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On principally right McCoy rings

Sang Jo Yun

Department of Mathematics,
Pusan National University,
Pusan 46241, Korea.

Abstract

We study a ring theoretic property which is a special case of right McCoy rings, introducing the concept of principally right McCoy rings. We study the basic properties of principally right McCoy rings, and ordinary ring extensions of principally right McCoy rings are considered.

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1. Principally right McCoy rings

Throughout this note every ring is associative with identity unless otherwise stated. Given a ring R, $r_R(-)$ ($l_R(-)$) is used for the right (left) annihilator in R. If S is a singleton, say $S = \{a\}$, then we write $r_R(a)$ ($\ell_R(a)$) in place of $r_R(\{a\})$ ($\ell_R(\{a\})$). We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Let $\mathbb{Z}$ ($\mathbb{Z}_n$) denote the ring of integers (modulo n). Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Let E_{ij} be the matrix with (i, j)-entry 1 and zeros elsewhere.

McCoy [7, Theorem 2] proved the following in 1942:

If R is a commutative ring and $r_{R[x]}(f(x)) \neq 0$, then $r_R(f(x)) \neq 0$.

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. Rege et al. [9] called a ring R *Armendariz* if $a_i b_j = 0$ for all i, j whenever $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$. Reduced rings are Armendariz by [2, Lemma 1]. A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [1, Theorem 6] or [5, Corollary 8].

Nielsen [8] called a ring R (possibly without identity) *right McCoy* provided that the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some nonzero $c \in R$, where $f(x), g(x)$ are nonzero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then the ring is called a *McCoy* ring. Armendariz rings are clearly McCoy.

A ring R will be called *principally right McCoy* provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the principal right ideal of R generated by a coefficient of $g(x)$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Principally left McCoy rings are defined similarly. If a ring is both principally left and right McCoy then the ring is called a *principally McCoy* ring.

A principally right McCoy ring is obviously right McCoy. Armendariz rings are clearly principally McCoy rings. And this implication is irreversible as the following example shows.

Example 1.1. We use the ring and argument in [5, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Let B be the subalgebra of A consist of all polynomials of zero constant term. Consider an ideal of A, say I, generated by $a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2, a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ and $r_1r_2r_3r_4$, where $r, r_1, r_2, r_3, r_4 \in B$. Note $B^4 \in I$.

Let next $R = A/I$. Consider $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) = 0$, but $a_1b_2 \neq 0$. So R is not Armendariz. We show next that R is principally right McCoy. $supp(-)$ means the support of a given polynomial. Take $0 \neq f(x) = \sum_{i=0}^{n} \alpha_i x^i, 0 \neq g(x) \sum_{j=0}^{m} \beta_j x^j \in R[x]$ with $f(x)g(x) = 0$. If $1 \in supp(\beta_j)$, then fix α' to be a monomial in the support of α_i of smallest degree. But this implies $\alpha' \cdot 1$ is in the support of $\sum \alpha_i \beta_j = 0$, a contradiction. Thus $1 \notin supp(\beta_j)$, and similarly, $1 \notin supp(\alpha_i)$. So $\alpha_i, \beta_i \in A$ for each i, j. Now we have $\beta_k \neq 0$ for some $0 \leq k \leq m$. Note that $A^4 = 0$. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Observe that H_n is finite for any n and that the ideal I of R is homogeneous

If β_k with smallest degree 1 exists, then $f_1 \in H_1$ and so we can find nonzero $r \in \beta_k R$ such that $f(x)r = 0$.

If β_k with smallest degree 2 exists, then $f_2 \in H_2$ and so we can find nonzero $r \in \beta_k R$ such that $f(x)r = 0$.

If β_k with smallest degree 3 exists, then $f_3 \in H_3$ and so $f_3 \in \beta_k R$ such that $f(x)f_3 = 0$.

Thus R is principally right McCoy.

Following [3], a ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. If given a ring is regular then we have the following equivalence.

Proposition 1.2. Given a regular ring R the following conditions are equivalent:

(1) R is reduced;

(2) R is Armendariz;

(3) R is principally right McCoy;

(4) R is right McCoy;

(5) R is Abelian.

Proof. The proof is done by help of [3, Theorem 3.2] and [6, Proposition 2.14], noting that Armendariz rings are principally right McCoy and principally right McCoy rings are right McCoy. ■

2. Extensions of principally right McCoy rings

In this section we examine some ring extensions, which have roles in ring theory, to be principally right McCoy. A ring R is usually called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Note that R is a right Ore ring if and only if the classical right quotient ring of R exists. There exist many reduced rings which are not right Ore as can be seen by the free algebra in two noncommuting indeterminates over a field. This kind of ring is a domain which cannot have its classical right quotient ring.

Theorem 2.1. Let R be a right Ore ring with the classical right quotient ring $Q_r(R)$. Then R is principally right McCoy if and only if so is $Q_r(R)$.

Proof. We adapt the proof of [4, Theorem 2.1] for the case of principally right McCoy rings. Let $Q = Q_r(R)$. Suppose $F(x)G(x) = 0$ for $0 \neq F(x), G(x) \in Q[x]$. We can write $F(x) = a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m$ and $G(x) = b_0v^{-1} + b_1v^{-1}x + \cdots + b_nv^{-1}x^n$, where u, v are regular. Since $F(x)G(x) = 0$, $(a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m)(b_0 + b_1x + \cdots + b_nx^n) = 0$ and this yields

$$(\dagger) \qquad a_0u^{-1}b_0 = 0, a_0u^{-1}b_1 + a_1u^{-1}b_0 = 0, \ldots, a_mu^{-1}b_n = 0.$$

For $u^{-1}b_0, u^{-1}b_1, \ldots, u^{-1}b_n$, there exist $c_0, c_1, \ldots, c_n$ and s regular such that $u^{-1}b_i = c_is^{-1}$ for all i. Then, from the equality (†), we get $a_0c_0 = 0, a_0c_1 + a_1c_0 = 0, \ldots, a_mc_n = 0$ and $f(x)g(x) = 0$, where $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = c_0 + c_1x + \cdots + c_nx^n$ in $R[x]$. Note that $f(x) \neq 0$ and $g(x) \neq 0$ because $F(x) \neq 0$ and $G(x) \neq 0$.

Since R is principally right McCoy, there exists nonzero $r \in c_iR$ for some i such that $f(x)r = 0$. Note that

$$r \in c_iR \subseteq c_iQ = c_is^{-1}Q = u^{-1}b_iQ.$$

Thus $ur \in b_iQ = b_iv^{-1}Q$ and $ur \neq 0$. Since $f(x)r = 0$, we have

$$\begin{aligned} 0 &= (a_0 + a_1x + \cdots + a_mx^m)r = (a_0 + a_1x + \cdots + a_mx^m)u^{-1}ur \\ &= (a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m)ur = F(x)ur. \end{aligned}$$

Therefore Q is principally right McCoy.

Conversely, let $0 \neq f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $0 \neq g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x]$ such that $f(x)g(x) = 0$. Then $f(x), g(x) \in Q[x]$. Since Q is principally right McCoy, there exists i such that $f(x)b_irs^{-1} = 0$ for some $0 \neq b_irs^{-1} \in b_iQ$. This implies $f(x)b_ir = 0$ and $b_ir \neq 0$. Since $b_ir \in b_iR$, R is principally right McCoy. ■

Let R be a ring R and $n \geq 2$. Following the literature, we usually consider the subring

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

of $U_n(R)$. Note that $D_n(R)[x]$ is isomorphic to $D_n(R[x])$.

Theorem 2.2. For a ring R, the following conditions are equivalent:

(1) R is principally right McCoy;

(2) $D_2(R)$ is principally right McCoy;

(3) $D_3(R)$ is principally right McCoy.

Proof. We apply the proof of [4, Theorem 2.2]. For the proof of (1) implying (2), suppose that R is principally right McCoy. Recall $(D_n(R))[x] \cong D_n(R[x])$. Let

$$0 \neq A(x) = \sum_{i=0}^{m} \begin{pmatrix} a_{1i} & b_{1i} \\ 0 & a_{1i} \end{pmatrix} x^i = \begin{pmatrix} f_1(x) & g_1(x) \\ 0 & f_1(x) \end{pmatrix}$$

and

$$0 \neq B(x) = \sum_{j=0}^{n} \begin{pmatrix} a_{2j} & b_{2j} \\ 0 & a_{2j} \end{pmatrix} x^j = \begin{pmatrix} f_2(x) & g_2(x) \\ 0 & f_2(x) \end{pmatrix},$$

where $f_1(x) = \sum_{i=0}^{m} a_{1i}x^i$, $g_1(x) = \sum_{i=0}^{m} b_{1i}x^i$, $f_2(x) = \sum_{j=0}^{n} a_{2j}x^j$, $g_2(x) = \sum_{j=0}^{n} b_{2j}x^j$.

Case 1. $(f_1(x) \neq 0,\ f_2(x) \neq 0)$

Note $f_1(x)f_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\alpha \in a_{2j}R$ for some $j \in \{1, 2, \ldots, n\}$ such that $f_1(x)\alpha = 0$, say $\alpha = a_{2j}\alpha_j$. So $A(x)\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{2j} & b_{2j} \\ 0 & a_{2j} \end{pmatrix}\begin{pmatrix} 0 & \alpha_j \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} a_{2j} & b_{2j} \\ 0 & a_{2j} \end{pmatrix} D_2(R).$$

Case 2. $(f_1(x) \neq 0$ and $f_2(x) = 0, g_2(x) \neq 0)$

Note $f_1(x)g_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\beta \in b_{2j}R$ for some $j \in \{1, 2, \ldots, n\}$ such that $f_1(x)\beta = 0$, say $\beta = b_{2j}\beta_j$. So $A(x)\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \beta_j & 0 \\ 0 & \beta_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} \\ 0 & 0 \end{pmatrix} D_2(R).$$

Case 3. $(f_1(x) = 0, g_1(x) \neq 0$, and $f_2(x) \neq 0)$

Let $p = a_{2i}p_i$ be any nonzero element for each $i = 1, 2, \ldots, n$. Then we get $A(x)\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{2i} & b_{2i} \\ 0 & a_{2i} \end{pmatrix}\begin{pmatrix} 0 & p_i \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} a_{2i} & b_{2i} \\ 0 & a_{2i} \end{pmatrix} D_2(R).$$

Case 4. $(f_1(x) = 0$ and $f_2(x) = 0, g_2(x) \neq 0)$

Let $q = b_{2i}q_i$ be any nonzero element for each $i = 1, 2, \ldots, n$. Then we get $A(x)\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q_j & 0 \\ 0 & q_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} \\ 0 & 0 \end{pmatrix} D_2(R).$$

By Cases 1, 2, 3 and 4, $D_2(R)$ is principally right McCoy.

For the proof of (1) implying (3), suppose that R is principally right McCoy and let

$$0 \neq A(x) = \sum_{i=0}^{m} \begin{pmatrix} a_{1i} & b_{1i} & c_{1i} \\ 0 & a_{1i} & d_{1i} \\ 0 & 0 & a_{1i} \end{pmatrix} x^i = \begin{pmatrix} f_1(x) & g_1(x) & h_1(x) \\ 0 & f_1(x) & i_1(x) \\ 0 & 0 & f_1(x) \end{pmatrix}$$

and

$$0 \neq B(x) = \sum_{j=0}^{m} \begin{pmatrix} a_{2i} & b_{2i} & c_{2i} \\ 0 & a_{2i} & d_{2i} \\ 0 & 0 & a_{2i} \end{pmatrix} x^j = \begin{pmatrix} f_2(x) & g_2(x) & h_2(x) \\ 0 & f_2(x) & i_2(x) \\ 0 & 0 & f_2(x) \end{pmatrix}$$

in $D_3(R)[x]$ such that $A(x)B(x) = O$, where $f_1(x) = \sum_{i=0}^{m} a_{1i}x^i$, $g_1(x) = \sum_{i=0}^{m} b_{1i}x^i$, $h_1(x) = \sum_{i=0}^{m} c_{1i}x^i$, $i_1(x) = \sum_{i=0}^{m} d_{1i}x^i$, $f_2(x) = \sum_{j=0}^{n} a_{2j}x^j$, $g_2(x) = \sum_{j=0}^{n} b_{2j}x^j$, $h_2(x) = \sum_{j=0}^{n} c_{2j}x^j$, $i_2(x) = \sum_{j=0}^{n} d_{2j}x^j$.

Case 1. $(f_1(x) \neq 0, f_2(x) \neq 0)$

Note $f_1(x)f_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\alpha \in a_{2j}R$ for some $j \in \{1, 2, \ldots, n\}$ such that $f_1(x)\alpha = 0$, say $\alpha = a_{2j}\alpha_j$. So $A(x)\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{2j} & b_{2j} & c_{2j} \\ 0 & a_{2j} & d_{2j} \\ 0 & 0 & a_{2j} \end{pmatrix}\begin{pmatrix} 0 & 0 & \alpha_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} a_{2j} & b_{2j} & c_{2j} \\ 0 & a_{2j} & d_{2j} \\ 0 & 0 & a_{2j} \end{pmatrix} D_3(R).$$

Case 2. $(f_1(x) \neq 0$ and $f_2(x) = 0, g_2(x) \neq 0)$

Note $f_1(x)g_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\beta \in b_{2j}R$ for some $j \in \{1, 2, \ldots, n\}$ such that $f_1(x)\beta = 0$, say $\beta = b_{2j}\beta_j$. So $A(x)\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_j \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Case 3. $(f_1(x) \neq 0$ and $f_2(x) = 0, g_2(x) = 0, h_2(x) \neq 0)$

Note $f_1(x)h_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\gamma \in c_{2j}R$ for some $j \in \{1, 2, \ldots, n\}$ such that $f_1(x)\gamma = 0$, say $\gamma = c_{2j}\gamma_j$. So $A(x)\begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{2j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma_j & 0 & 0 \\ 0 & \gamma_j & 0 \\ 0 & 0 & \gamma_j \end{pmatrix} \in \begin{pmatrix} 0 & 0 & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Case 4. $(f_1(x) \neq 0$ and $f_2(x) = 0, g_2(x) = 0, i_2(x) \neq 0)$

Note $f_1(x)i_2(x) = 0$. Then since R is principally right McCoy, there exists nonzero $\delta \in d_{2j}R$ for some $j \in \{1, 2, \dots, n\}$ such that $f_1(x)\delta = 0$, say $\delta = d_{2j}\delta_j$. So $A(x)\begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta_j & 0 & 0 \\ 0 & \delta_j & 0 \\ 0 & 0 & \delta_j \end{pmatrix} \in \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Case 5. $(f_1(x) = 0,\ f_2(x) \neq 0)$

Note $f_1(x)f_2(x) = 0$. Then, for any nonzero $p = a_{2i}p_i$ for each $i = 1, 2, \dots, n$, $A(x)\begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{2j} & b_{2j} & c_{2j} \\ 0 & a_{2j} & d_{2j} \\ 0 & 0 & a_{2j} \end{pmatrix}\begin{pmatrix} 0 & 0 & p_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} a_{2j} & b_{2j} & c_{2j} \\ 0 & a_{2j} & d_{2j} \\ 0 & 0 & a_{2j} \end{pmatrix} D_3(R).$$

Case 6. $(f_1(x) = 0,\ f_2(x) = 0)$

Subcase 1. $(g_1(x) \neq 0)$

If $g_2(x) \neq 0$, then for any nonzero $q = b_{2i}q_i$ for each $i = 1, 2, \dots, n$ we get $A(x)\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_j \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $h_2(x) \neq 0$, then for any nonzero $r = c_{2i}r_i$ for each $i = 1, 2, \dots, n$ we get $A(x)\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r_j & 0 & 0 \\ 0 & r_j & 0 \\ 0 & 0 & r_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $i_2(x) \neq 0$, then since $g_1(x)i_2(x) = 0$ and R is principally right McCoy, there exists nonzero $\zeta \in d_{2j}R$ for some $j \in \{1, 2, \dots, n\}$ such that $f_1(x)\zeta = 0$, say $\zeta = d_{2j}\zeta_j$. So

$A(x)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \zeta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \zeta_j & 0 & 0 \\ 0 & \zeta_j & 0 \\ 0 & 0 & \zeta_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Subcase 2. $(g_1(x) = 0, h_1(x) \neq 0)$

If $g_2(x) \neq 0$, then for any nonzero $q = b_{2i}q_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_j \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $h_2(x) \neq 0$, then for any nonzero $r = c_{2i}r_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r_j & 0 & 0 \\ 0 & r_j & 0 \\ 0 & 0 & r_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $i_2(x) \neq 0$, for any nonzero $s = d_{2i}s_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_j & 0 & 0 \\ 0 & s_j & 0 \\ 0 & 0 & s_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Subcase 3. $(g_1(x) = 0, h_1(x) = 0, i_1(x) \neq 0)$

If $g_2(x) \neq 0$, then for any nonzero $q = b_{2i}q_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_j \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $h_2(x) \neq 0$, then for any nonzero $r = c_{2i}r_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r_j & 0 & 0 \\ 0 & r_j & 0 \\ 0 & 0 & r_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

If $i_2(x) \neq 0$, for any nonzero $s = d_{2i}s_i$ for each $i = 1, 2, \ldots, n$ we get $A(x)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = 0$, noting that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s_j & 0 & 0 \\ 0 & s_j & 0 \\ 0 & 0 & s_j \end{pmatrix} \in \begin{pmatrix} 0 & b_{2j} & c_{2j} \\ 0 & 0 & d_{2j} \\ 0 & 0 & 0 \end{pmatrix} D_3(R).$$

Therefore $D_3(R)$ is principally right McCoy.

For the proof of (2) implying (1), suppose that $D_2(R)$ is principally right McCoy, and let $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i$, $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Then, letting

$$A(x) = \sum_{i=0}^{m} \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} x^i \text{ and } B(x) = \sum_{j=0}^{n} \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} x^j,$$

we have $A(x) = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} B(x) = \begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix} x^j$ with $A(x)B(x) = O$. Since $D_2(R)$ is principally right McCoy, there exists nonzero $C \in \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} D_2(R)$ such that $A(x)C = O$, say $C = \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix}\begin{pmatrix} c_{1j} & c_{2j} \\ 0 & c_{1j} \end{pmatrix}$. Note that $b_j c_{1j} \neq 0$ or $b_j c_{2j} \neq 0$. Since $f(x)b_j c_{1j} = 0$ and $f(x)b_j c_{2j} = 0$, R is principally right McCoy. The proof of (3) implying (1) is similar. ■

Considering Theorem 2.2, one can ask whether $D_n(R)$ may be also principally right McCoy for $n \geq 4$ over a principally right McCoy ring R. However the following shows that the answer is negative.

Example 2.3. Let R be any ring and consider $D_n(R)$ for $n \geq 4$. We use the polynomials

$$f(x) = E_{12} + E_{13}x \text{ and } g(x) = -E_{3n} + E_{2n}x \in D_n(R)[x]$$

in Remark (1) after [4, Theorem 2.2]. Then $f(x)g(x) = 0$. Consider the right ideals $I_1 = E_{2n}D_n(R)$ and $I_2 = E_{3n}D_n(R)$ of $D_n(R)$. Let $r_1 = aE_{2n}$ and $r_2 = bE_{3n}$ be any nonzero elements in I_1 and I_2, respectively. Then

$$f(x)r_1 = aE_{1n} \neq 0 \text{ and } f(x)r_2 = bE_{1n}x \neq 0$$

and so $D_n(R)$ is not principally right McCoy.

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