^{(\nu)} \tag{26}$$

Each iteration as ν require the solution of the system (22) by sweep method [12]. We stop the iteration procedure on ν when $\left|\left(u_i^j\right)^{(\nu+1)}-\left(u_i^j\right)^{(\nu)}\right|<\varepsilon$ where $\varepsilon>0$ is given number. Similarly, we define a solution Θ_i^j to the difference adjoint problem (11)-(13).

Now, we give a discretization of the modified functional (8) and its gradient formulae as follows:

$$I(v)=\beta_0\,\tau\sum_{j=1}^{M_t}\ [u_0^j-f_0^j]^2+\beta_1\ \tau\sum_{j=1}^{M_t}\ [u_{N_x}^j-f_1^j]^2+\alpha\sum_{s=1}^{N}[v_s-\omega_s]^2$$

$$+h\ \tau\ A_k\ \sum_{i=0}^{N_x-1}\sum_{j=1}^{M_t}[\left(S^1\right)_i^j+\left(S^2\right)_i^j+\left(Q_1\right)_i^j+\left(Q_2\right)_i^j] \tag{27}$$

and

$$\frac{\partial\Phi}{\partial v_s}=-h\ \tau\ \sum_{i=0}^{N_x-1}\sum_{j=1}^{M_t}\left[\frac{\partial\lambda_i^j(u_i^j,v_s)}{\partial v_s}\frac{\partial\Theta_i^j}{\partial x}\ \frac{\partial u_i^j}{\partial x}-\frac{\partial B_i^j(u_i^j,v_s)}{\partial v_s}\Theta_i^j\ \frac{\partial u_i^j}{\partial x}\right]$$

$$-h\ \tau\ A_k\ \sum_{i=0}^{N_x-1}\sum_{j=1}^{M_t}[\frac{\partial\left(S^1\right)_i^j(u_i^j,v_s)}{\partial v_s}+\frac{\partial\left(S^2\right)_i^j(u_i^j,v_s)}{\partial v_s}]$$

$$-2\alpha\ \sum_{s=1}^{N}\left[v_s-\omega_s\right],\ s=\overline{1,N} \tag{28}$$

4. Numerical Algorithm

With the gradient obtained $\frac{\partial\Phi(v)}{\partial v_s}\cong\frac{\partial\Phi}{\partial v}$, the following gradient type algorithm can be developed for the optimal control values of $v\in V$ based on the Fletcher-Reeves method [13]. The outlined of the algorithm for solving the optimal control problem are as follows:

Step 1: Choose an initial control $v^{(0)}, \epsilon' > 0, A_0 > 0, \epsilon > 0$.
If $\frac{\partial \Phi(v)}{\partial v}|_{v=v^{(0)}} = 0$, $v^{(0)}$ is the optimal solution of the problem.

Step 2: Set the first searching direction $S^{(0)} = -\frac{\partial \Phi(v)}{\partial v}|_{v=v^{(0)}}$.

Step 3: Set $v^{(1)} = v^{(0)} + \alpha^{(0)} S^{(0)}$, with $\alpha^{(0)}$ being the optimal step length in the searching direction $S^{(0)}$.
Set $It = 1$ and go to step 4.

Step 4: Find $\frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It)}}$, by solving the state and adjoint systems and then, set $S^{(It)} = -\frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It)}} + \beta^{It} S^{(It-1)}$, with

$$\beta^{(It)} = \frac{\left\langle \frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It)}}, \frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It)}} \right\rangle_{E_N}}{\left\langle \frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It-1)}}, \frac{\partial \Phi(v)}{\partial v}|_{v=v^{(It-1)}} \right\rangle_{E_N}}$$

Step 5: Compute the optimum step length $\alpha^{(It)}$ in the searching direction $S^{(It)}$ and update $v^{(It)}$ by $v^{(It+1)} = v^{(It)} + \alpha^{(It)} S^{(It)}$.

Step 6: Test the optimality of $v^{(It+1)}$. If $v^{(It+1)}$ is optimum, stop the process. Otherwise, set $It = It + 1, A_{It} = \epsilon' A_{It-1}$ and go to Step 4.

The following theorem represents the main contribution of the convergence theory for the control sequence, generated by the above numerical algorithm for the unconstrained optimal control problem (UOCP).

Theorem 4.1:

Let $v^{(It)}$ be a sequence of minimizers to the UOCP problem which generated by the above numerical algorithm for any increasing sequence of values A_{It}. Then $v^{(It)}$ converge to the optimum solution v^* of the constrained optimal control problem (COCP) as $\lim_{It \to \infty} A_{It} = +\infty$.

The proof is similar to that of Theorem 7.4 [7].

The following theorem represents the main contribution of the convergence theory for the control sequence, generated by the above numerical algorithm for the unconstrained optimal control problem (UOCP).

Theorem

Let $\{v^{It}\}$ be a sequence of minimizers to the UOCP problem which generated by the above numerical algorithm for any increasing sequence of values A_{It}. then $\{v^{It}\}$ converge to the optimum solution v^* of the constrained optimal control problem (COCP) as $\lim_{It\to\infty} A_{It} = +\infty$.

5. Numerical results and discussion

The considering problem is considered as one of the identification problems on definition of unknown coefficients of quasilinear parabolic equation type. The numerical results were carried out for the following examples:

1. $u(x,t) = x + t$, $\lambda(u,v) = \frac{u}{1+u}$, $B(u,v) = \tan^{-1}(u)$, $x \in [0, 0.8]$, $t \in [0, 0.001]$.

2. $u(x,t) = x + t$, $\lambda(u,v) = \tan^{-1}(u)$, $B(u,v) = \frac{u^2}{(1-u^2)}$, $x \in [0, 0.9]$, $t \in [0, 0.001]$.

3. $u(x,t) = x + t$, $\lambda(u,v) = e^{-u}\sin(u)$, $B(u,v) = \log(1+u)$, $x \in [0, 0.7]$, $t \in [0, 0.001]$.

4. $u(x,t) = x + t$, $\lambda(u,v) = \log(\frac{1}{1-u})$, $B(u,v) = e^{\sin(u)}$, $x \in [0, 0.9]$, $t \in [0, 0.001]$.

5. $u(x,t) = \tan(3\pi x + t)$, $\lambda(u,v) = \tan^{-1}(u)$, $B(u,v) = \frac{u^2}{(1-u^2)}$, $x \in [0, 0.8]$, $t \in [0, 0.001]$.

In Table 1, the input data of parameters in our optimal control problem is given as follows:

Table 1

	ν_0	ν_1	μ_0	μ_1	r_1	r_2	β_0, β_1, α	ϵ
Example:1	0	0.445	0	38.69	0	0.801	0.9	10^{-6}
Example:2	0	42.02	0	0.08	0	0.901	0.9	10^{-6}
Example:3	0	0.006	0	0.23	0	0.701	0.9	10^{-6}
Example:4	0	1.004	1	1.016	0	0.901	0.9	10^{-6}
Example:5	0	7.5438	0	.0179	0	0.1324	0.9	10^{-6}

<u>The Numerical study has given the following results</u>:

1) Knowing the computed optimal control values v^* obtained by the above numerical algorithm, we can calculate the approximate values of the unknown coefficients $\lambda(u,v), B(u,v)$, each one can be represented in a series according to every example.

2) For examples 1 and 2, in Figures 1 and 3 the curves are denoted by $L4(u,v^*),\cdots$ and $B4(u,v^*), B6(u,v^*),\cdots$ are the approximate values of $\lambda(u,v), B(u,v)$ with v^* while $\lambda_{EXACT}, B_{EXACT}$ are the exact values of $\lambda(u,v), B(u,v)$.

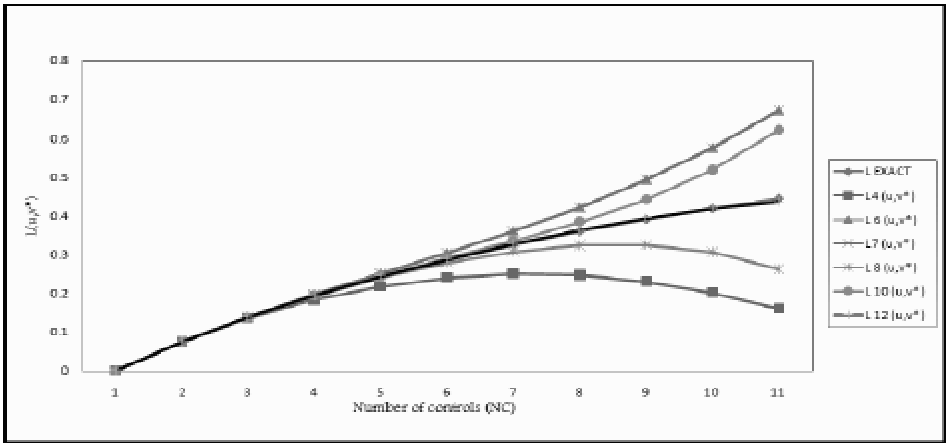

Figure 1: Values of L(u,v*) with numbers of controls

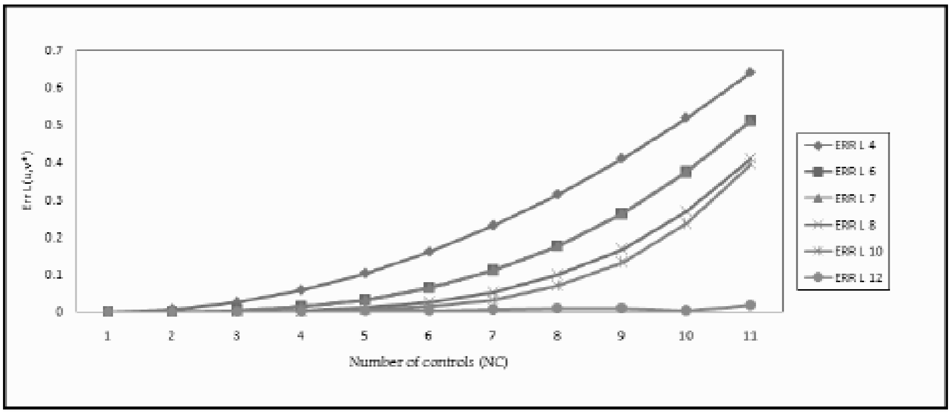

Figure 2: Errors of L(u,v*) with numbers of controls

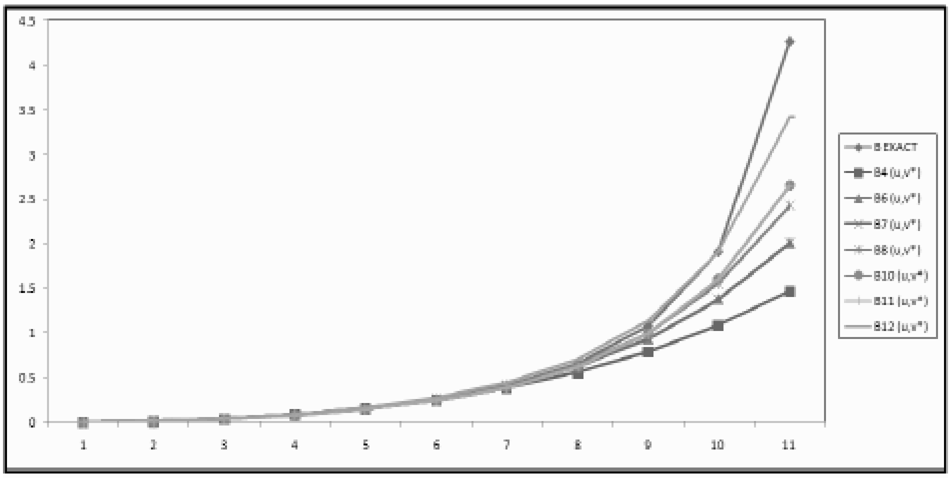

Figure 3: Values of B(u,v*) with numbers of controls

Obviously, by increasing the number of controls NC, the approximate values of the coefficients $\lambda(u,v), B(u,v)$ are agreed with the exact values. Also, in Figures 2 and 4 the curves are denoted by $ERR\,L4, ERR\,L6, \cdots$ and $ERRB4, ERRB6, \cdots$ are the absolute errors of $\lambda(u,v), B(u,v)$. It is clear that the absolute errors are decreased by increasing the number of controls NC.

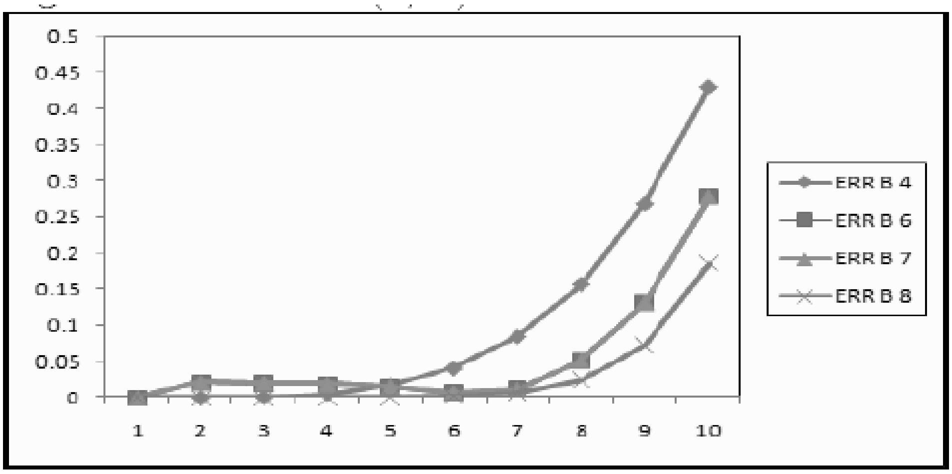

Figure 4: Errors of B(u,v*) with numbers of controls

3) For example 2, the curves of the initial, optimal and computed optimal control by the above numerical algorithm versus the iteration numbers IT are displayed in Figure 5.

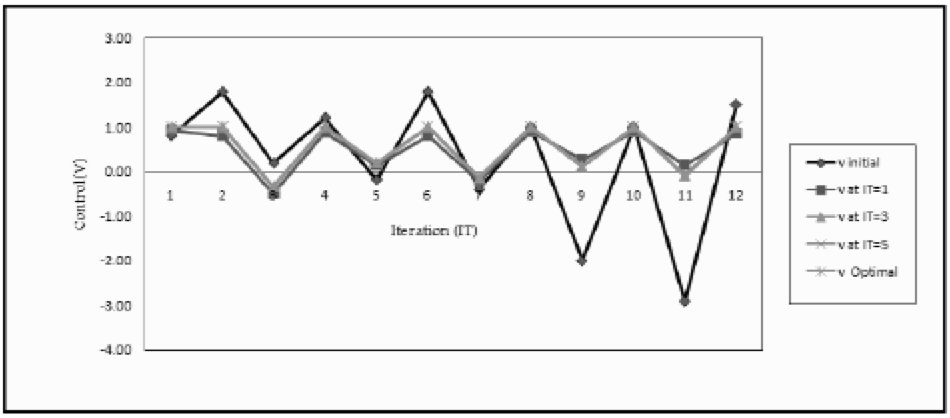

Figure 5: Control values various iterations

4) For example 3, in table 2, we report the iteration number IT, the computed values of the modified functional $\Phi_{\alpha,k}$, the objective function $f_\alpha(v)$, the penalty function $P_k(v)$ and the penalty parameter $A_k(v)$. The numerical algorithm in this chapter takes six iterations for decreasing $\Phi(v)$ to the value $0.150067E-3$.

Table 2				
It	$\Phi_{(v)}$	$f_\alpha(v)$	$P_k(v)$	A_k
0	0.471368E+01	0.473008E+01	0.000000E+00	0.000000E+00
1	0.506609E+00	0.506597E+00	0.114162E-04	0.440000E+00
2	0.690376E-01	0.690360E-01	0.160320E-05	0.880000E-01
3	0.544472E-03	0.544170E-03	0.302103E-06	0.176000E-01
4	0.150208E-03	0.150148E-03	0.601181E-07	0.352000E-02
5	0.150067E-03	0.150055E-03	0.120227E-07	0.704000E-03

5) For examples 3 and 4, Figures 6 and 7 shows the values of the components of the gradient modified function $\frac{\partial \Phi_{\alpha,k}(v)}{\partial v}$ which are calculated by the following forward and central formulae

$$\frac{\partial \Phi_{\alpha,k}(v)}{\partial v} = \frac{\Phi_{\alpha,k}(v+h) - \Phi_{\alpha,k}(v)}{h} + O(h)$$

versus

$$\frac{\partial \Phi_{\alpha,k}(v)}{\partial v} = \frac{\Phi_{\alpha,k}(v+h) - \Phi_{\alpha,k}(v-h)}{h} + O(h^2)$$

the iteration numbers IT .

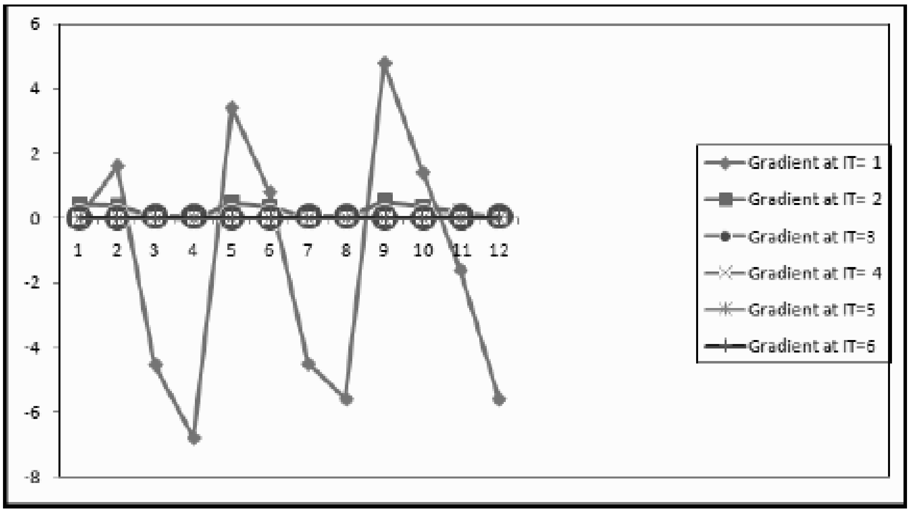

Figure 6: Values of forward gradient function various Iterations

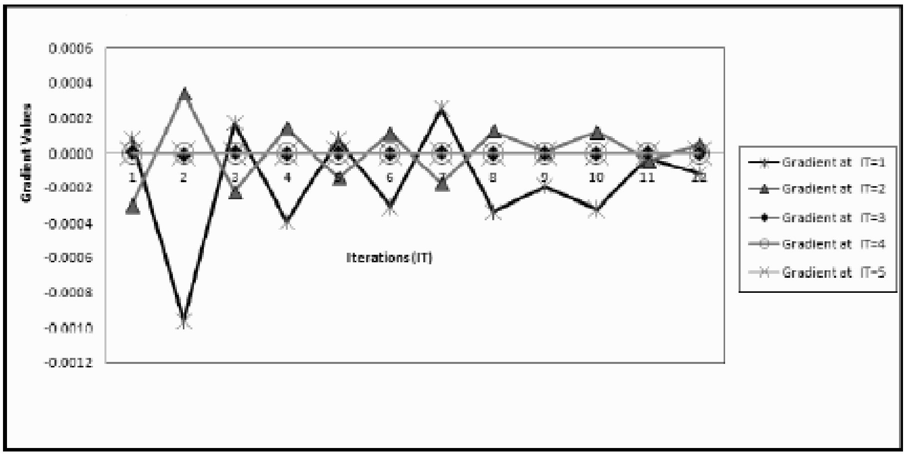

Figure 7: Values of central gradient function various Iterations

6) For example 4, the maximum absolute errors of $\lambda(u,v), B(u,v)$ versus the number of controls NC are displayed in table 5 which are calculated by the following formulae $MAE_\lambda = \max\left|\lambda_{Exact} - \lambda^*_{\frac{NC}{2}}\right|$ and $MAE_B = \max\left|B_{Exact} - B^*_{\frac{NC}{2}}\right|$. It is clear that the maximum absolute errors are decreased by increasing the number of controls.

Table 3		
NC	$MAE_\lambda = \max\|\lambda_{Exact} - \lambda^*_{\frac{NT}{2}}\|$	$MAE_B = \max\|B_{Exact} - B^*_{\frac{NT}{2}}\|$
3	1.4026E+00	1.1887E+00
5	8.3558E-01	1.5795E-01
6	7.29556E-01	1.28055E-01
8	4.6555E-01	3.7993E-02
10	3.2946E-01	2.9703E-02
12	2.6045E-01	5.4946E-03

7) For examples 5, Figures 8 and 9, we display the exact and the approximate solutions $u(x,t;v^*)$ of the state equation (3.1)-(3.3) at the computed optimal control values. While Figure 10 show the error between the exact and the approximate solutions $u(x,t;v^*)$ of the state equation (3.1)-(3.3).

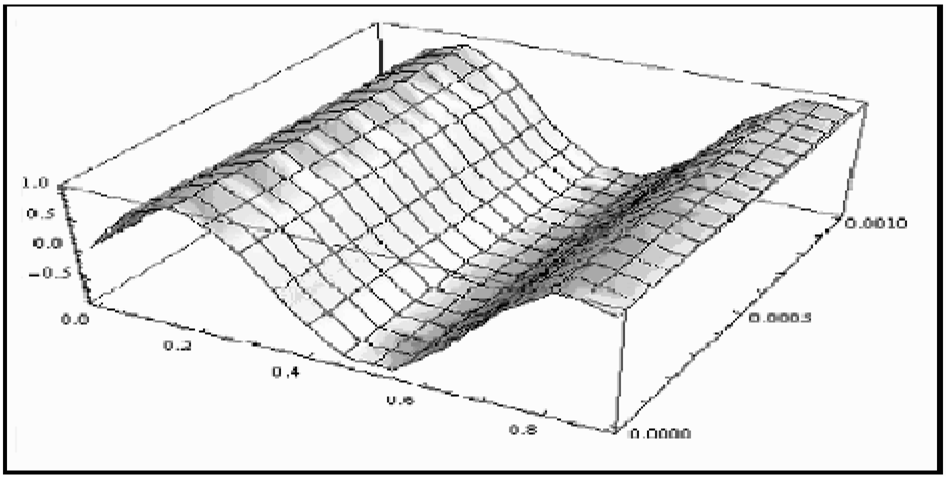

Figure 8: Exact solution of $u(x,t) = \tan(3\pi x + t)$

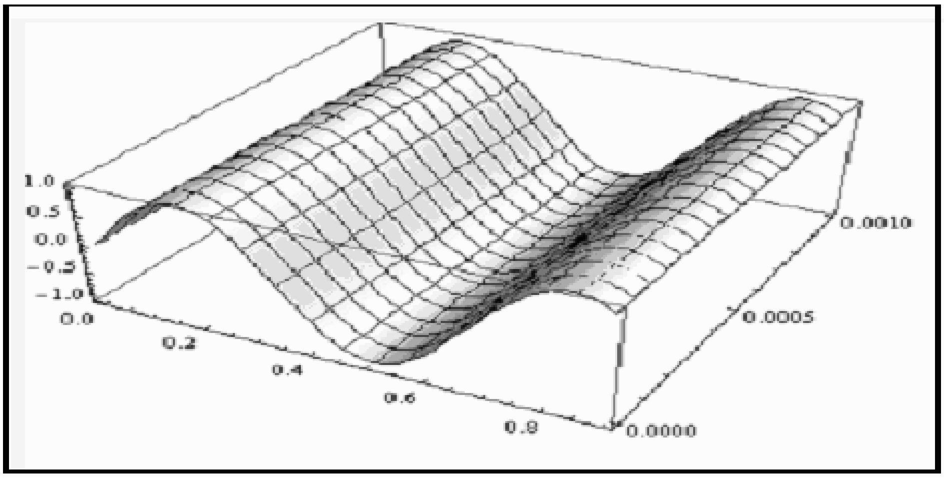

Figure 9: Approximate solution of $u(x,t) = \tan(3\pi x + t)$

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