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An improvement of Steffensen's method for solving nonlinear equations

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Abstract

In this work we present an improvement of Steffensen's method [10] for computing numerical approximation of nonlinear equations $f(x) = 0$. The order of convergence of this new iterative method (with two-steps) is "p^2", knowing that the method of Steffensen (with only one step) is of order "$2p - 1$".

Keywords: Nonlinear equations; Iterative method; Steffensen's method; Aitken's method; Order of convergence.

Introduction

Solving nonlinear equations of the form $f(x) = 0$ can be considered to be one of the most significant problems arising in the numerical analysis. In the last few years several methods were proposed. The purpose of which is to rise the convergence order while being based on the iterative methods [2], [3], [4], [5] and [7]. In this work we would like to improve the order of convergence of Steffensen's method.

Acceleration of Convergence

From a given convergent sequence, one can build several iterative methods which converge more quickly towards the same limit. One of the traditional processes is given by: Aitken.

Theorem 1 (Δ^2 Aitken) [6].
If (x_n) $n \in N$ is a sequence which converges towards the exact solution x_* of the nonlinear equation with order one, the sequence defined by: $x'_n = x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n}$ *converges more quickly towards x_*, such as $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$*

To avoid the heaviness of this classical iterative method, one of the processes is given by the method of Steffensen [6].

Consequently, being given an iterative method generated by F, starting from the sequence $y_n = F(x_n)$ and $z_n = F(y_n)$ one can obtain the recurrence sequence defined by the following iteration function φ :

$$\varphi(x) = \frac{\left(x - F(x)\right)^2}{F\left(F(x)\right) - 2F(x) + x}$$

Theorem 2 (Steffensen) [10]
Let x_* be a simple root of the equation $x = F(x)$.
If the method generated by F is of order 1, then the method of Steffensen is of order at least 2
If the method generated by F is of order $p > 1$, then the method of Steffensen is of order $2p - 1$.

We present a new iterative method with two steps, an improvement of Steffensen's method given by the following algorithm:

Algorithm

$$\varphi_n = x_n - \frac{\left(x_n - F(x_n)\right)^2}{F\left(F(x_n)\right) - 2F(x_n) + x_n}$$

$$x_{n+1} = \frac{\varphi_n F\left(F(x_n)\right) - \left(F(\varphi_n)\right)^2}{F\left(F(x_n)\right) - 2F(\varphi_n) + \varphi_n} \qquad (1)$$

Convergence Analysis of the New Method

In this section we compute the convergence order of the proposed method given by (1).

Theorem 3 *Let x_* be a simple root of the equation $x = F(x)$.*
If the method generated by F is of order 1, then the method (1) is of order at least 2.
If the method generated by F is of order $p > 1$, then the method (1) is of order "p^2".

Proof. For $F \in C^{p+1}$, in the neighborhood of the solution x_*, we have the Taylor expansion:

$$F(x_* + e_n) = x_* + e_n F'(x_*) + \frac{e_n^2}{2} F''(x_*) + \cdots + \frac{e_n^p}{p!} F^{(p)}(x_*) + O\left(e_n^{p+1}\right), = x_* + \alpha_1 e_n + \alpha_2 e_n^2 + \cdots + \alpha_p e_n^p + O\left(e_n^{p+1}\right), \quad (2)$$

where

$$\alpha_p = \frac{F^{(p)}(x_*)}{p!} \quad (3)$$

- If the order of the method is 1, i.e. $\alpha_1 \neq 0$, with $|\alpha_1| < 1$, we have:

$$F(x_* + e_n) = x_* + \alpha_1 e_n + \alpha_2 e_n^2 + \cdots + \alpha_p e_n^p + O\left(e_n^{p+1}\right) \quad (4)$$

$$F\big(F(x_* + e_n)\big) = x_* + \alpha_1^2 e_n + (\alpha_1\alpha_2 + \alpha_1^2\alpha_2) e_n^2 + (\alpha_1\alpha_3 + 2\alpha_1\alpha_2^2 + \alpha_1^3\alpha_3) e_n^3 + O(e_n^4) \quad (5)$$

Thus:

$$\varphi(x_* + e_n) = x_* + \frac{\alpha_1\alpha_2}{1+\alpha_1} e_n^2 + O(e_n^3). \quad (6)$$

Then:

$$F\big(\varphi(x_* + e_n)\big) = x_* + \frac{\alpha_1^2\alpha_2}{1+\alpha_1} e_n^2 + \frac{\alpha_1^2\alpha_2^3}{(1+\alpha_1)^2} e_n^4 + O(e_n^5). \quad (7)$$

We let:

$$H(x) = \frac{\varphi(x) F\big(F(x)\big) - \Big(F\big(\varphi(x)\big)\Big)^2}{F\big(F(x)\big) - 2F\big(\varphi(x)\big) + \varphi(x)}, \quad (8)$$

then

$$H(x_* + e_n) = x_* + \frac{\alpha_1\alpha_2}{1+\alpha_1} e_n^2 + O(e_n^3), \quad (9)$$

Hence

$$e_{n+1} = O(e_n^2).$$

- If the method is of order $p > 1$, i.e. $\alpha_1 = \alpha_2 = \cdots = \alpha_{p-1} = 0$, and $\alpha_p \neq 0$.

We have:

$$F(x_* + e_n) = x_* + \alpha_p e_n^p + O\left(e_n^{p+1}\right). \quad (10)$$

$$F\big(F(x_* + e_n)\big) = x_* + \alpha_p^{p+1} e_n^{p^2} + O\left(e_n^{p^2+1}\right). \quad (11)$$

Thus:

$$\varphi(x_* + e_n) = x_* - \alpha_p^2 e_n^{2p-1} + O\left(e_n^{2p}\right). \quad (12)$$

Using (10) and (12) we get:

$$F\big(\varphi(x_* + e_n)\big) = x_* + (-1)^p \alpha_p^{2p+1} e_n^{2p^2-p} + O\left(e_n^{2p^2-p+1}\right). \quad (13)$$

Thus:

$$H(x_* + e_n) = x_* + \frac{-\alpha_p^{p+3} e_n^{p^2+2p-1} + O\left(e_n^{p^2+2p}\right)}{-\alpha_p^2 e_n^{2p-1} + \alpha_p^{p+1} e_n^{p^2} + O\left(e_n^{p^2+1}\right)}$$

$$= x_* + \alpha_p^{p+1} e_n^{p^2} + O\left(e_n^{p^2+1}\right)$$

Finally we have:

$$e_{n+1} = \alpha_p^{p+1} e_n^{p^2} + O\left(e_n^{p^2+1}\right).$$

Numerical Examples

We present few examples to illustrate the efficiency of the suggested iterative method. We make a comparison between the methods of Newton (of order *2*) [8], Frontini (of order 3) [5], Kou (of order 5) [4], Chun2 (of order 6) [3] and Steffensen with the new method.

We use the following test functions:

$f_1(x) = 7x^3 - 22x^2 + 7x - 22,$

$f_2(x) = \frac{2}{x} - 1,$

$f_3(x) = x - 2 - \exp(-x)$, in [1],

$f_4(x) = x^3 + 3x + 5,$

$f_5(x) = \exp(x-1) + 2x - 3.$

Table 1: Comparison of different methods with our method: Number of iterations and computed solution

$f_1(x) = 7x^3 - 22x^2 + 7x - 22$; $x_0 = 3.5$		
Method	Number of iterations	Computed solution
Newton	5	with 23 significant digits
Steffensen	4	with 59 significant digits
New method	4	with 186 significant digits
Frontini	3	with 20 significant digits
Steffensen	2	with 18 significant digits
New method	2	with 62 significant digits
Kou	2	with 20 significant digits
Steffensen	2	with 68 significant digits
New method	2	with ≥300 significant digits
Chun 2	4	with 22 significant digits
Steffensen	3	with 37 significant digits
New method	3	with 93 significant digits
The exact solution is $x_*=\frac{22}{7}$		

Table 2: Comparison of different methods with our method: Number of iterations and computed solution

$f_2(x) = \frac{2}{x} - 1$; $x_0 = 1.9$		
Method	Number of iterations	Computed solution
Newton	4	with 20 significant digits
Steffensen	3	with 34 significant digits
New method	3	with 82 significant digits
Frontini	3	with 36 significant digits
Steffensen	2	with 33 significant digits
New method	2	with 109 significant digits
Kou	3	with 159 significant digits
Steffensen	2	with 102 significant digits
New method	2	with ≥500 significant digits
Chun 2	4	with 35 significant digits
Steffensen	3	with 60 significant digits
New method	3	with 143 significant digits
The exact solution is x_*=2		

Table 3: Comparison of different methods with our method: Number of iterations and computed solution

$f_3(x) = x - 2 - \exp(-x)$;$x_0 = 2$		
Method	Number of iterations	Computed solution
Newton	6	with 137 significant digits
Steffensen	4	with 174 significant digits
New method	4	with ≥500 significant digits
Frontini	3	with 61 significant digits
Steffensen	2	with 56 significant digits
New method	2	with 187 significant digits
Kou	2	with 49 significant digits
Steffensen	1	with 16 significant digits
New method	1	with 52 significant digits
Chun 2	2	with 8 significant digits
Steffensen	2	with 22 significant digits
New method	2	with 43 significant digits
The exact solution is x_*=2.12002823898764122948468797527184924493914736613686...		

Table 4: Comparison of different methods with our method: Number of iterations and computed solution

$f_4(x) = x^3 + 3x + 5;\ x_0 = -1$		
Method	Number of iterations	Computed solution
Newton	4	with 15 significant digits
Steffensen	3	with 20 significant digits
New method	3	with 70 significant digits
Frontini	3	with 30 significant digits
Steffensen	2	with 27 significant digits
New method	2	with 92 significant digits
Kou	2	with 27 significant digits
Steffensen	2	with 92 significant digits
New method	2	with ≥427 significant digits
Chun 2	4	with 33 significant digits
Steffensen	3	with 58 significant digits
New method	3	with 138 significant digits
The exact solution is x_*=-1.1541714951814412673711179139254502781075053 68...		

Table 5: Comparison of different methods with our method: Number of iterations and computed solution

$f_5(x) = \exp(x-1) + 2x - 3;\ x_0 = 1.1$		
Method	Number of iterations	Computed solution
Newton	4	with 27 significant digits
Steffensen	3	with 46 significant digits
New method	3	with 112 significant digits
Frontini	3	with 50 significant digits
Steffensen	2	with 46 significant digits
New method	2	with 154 significant digits
Kou	2	with 42 significant digits
Steffensen	1	with 14 significant digits
New method	1	with 51 significant digits
Chun 2	6	with 168 significant digits
Steffensen	3	with 69 significant digits
New method	3	with 168 significant digits
The exact solution is x_*=1		

Conclusion

In this article, from the tables 1, 2, 3, 4, 5, we can observe that our new proposed method can be considered as an improvement of Steffensen's method. In addition, it is applicable to all the known iterative methods. Besides the number of iterations proposed, this method gives better approximations in the first iterations.

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