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Beta Exponentiated Mukherjii-Islam Distribution: Mathematical Study of Different Properties

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Abstract

Mukherji-Islam discussed a distribution for the study of reliability through failure data, that distribution is useful in reliability analysis. The same distribution is used here to develop, a new beta exponentiated distribution with the name ***Beta-Exponentiated Mukherjee-Islam distribution.*** *It is* a generalized form of the Mukherji-Islam distribution. In this article we have studied different mathematical properties in detail. We have obtained quantile, moments, moments generating function, characteristic function of the distribution and the pdf of r^{th} order statistics. We have also obtained maximum likelihood estimates of all parameters of the new distribution. A relation among three shape parameters of the distribution is also obtained, which gives a view how one shape parameter varies according to other shape parameter. In the last we have obtained R'enyi and q-entropies to measure the disorder of the system.

Keywords: Beta Exponentiated Mukherji-Islam Distribution (BEMI), Beta function, Moment generating function, Characteristic function, r^{th} order statistics, maximum likelihood estimation, R'enyi and q-Entropies.

1. Introduction

Mathematical study of theoretical probability distributions has been a choice of many researchers, a number of research workers have worked on developing new distributions to be used in different fields of applications in real life data, few of them are, Weibull (1951), Folks and Chikara(1978), Mukherji and Islam (1983), Siddiqui et al. (1992, 1994, 1995), Cha (2007), Mandel and Yaakov(2010).

Eugene et. al (2002) pioneered the class of beta generated probability distributions. Eugene et al (2002) used beta distribution with shape parameter α and β to develop the beta generated distribution.

The cumulative distribution function (cdf) of a beta generated random variable X is defined as,

$$F(x) = \int_0^{G(x)} b(t)dt \tag{1}$$

Where b(t) is the probability density function (pdf) of the beta random variable and $F(x)$ is the cdf of any random variable X.

Eq. (1) can be expressed in the form,

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1}\, dw \tag{2}$$

Where a>0 $and\ b > 0$ are two additional parameters to introduce skewness and to vary tail weight.

Here,

$B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}\, dw$, is a beta function.

Probability density function (pdf) of this function will be;

$$f(x) = \frac{g(x)}{B(a,b)} F(x)^{a-1}\{1-F(x)\}^{b-1} \tag{3}$$

If x is discrete, the probability mass function is given by

$$g(x) = G(x) - G(x-1) \tag{4}$$

The important property of this class is that it is a generalization of the distribution of order statistics for the random variable X with cdf $F(x)$ as provided by Eugene et al. (2002) and Jones (2004). Eugene el al. (2002) defined and studied normal distribution, on the basis of that Beta Frechet (Nadaraja and Gupta, 2004), Beta Weibull (Famoye et al. 2005), Beta-Pareto (Akinsete et al. 2008), Beta-Birnbaum-Saunders (Cordeiro and Leomonte, 2011), and Beta-Cauchy (Al-Showarbehetal, 2012) appeared in this class.

We have tried to enlarge the class of exponentiated distributions and developed here a new distribution with its important properties.

1. 1 Short Details of Mukherji-Islam Distribution

The cdf of Mukherji-Islam Distribution (1983) of a random variable X is given by,

$$G(x) = \frac{x^p}{\theta^p}, where\ p, \theta > 0, \theta \geq x > 0$$

Or $= 0\ otherwise$ (5)

Here θ and p are scale & shape parameters, the pdf of above cdf (5) will be,

$$g(x) = \frac{px^{p-1}}{\theta^p},$$

Where, $p, \theta > 0, \theta \geq x > 0$ (6)

The reliability function $R(x)$ and hazard rate function $h(x)$ of the distribution are;

$$R(x) = 1 - \left(\frac{x}{\theta}\right)^p,$$

Where, $p, \theta > 0, \theta \geq x > 0,$ (7)

$h(x) = \frac{px^{p-1}}{1-\left(\frac{x}{\theta}\right)^p},$ (8)

2. Beta Exponentiated Mukherji-Islam Distribution (BEMI)

Now we define the cdf of the new distribution Beta Exponentiated Mukherjee-Islam distribution,

$F(x) = \frac{1}{B(a,b)} \int_0^{\frac{x^p}{\theta^p}} w^{a-1}(1-w)^{b-1} dw$ (9)

Where $a, b, p, \theta > 0,$ and $\theta \geq x > 0,$

The pdf will be from equ (6)

$f(x) = \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} \left(\frac{x^p}{\theta^p}\right)^{a-1} \left(1 - \frac{x^p}{\theta^p}\right)^{b-1}, x > 0$ (10)

$h$azard rate function will be,

$h(x) = \frac{f(x)}{1-F(x)} = \frac{px^{p-1}\left(\frac{x^p}{\theta^p}\right)^{a-1}\left(1-\frac{x^p}{\theta^p}\right)^{b-1}}{\theta^p B(a,b) I_{\left[1-\frac{x^p}{\theta^p}\right]}(a,b)}$ (11)

where, $F(x) = I_{G(x)}(a,b)$ and $I_y(a,b) = \frac{1}{B(a,b)} \int_0^1 w^{a-1}(1-w)^{b-1} dw$

The probability density function in equation (10) does not involve any complicated function. If X is a random variable with pdf (10), we can write

$$X \sim BEMI(a, b, p, \theta)$$

We found BEMI distribution with same special case as follows

If $a = b = 1$, we get back the main Mukherjee-Islam distribution.

Now, in coming sections, we demonstrate that BEMI density function can be expressed as a linear combination of the Exponentiated Mukherjee-Islam distribution. Result of above density function is important in giving different mathematical properties of BEMI distribution. In section 3 we discussed some important statistical properties of BEMI as quantile, moments and moment generating function and the idea of order statistics. Section 4 deals with the maximum likelihood estimates of the four parameters. In section 5 the relationship among the parameters is shown and section 6 deals with the entropies.

2.1 Expansion for the Density Function:

Here, we will show the two results of the cdf of BEMI distribution, depending on two cases:

Case 1: if b is a real non-integer. It follows

$$F(x) = \frac{1}{B(a,b)} \int_0^{\frac{x^p}{\theta^p}} w^{a-1}(1-w)^{b-1} dw$$

By Binomial expansion we know

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j = \frac{(-1)^j \,\Gamma \mathrm{b}}{\mathrm{j}!\ \Gamma(\mathrm{b}-1)} z^j \qquad (12)$$

Where |z|< 1

Appling equation (12) in cdf we get

$$F(x) = \frac{1}{B(a,b)} \int_0^{\frac{x^p}{\theta^p}} w^{a-1} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma \mathrm{b}}{\mathrm{j}!\ \Gamma(\mathrm{b}-1)} w^j \, dw$$

$$F(x) = \frac{\Gamma \mathrm{b}}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\mathrm{j}\ \Gamma(\mathrm{b}-1)} \int_0^{\frac{x^p}{\theta^p}} w^{a+j-1} dw$$

$$F(x) = \frac{\Gamma(\mathrm{a+b})}{\Gamma \mathrm{a}} \sum_{j=0}^{\infty} (-1)^j \frac{(-1)^j}{\mathrm{j}!\ \Gamma(\mathrm{b}-1)(a+j)} \left[\frac{x^p}{\theta^p}\right]^{a+j} \qquad (13)$$

Where $B(a,b) = \frac{(\Gamma \mathrm{a})(\Gamma \mathrm{b})}{\Gamma(\mathrm{a+b})}$, a beta function.

Equation (13) shows that cdf of BEMI distribution can be expressed as an infinite weighted sum of MI distribution. Result can be expressed as

$$F(x) = \frac{\Gamma(\mathrm{a+b})}{\Gamma \mathrm{a}} \sum_{j=0}^{\infty} (-1)^j \, G\big(x, \theta, p, (a+j)\big)$$

Case 2:$If b$ is an integer,

Applying binomial expansion in cdf of BEMI we get

$$F(X) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{(a+j)} \qquad (14)$$

Now, equation (14) holds the same property like (13), but here sum is finite.

If, we follow the same expansion method for pdf of BEMI we get two results, depends on two cases, same as above

1. When b is a real non-integer:

$$f(x) = \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} G\big(x,\theta,p(a-1)\big) \sum_{j=0}^{\infty} \frac{(-1)^j}{\text{j! } \Gamma(\text{b}-\text{j})} G(x\,,\theta,pj),$$

Where, $x > 0$ .... (15)

2. When b is an integer:

$$f(x) = \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} G\big(x,\theta,p(a-1)\big) \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^j G(x\,,\theta,pj), x > 0 \quad (16)$$

3. Statistical Properties

Statistical properties include study of quantile function, moments and mgf of BEMI distribution.

3.1 Quantile Function:

Quantile function x_q is defined as

$$F(x_q) = I_{\left[1-\frac{x^p}{\theta^p}\right]}(a,b)$$

$$(x_q)_{BEMI} = F^{-1}(u)$$

3.1.1 Moments:

Moments are used to study the tendency, dispersion, skewness and kurtosis, which are the most important features and characteristic of a distribution.

If X be a random variable with density function

$$f(x) = \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} \left(\frac{x^p}{\theta^p}\right)^{a-1} \left(1-\frac{x^p}{\theta^p}\right)^{b-1}, x > 0$$

The r^{th} moment of the BEMI distribution is given by,

$$\mu_r(x) = E(x^r) = \int_0^{\theta} x^r f(x) dx$$

$$= \frac{p}{\theta^p B(a,b)} \int_0^{\theta} x^{r+p-1} \left(\frac{x^p}{\theta^p}\right)^{a-1} \left(1-\frac{x^p}{\theta^p}\right)^{b-1} dx$$

By using binomial expansion, we have,

$$= \frac{p}{\theta^p B(a,b)} \int_0^{\theta} x^{r+p-1} \left(\frac{x^p}{\theta^p}\right)^{a-1} \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \left(\frac{x^p}{\theta^p}\right)^{j} dx$$

$$= \frac{p}{\theta^p B(a,b)} \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \int_0^{\theta} x^{r+p-1} \left(\frac{x^p}{\theta^p}\right)^{a+j-1} dx = \frac{p}{\theta^p B(a,b)} \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \int_0^{\theta} x^{r+p-1} \left(\frac{x^p}{\theta^p}\right)^{a+j-1} \frac{x^{r+p(a+j)-1}}{\theta^{p(a+j-1)}} dx$$

$$\mu_r(x) = E(x^r) = \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \frac{p\theta^r}{r + p(a+j), B(a,b)}$$

Or we can write,

$$\mu_r(x) = E(x^r) = t_j \theta^r \ \ldots \tag{17}$$

Where,

$$t_j = \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \frac{1}{\frac{r}{p} + (a+j), B(a,b)}$$

Eq. (17) gives the moments of BEMI distribution.

By putting r=1 and r=2 in Eq. (17), we can get the mean and variance by the following way,

$\mu'_{1=} t_j \theta^1$ and $\mu'_{2=} t_j \theta^2$

$where,$

$$t_j = \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \frac{1}{\frac{r}{p} + (a+j), B(a,b)}$$

So mean = $t_j \theta^1$ and

Variance $= \mu'_2 - \left(\mu'_1\right)^2 = t_j\theta^2 - {t_j}^2\theta^2 = t_j\theta^2(1 - t_j)$

Based on the first four moments of the BEMI distribution, the measures of skewnessA(Φ) and Kurtosis K(Φ) of BEMI distribution can be obtained as:

$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^{\frac{3}{2}}}$

And $K(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{[\mu_2(\theta) - \mu_1^2(\theta)]^2}$

3.1.1.1 Moments Generating Function(mgf)

Moment generating function is an important statistical property of a probability distribution as it provides the basis of an alternative route to analytical results compared with compared to working directly with pdf or cdf.

If some random variable X has BEMI distribution, then the moment generating function will be according to the definition of Mgf, as follows:

$$M_x(t) = E(e^{tx}) = \int_0^{\theta} e^{tx} f_{BEMI}(x) dx,$$

$$where, p, \theta > 0, \theta \geq x > 0,$$

So we will have Mgf of BEMI distribution

$$M_x(t) = E(e^{tx}) = \int_0^{\theta} e^{tx} \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} \left(\frac{x^p}{\theta^p}\right)^{a-1} \left(1 - \frac{x^p}{\theta^p}\right)^{b-1} = \frac{p}{\theta^{p(a+j)} B(a,b)} \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \int_0^{\theta} e^{tx} x^{p(a+j)-1} dx$$

$=\frac{p}{\theta^{p(a+j)}B(a,b)}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j$

$-\,\Gamma[(a+j),-t\theta]\theta^{p(a+j)}(-t\theta)^{-(a+j)p}$

$$M_x(t)=\ \Gamma\left[\left(a+\frac{p}{B(a,b)}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j-j),-tx\right)\right](-t\theta)^{-(a+j)p} \quad (18)$$

Which is the desired Moment generating function.

3.1.1.1.1 Characteristic function (cf)

$$\Phi_x(t)=E\left(e^{itx}\right)=\int_0^{\theta}e^{itx}f_{BEMI}\,(x)dx$$

Where, $p,\theta>0,\theta\geq x>0,$

Above definition of characteristic function gives us the result as ;

$$\Phi_x(t)=\frac{p}{B(a,b)}\sum_{j=0}^{\infty}\binom{b-1}{j}(-1)^j-\Gamma[(a+j),-itx)](-it\theta)^{-(a+j)p} \quad (19)$$

3.1.1.1.1.1.1 Distribution of order statistics

Let $X_{1,}X_2\ldots\ldots\ldots.X_n$ be a simple random sample from BEMI distribution with pdf and cdf given by Eqs. (9) and (10) consecutively,

$$F(x)=\frac{1}{B(a,b)}\int_0^{\frac{x^p}{\theta^p}}w^{a-1}(1-w)^{b-1}dw$$

$$f(x)=\frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)}\left(\frac{x^p}{\theta^p}\right)^{a-1}\left(1-\frac{x^p}{\theta^p}\right)^{b-1},x>0$$

Let $X_{1,}X_2\ldots\ldots\ldots.X_n$ denote the order statistics obtained from this sample. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\emptyset)=\frac{1}{B(r,n-r+1)}[F(x,\emptyset)]^{r-1}[1-F(x,\emptyset)]^{n-r}*f(x,\emptyset) \quad (20)$$

Where $F(x,\emptyset)$ and $f(x,\emptyset)$ are the cdf and pdf of BEMI order statistics.

Here, $0<F(x,\emptyset)<1$ for $x>0,$

binomial series expansion of $[1-F(x,\emptyset)]^{n-r}$will give us,

$[1-F(x,\emptyset)]^{n-r}=\binom{n-r}{j}[F(x,\emptyset)]^j\sum_{j=0}^{\infty}(-1)^j$

Putting this in Eq. (20) we will get,

$$f_{r:n}(x,\emptyset)=\frac{1}{B(r,n-r+1)}[F(x,\emptyset)]^{r+j-1}*f(x,\emptyset)$$

Substituting cdf and pdf of BEMI into above equation we can express k_{th} ordinary moment of the r_{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$as a linear combination of the k_{th} moments of the BEMI distribution with different shape parameter. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

4. Estimation

4.1 Maximum Likelihood Estimation (mle):

Let $X_{1,}X_2 \dots\dots\dots . X_n$ be a random sample of size n from BEMI (p,θ,a,b) the likelihood function for the vectors of parameters, ϕ=(p,θ,a,b) can be written as,

$L(x_{i,}\,\Phi)=\prod_{i=1}^{n} f\left(x_{i,}\Phi\right)$

$$=\prod_{i=1}^{n}\frac{\frac{p x_i^{p-1}}{\theta^p}}{B(a,b)}\left(\frac{x_i^p}{\theta^p}\right)^{a-1}\left(1-\frac{x_i^p}{\theta^p}\right)^{b-1}$$

$$=\left(\frac{p}{\theta^p}\right)^n\frac{\prod_{i=1}^{n} x_i^{p-1}}{[B(a,b)]^n}*\left(\frac{1}{\theta^p}\right)^{n(a-1)}*\prod_{i=1}^{n} x_i^{p(a-1)}*\prod_{i=1}^{n}\left(1-\frac{x_i^p}{\theta^p}\right)^{b-1}$$

Taking log-likelihood of above for the vector of parameters ϕ=(p,θ,a,b), we get,

$$\log L = n\log p - np\log\theta + n\log\,\Gamma(a+b) - n\log\,\Gamma a - n\log\,\Gamma b - np(a-1)\log\theta + \sum_{i=1}^{n}\log x_i^{pa-1} + (b\,-1)\sum_{i=1}^{n}\log\left(1-\frac{x_i^p}{\theta^p}\right)$$

The log-likelihood can be maximized either directly or by solving the non-linear likelihood equations obtained by differentiating above equation

The components of the score vectors is given by,

$$\frac{\partial\log L}{\partial\theta}=\frac{-np\alpha}{\theta}+\frac{p(b-1)\theta^{p-1}\sum_{i=1}^{n}x_i^p}{\left[1-\frac{x_i^p}{\theta^p}\right]} \tag{21}$$

The mle of θ will of be,

$$\hat{\theta}=\left[\frac{na}{b-1}\sum_{i=1}^{n}\left[\frac{1}{x_i^p}-\frac{1}{\theta^p}\right]\right]^{\frac{1}{p}} \tag{22}$$

$$\frac{\partial\log L}{\partial p}=\frac{n}{p}-na\log\theta+a\sum_{i=1}^{n}\log x_i$$

$$-(b-1)\sum_{i=1}^{n}\frac{x_i^p}{\theta^p}*\frac{\log\left[1-\frac{x_i}{\theta}\right]}{\left[1-\frac{x_i^p}{\theta^p}\right]} \tag{23}$$

The mle of p, will be,

$$\hat{p}=\frac{n}{na\log\theta-a\sum_{i=1}^{n}\log x_i+(b-1)\sum_{i=1}^{n}\frac{x_i^p}{\theta^p}*\frac{\log\left[1-\frac{x_i}{\theta}\right]}{\left[1-\frac{x_i^p}{\theta^p}\right]}} \tag{24}$$

$$\frac{d\log L}{da}=n\Psi(a+b)-n\Psi(a)-np\log\theta+p\sum_{i=1}^{n}\log x_i \tag{25}$$

$$\Psi(a)=\{\Psi(a+b)-p\log\theta\}+\frac{p}{n}\sum_{i=1}^{n}\log x_i \tag{26}$$

$$\frac{d\log L}{db}=n\Psi(a+b)-n\Psi(b)+\sum_{i=1}^{n}\log\left(1-\frac{x_i^p}{\theta^p}\right) \tag{27}$$

$$\Psi(b)=\Psi(a+b)+\frac{1}{n}\sum_{i=1}^{n}\log\left(1-\frac{x_i^p}{\theta^p}\right) \tag{28}$$

Where $\Psi(.)$is a digamma function.

5. Relation Between all three Shape Parameters(p, a, b)

We used three shape parameters in BEMI distribution. Relation between all three shape parameters p, a, b can be established as follows,
The pdf of BEMI distribution is,

$$f(x) = \frac{\frac{px^{p-1}}{\theta^p}}{B(a,b)} \left(\frac{x^p}{\theta^p}\right)^{a-1} \left(1 - \frac{x^p}{\theta^p}\right)^{b-1}, x > 0$$

Taking logarithm of both side of above equation of pdf of BEMI distribution,

$$\log f(x) = \log p + (p-1)\log x - p\log\theta - \log B(a,b)$$
$$+(a-1)p[\log x - \log\theta] - p(b-1)[\log x - \log\theta] \quad (29)$$

Taking logarithm of both sides and equating it to zero we get,

$$\frac{d\log L}{dx} = \frac{(p-1)}{x} + \frac{p(a-1)}{x} - \frac{p(b-1)}{x} = 0$$

$$p = \frac{1}{(1+a-b)} \quad (30)$$

Above relation shows how three shape parameters varies according to each other.

6. R'enyi and q-Entropies

6.1 R'enyi Entropy:

The entropy of a random variable X is a measure of the uncertain variations in the system. The R'enyi entropy is defined by,

$$I_R(\delta) = \frac{1}{1-\delta}\log|I(\delta)| \quad (31)$$

Where $I(\delta) = \int_R f^\delta(x)\,dx$ for$\delta > 0$ $and\delta \neq 1$

$$f^\delta(x) = \frac{p^\delta (x^{p-1})^\delta}{(\theta^p)^\delta B^\delta(a,b)} \left(\frac{x^p}{\theta^p}\right)^{\delta(a-1)} \left(1 - \frac{x^p}{\theta^p}\right)^{\delta(b-1)},$$
$$\theta \geq x > 0$$

$$= \frac{g^\delta(x)}{B^\delta(a,b)} * G(x)^{\delta(a-1)} * \left(1 - G(x)\right)^{\delta(b-1)}$$

Applying the power series used before, we obtain

$$f^\delta(x) = \frac{g^\delta(x)}{B^\delta(a,b)} G(x)^{\delta(a-1)} \sum_{k_1}^{\delta(b-1)} (-1)^{k_1} \binom{\delta(b-1)}{k_1} \left(G(x)\right)^{k_1}$$

$$= \frac{g^\delta(x)}{B^\delta(a,b)} \sum_{k_1}^{\delta(b-1)} (-1)^{k_1} \binom{\delta(b-1)}{k_1} \left(G(x)\right)^{k_1+\delta(a-1)}$$

$$= \frac{g^\delta(x)}{B^\delta(a,b)} \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} (-1)^{k_1+k_1} \binom{\delta(b-1)}{k_1} \binom{k_2+\delta(b-1)}{k_2} \left(G(x)\right)^{k_2}$$

Now,

$$f^{\delta}(x) = {v^*}_{k_1,k_2} g^{\delta}(x)\big(G(x)\big)^{k_2}$$

Where,

$${v^*}_{k_1,k_2} = \frac{1}{B^{\delta}(a,b)}$$

$$\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty}(-1)^{k_1+\,k_1}\binom{\delta(b-1)}{k_1}\binom{k_2+\delta(b-1)}{k_2}$$

Now,

$$I(\delta) = \int_R f^{\delta}(x)\,dx = {v^*}_{k_1,k_2}\int_0^{\theta} g^{\delta}(x)\big(G(x)\big)^{k_2} \tag{32}$$

Since for BEMI distribution limit for *x* varies like $\theta \geq x > 0$.

Now putting values of cdf and pdf of BEMI distribution in Eq. (32), we get

$$\int_R f^{\delta}(x)\,dx = {v^*}_{k_1,k_2}\int_0^{\theta}\frac{p^{\delta}(x^{p-1})^{\delta}}{(\theta^p)^{\delta}B^{\delta}(a,b)}\left(\frac{x^p}{\theta^p}\right)^{k_2}$$

$$\int_R f^{\delta}(x)\,dx = {v^*}_{k_1,k_2}\frac{p^{\delta}}{\theta^{p(k_2+\delta)}B^{\delta}(a,b)}\int_0^{\theta} x^{p(k_2+\delta)-\delta}\,dx$$

$$I(\delta) = \int_R f^{\delta}(x)\,dx = {v^*}_{k_1,k_2}\frac{p^{\delta}}{B^{\delta}(a,b)}\frac{\theta^{1-\delta}}{p(k_2+\delta)-\delta+1}$$

Hence R'enyi entropy reduces to

$$I_R(\delta) = \frac{1}{1-\delta}\log\left|{v^*}_{k_1,k_2}\frac{p^{\delta}}{B^{\delta}(a,b)}\frac{\theta^{1-\delta}}{p(k_2+\delta)-\delta+1}\right| \tag{33}$$

Where, ${v^*}_{k_1,k_2} = \frac{1}{B^{\delta}(a,b)}\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty}(-1)^{k_1+\,k_1}\binom{\delta(b-1)}{k_1}\binom{k_2+\delta(b-1)}{k_2}$

6.1.1 q-Entropy:

q-entropy say $H_q(f)$ is defined by,

$$H_q(f) = \frac{1}{q-1}\log\left|1 - I_q(f)\right|$$

Where $I_q(f) = \int_R f^q(x)\,dx$ for $q > 0$ *and* $q \neq 1$

Same like above Eq. (33), we can easily obtain,

$$H_q(f) = \frac{1}{q-1}\log\left|1 - {v^*}_{k_1,k_2}\frac{p^{q}}{B^{q}(a,b)}\frac{\theta^{1-q}}{p(k_2+\delta)-q+1}\right|$$

Where, ${v^*}_{k_1,k_2} = \frac{1}{B^{q}(a,b)}\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty}(-1)^{k_1+\,k_1}\binom{q(b-1)}{k_1}\binom{k_2+q(b-1)}{k_2}$

Conclusion

In this paper, we proposed a new four-parameter ***Beta Exponentiated Mukherjee-Islam (BEMI) distribution.*** We studied some of its structural properties including an expansion for the density function and explicit expressions for the quantile function, moments generating function and characteristic function. We depict a relation between three shape parameters to give a picture of their variation according to each other. The maximum likelihood method is employed for estimating the model parameters. Two entropies R'enyi and q-entropy are also obtained. We also obtain observed information matrix.

Appendix

The elements of the 4 × 4 observed information matrix $J(\Theta) = \{BM_{m,n}\}$ $= \{form, n = p, \theta, a, b\}$are given by,

$$BM_{pp} = -\frac{n}{p^2} - (b-1)\sum_{i=1}^{n} \log\left(1 - \frac{x_i}{\theta}\right) \frac{p{x_i}^{p-1}}{\theta^{p-1}\left[1 - \frac{x_i^p}{\theta^p}\right]^2}$$

$$BM_{p\theta} = -\frac{na}{\theta} - p(b-1)\sum_{i=1}^{n} \frac{x_i^{p-1}\log\left(1 - \frac{x_i}{\theta}\right)}{\theta^{p-1}\left(1 - \frac{x_i^p}{\theta^p}\right)} - (b-1)\sum_{i=1}^{n} \frac{x_i^{p+1}}{(\theta^p)^2\left(1 - \frac{x_i}{\theta}\right)\left(1 - \frac{x_i^p}{\theta^p}\right)}$$

$$-p(b-1)\sum_{i=1}^{n} \frac{\left(x_i^p\right)^2 \log\left(1 - \frac{x_i}{\theta}\right)}{\theta^{2p+1}\left(1 - \frac{x_i^p}{\theta^p}\right)^2}$$

$BM_{pa} = \sum_{i=1}^{n} \log x_i - n\log\theta$,

$$BM_{pb} = -\sum_{i=1}^{n} \frac{\frac{x_i^p}{\theta^p}\log\left(1 - \frac{x_i}{\theta}\right)}{\left(1 - \frac{x_i^p}{\theta^p}\right)}$$

$$BM_{\theta P} = -\frac{na}{\theta} - p(b-1)\sum_{i=1}^{n} \frac{x_i^{p-1}\log\left(1 - \frac{x_i}{\theta}\right)}{\theta^{p-1}\left(1 - \frac{x_i^p}{\theta^p}\right)}$$

$$-(b-1)\sum_{i=1}^{n} \frac{x_i^{p+1}}{(\theta^p)^2\left(1 - \frac{x_i}{\theta}\right)\left(1 - \frac{x_i^p}{\theta^p}\right)}$$

$$-p(b-1)\sum_{i=1}^{n} \frac{\left(x_i^p\right)^2 \log\left(1 - \frac{x_i}{\theta}\right)}{\theta^{2p+1}\left(1 - \frac{x_i^p}{\theta^p}\right)^2}$$

$$BM_{\theta\theta} = \frac{npa}{\theta^2} - p(b-1)(p+1)\sum_{i=1}^{n}\frac{x_i^p}{\theta^{p+2}}\left(1-\frac{x_i^p}{\theta^p}\right) - p^2(b-1)\sum_{i=1}^{n}\frac{\left(x_i^p\right)^2}{(\theta^{p+1})^2}$$

$$BM_{\theta a} = -\frac{np}{\theta}$$

$BM_{\theta b} = \sum_{i=1}^{n}\frac{px_i^p}{\theta^{p+1}\left(1-\frac{x_i^p}{\theta^p}\right)}$, $BM_{ap} = -n\log\theta + \sum_{i=1}^{n}\log x_i$

$BM_{a\theta} = -\frac{np}{\theta}$; $BM_{aa} = n\Psi'(a+b) - n\Psi'(a)$

$$BM_{ab} = n\Psi'(a+b)$$

$$BM_{bp} = -\sum_{i=1}^{n}\frac{x_i^p\log\left(1-\frac{x_i}{\theta}\right)}{\theta^p\left(1-\frac{x_i^p}{\theta^p}\right)} BM_{b\theta} = \sum_{i=1}^{n}\frac{px_i^p}{\theta^{p+1}\left(1-\frac{x_i^p}{\theta^p}\right)}$$

$$BM_{ba} = n\Psi'(a+b)$$

$$BM_{bb} = n\Psi'(a+b) - n\Psi'(b)$$

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