Abstract

In this paper, we introduce soft generalized closed sets in soft Čech closure spaces which are defined over an initial universe with a fixed set of parameters. Also we investigate the behavior relative to union, intersection and soft subspaces of soft g-closed sets as well as soft g-open sets. In the study of soft g-closed sets, we introduce a new separation axiom, namely $T_{1/2}$-space which lie between $T_0$ and $T_1$.

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1. Introduction

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. In 1970, N. Levine [7] initiated the study of generalized closed sets in topological space in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform space are inherited by g-closed subsets.
E. Čech [2] introduced the concept of closure spaces. In Čech’s approach the operator satisfies idempotent condition among Kuratowski axioms. This condition need not hold for every set $A$ of $X$. When this condition is also true, the operator becomes topological closure operator. Thus, the concept of closure space is the generalisation of a topological space. In 1999, D. Molodtsov [8] introduced the notion of soft set to deal with problems of incomplete information. Later, he applied this theory to several directions [9] and [10]. Generalized closed sets in Čech closed spaces were introduced by Chawalit Boonpok [5]. R. Gowri and G. Jegadeesan [6] introduced and studied the concept of lower separation axioms in soft Čech closure spaces.

In this paper, we introduce soft generalized closed sets in soft Čech closure spaces which are defined over an initial universe with a fixed set of parameters. Also we investigate the behavior relative to union, intersection and soft subspaces of soft g-closed sets as well as soft g-open sets. In the study of soft g-closed sets, we introduce a new separation axiom, namely $T_{1/2}$-space which lie between $T_0$ and $T_1$.

2. Preliminaries

In this section, we recall the basic definitions of soft Čech closure spaces.

**Definition 2.1.** Let $X$ be an initial universe set, $A$ be a set of parameters. Then the function $k : P(X_{F_A}) \rightarrow P(X_{F_A})$ defined from a soft power set $P(X_{F_A})$ to itself over $X$ is called Čech Closure operator if it satisfies the following axioms:

(C1) $k(\emptyset_A) = \emptyset_A$.

(C2) $F_A \subseteq k(F_A)$.

(C3) $k(F_A \cup G_A) = k(F_A) \cup k(G_A)$.

Then $(X, k, A)$ or $(F_A, k)$ is called a soft Čech closure space.

**Definition 2.2.** A soft subset $U_A$ of a soft Čech closure space $(F_A, k)$ is said to be soft $k$-closed (soft closed) if $k(U_A) = U_A$.

**Definition 2.3.** A soft subset $U_A$ of a soft Čech closure space $(F_A, k)$ is said to be soft $k$-open (soft open) if $k(U_A^C) = U_A^C$.

**Definition 2.4.** A soft set $Int(U_A)$ with respect to the closure operator $k$ is defined as $Int(U_A) = F_A - k(F_A - U_A) = [k(U_A^C)]^C$. Here $U_A^C = F_A - U_A$.

**Definition 2.5.** A soft subset $U_A$ in a soft Čech closure space $(F_A, k)$ is called Soft neighbourhood of $e_F$ if $e_F \in Int(U_A)$.

**Definition 2.6.** If $(F_A, k)$ be a soft Čech closure space, then the associate soft topology on $F_A$ is $\tau = \{U_A^C : k(U_A) = U_A\}$. 
Definition 2.7. Let \((F_A, k)\) be a soft Čech closure space. A soft Čech closure space \((G_A, k^*)\) is called a soft subspace of \((F_A, k)\) if \(G_A \subseteq F_A\) and \(k^*(U_A) = k(U_A) \cap G_A\), for each soft subset \(U_A \subseteq G_A\).

3. Soft generalized closed sets

In this section, we introduce and characterize a new class of soft generalized closed sets in Soft Čech closure spaces.

Definition 3.1. Let \((F_A, k)\) be a soft Čech closure space. A soft subset \(U_A \subseteq F_A\) is called a soft generalized closed set, briefly a soft g-closed set if \(k(U_A) \subseteq G_A\) whenever \(G_A\) is an soft open subset of \((F_A, k)\) with \(U_A \subseteq G_A\). A soft subset \(U_A \subseteq F_A\) is called a soft generalized open set, briefly a soft g-open set, if its complement is soft g-closed.

Example 3.2. Let the initial universe set \(X = \{u_1, u_2\}\) and \(E = \{x_1, x_2, x_3\}\) be the parameters. Let \(A = \{x_1, x_2\} \subseteq E\) and \(F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_1, u_2\})\}\). Then \(P(X_{FA})\) are

\[
F_{1A} = \{(x_1, \{u_1\})\}, \quad F_{2A} = \{(x_1, \{u_2\})\},
\]
\[
F_{3A} = \{(x_1, \{u_1, u_2\})\}, \quad F_{4A} = \{(x_2, \{u_1\})\},
\]
\[
F_{5A} = \{(x_2, \{u_2\})\}, \quad F_{6A} = \{(x_2, \{u_1, u_2\})\},
\]
\[
F_{7A} = \{(x_1, \{u_1\}), (x_2, \{u_1\})\}, \quad F_{8A} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\},
\]
\[
F_{9A} = \{(x_1, \{u_2\}), (x_2, \{u_1\})\}, \quad F_{10A} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\},
\]
\[
F_{11A} = \{(x_1, \{u_1\}), (x_2, \{u_1, u_2\})\}, \quad F_{12A} = \{(x_1, \{u_2\}), (x_2, \{u_1, u_2\})\},
\]
\[
F_{13A} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_1\})\}, \quad F_{14A} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\},
\]
\[
F_{15A} = F_A, \quad F_{16A} = \emptyset_A.
\]

An operator \(k : P(X_{FA}) \rightarrow P(X_{FA})\) is defined from soft power set \(P(X_{FA})\) to itself over \(X\) as follows.

\[
k(F_{1A}) = F_{1A}, k(F_{2A}) = F_{12A}, k(F_{4A}) = F_{4A},
\]
\[
k(F_{5A}) = F_{14}, k(F_{7A}) = F_{7A}, k(F_{8A}) = F_{14A},
\]
\[
k(F_{9A}) = F_{12A}, k(F_{3A}) = k(F_{6A}) = k(F_{10A}) = k(F_{11A}) = k(F_{13A}) = k(F_{14A}) = k(F_A) = F_A, k(\emptyset_A) = \emptyset_A.
\]

Here, the soft g-closed sets are \(\emptyset_A, F_{1A}, F_{4A}, F_{7A}, F_{8A}, F_{9A}, F_{11A}, F_{13A}, F_A\).

Remark 3.3. Every soft closed sets are soft g-closed, but the converse is not true as shown in the following example.

Example 3.4. In the example 3.2, \(F_{8A}, F_{9A}, F_{11A}, F_{13A}\) are soft g-closed, but not soft closed.
Theorem 3.5. Let \((F_A, k)\) be a soft \v{C}ech closure space. If \(U_A\) and \(V_A\) are soft g-closed subsets of \((F_A, k)\), then \(U_A \cup V_A\) is soft g-closed.

Proof. Let \(G_A\) be a soft open subset of \((F_A, k)\) such that \(U_A \cup V_A \subseteq G_A\). Then \(U_A \subseteq G_A\) and \(V_A \subseteq G_A\). Since, \(U_A\) and \(V_A\) are soft g-closed, \(k[U_A] \subseteq G_A\) and \(k[V_A] \subseteq G_A\). Then, \(k[U_A \cup V_A] = k[U_A] \cup k[V_A] \subseteq G_A\). Therefore, \(U_A \cup V_A\) is soft g-closed. ■

Result 3.6. The following example shows that, the intersection of two soft g-closed sets need not be a soft g-closed set.

Example 3.7. Let us consider the soft subsets of \(F_A\) that are given in example 3.2. An operator \(k : P(X_{F_A}) \rightarrow P(X_{F_A})\) is defined from soft power set \(P(X_{F_A})\) to itself over \(X\) as follows.

\[

g(F_1A) = k(F_7A) = k(F_8A) = k(F_{11}A) = F_{11}A, k(F_2A) = F_{10}A,
\]
\[
g(F_4A) = k(F_5A) = k(F_6A) = F_6A, k(F_3A) = k(F_{13}A) = k(F_{14}A) = F_{14}A,
\]
\[
g(0A) = \emptyset A.
\]

Here, take the soft g-closed sets, \(U_A = F_7A\) and \(V_A = F_8A\). Then \(U_A \cap V_A = F_1A\), which is not a soft g-closed set.

Theorem 3.8. Let \((F_A, k)\) be a soft \v{C}ech closure space. If \(U_A\) is soft g-closed and \(V_A\) is soft closed in \((F_A, k)\), then \(U_A \cap V_A\) is soft g-closed.

Proof. Let \(G_A\) be a soft open subset of \((F_A, k)\) such that \(U_A \cap V_A \subseteq G_A\). Then \(U_A \subseteq G_A \cup V_A^c\) and \(k[U_A] \subseteq G_A \cup V_A^c\). That is, \(U_A \subseteq G_A \cup (F_A - V_A)\) and \(k[U_A] \subseteq G_A \cup (F_A - V_A)\). Then, \(k[U_A] \cap V_A \subseteq G_A\). Since, \(V_A\) is soft closed. Therefore, \(k[U_A \cap V_A] \subseteq G_A\). Hence, \(U_A \cap V_A\) is soft g-closed. ■

Theorem 3.9. Let \((H_A, k^*)\) be a soft closed subspace of \((F_A, k)\). If \(V_A\) is a soft g-closed subset of \((H_A, k^*)\), then \(V_A\) is a soft g-closed subset of \((F_A, k)\).

Proof. Let \(G_A\) be a soft open subset of \((F_A, k)\) such that \(V_A \subseteq G_A\). Then, \(V_A \subseteq G_A \cap H_A\). Since, \(V_A\) is soft g-closed and \(G_A \cap H_A\) is soft open in \((H_A, k^*)\). Therefore, \(k[V_A] \cap H_A = k^*[V_A] \subseteq G_A\). But \(H_A\) is a soft closed subset of \((F_A, k)\) and \(k[V_A] \subseteq G_A\). Hence, \(V_A\) is a soft g-closed subset of \((F_A, k)\). ■

Theorem 3.10. Let \((F_A, k)\) be a soft \v{C}ech closure space and let \(U_A \subseteq F_A\). If \(U_A\) is both soft open and soft g-closed, then \(U_A\) is soft closed.

Proof. It is obvious. ■

Theorem 3.11. Let \((F_A, k)\) be a \v{C}ech closure space and let \(U_A \subseteq F_A\). If \(U_A\) is soft g-closed, then \(k[U_A] - U_A\) has no non-empty soft closed subset.
Theorem 3.15. Let $V_A$ be a soft closed subset of $k[U_A] - U_A$. Then $V_A \subseteq k[U_A] \cap (F_A - U_A)$ and so $U_A \subseteq F_A - V_A$. Consequently, $V_A \subseteq F_A - k[U_A]$. Since, $V_A \subseteq k[U_A]$, $V_A \subseteq k[U_A] \cap (F_A - k[U_A]) = \emptyset_A$. Thus $V_A = \emptyset_A$. Therefore, $k[U_A] - U_A$ contains no nonempty soft closed set.

**Remark 3.12.** The converse of the above theorem 3.11 is not true as shown in the following example.

**Example 3.13.** Let us consider the soft subsets of $F_A$ that are given in example 3.2. An operator $k : P(X_{F_A}) \to P(X_{F_A})$ is defined from soft power set $P(X_{F_A})$ to itself over $X$ as follows.

\[
\begin{align*}
    k(F_{1A}) &= k(F_{5A}) = F_{8A}, k(F_{3A}) = k(F_{9A}) = k(F_{13A}) = F_{13A}, \\
    k(F_{2A}) &= F_{3A}, k(F_{4A}) = F_{4A}, k(F_{6A}) = k(F_{8A}) = k(F_{11A}) = F_{11A}, \\
    k(F_{7A}) &= F_{7A}, k(F_{10A}) = F_{14A}, k(F_{12A}) = k(F_{14A}) = k(F_A) \\
    &= F_A, k(\emptyset_A) = \emptyset_A.
\end{align*}
\]

Here, take $U_A = F_{2A}$. Then, $k[U_A] - U_A = k[F_{2A}] - F_{2A} = F_{1A}$, which does not contain non-empty soft closed subset. But, $F_{2A}$ is not soft $g$-closed.

**Corollary 3.14.** Let $(F_A, k)$ be a soft Čech closure space and let $U_A$ be a soft $g$-closed subset of $(F_A, k)$. Then $U_A$ is soft closed if and only if $k[U_A] - U_A$ is soft closed.

**Proof.** Let $U_A$ be a soft $g$-closed subset of $(F_A, k)$. If $U_A$ is soft closed, then $k[U_A] - U_A = \emptyset_A$. Since, $\emptyset_A$ is soft closed. Therefore, $k[U_A] - U_A$ is also soft closed. Conversely, Suppose that $k[U_A] - U_A$ is soft closed. Since, $U_A$ is soft $g$-closed, by theorem 3.11, $k[U_A] - U_A = \emptyset_A$. This implies, $k[U_A] = U_A$. Hence, $U_A$ is soft closed.

**Theorem 3.15.** Let $(F_A, k)$ be a soft Čech closure space. A soft set $U_A \subseteq F_A$ is soft $g$-open if and only if $V_A \subseteq F_A - k[F_A - U_A]$ whenever $V_A$ is soft closed and $V_A \subseteq U_A$.

**Proof.** Suppose that $U_A$ is soft $g$-open and $V_A$ be a soft closed subset of $(F_A, k)$ such that $V_A \subseteq U_A$. Then $F_A - U_A \subseteq F_A - V_A$. But, $F_A - U_A$ is soft $g$-closed and $F_A - V_A$ is soft open. This implies that, $k[F_A - U_A] \subseteq F_A - V_A$. Therefore, $V_A \subseteq F_A - k[F_A - U_A]$. Conversely, Let $G_A$ be a soft open subset of $(F_A, k)$ such that $F_A - U_A \subseteq G_A$. Then $F_A - G_A \subseteq U_A$. Since, $F_A - G_A$ is soft closed, $F_A - G_A \subseteq F_A - k[F_A - U_A]$. This implies, $k[F_A - U_A] \subseteq G_A$. Therefore, $F_A - U_A$ is soft $g$-closed. Hence, $U_A$ is soft $g$-open.

**Remark 3.16.** The following example shows that, the union of two soft $g$-open sets need not be a soft $g$-open.

**Example 3.17.** In example 3.7, take soft $g$-open sets $U_A = F_{4A}$ and $V_A = F_{5A}$. Then, $U_A \cup V_A = F_{6A}$, which is not a soft $g$-open.
Theorem 3.18. Let \((F_A, k)\) be a soft Čech closure space. If \(U_A\) is soft g-open and \(V_A\) is soft open in \(F_A\), then \(U_A \cup V_A\) is soft g-open.

Proof. Let \(G_A\) be a soft closed subset of \((F_A, k)\) such that \(G_A \subseteq U_A \cup V_A\). Then \(F_A - (U_A \cup V_A) \subseteq F_A - G_A\). Hence, \((F_A - U_A) \cap (F_A - V_A) \subseteq F_A - G_A\). By theorem 3.8, \((F_A - U_A) \cap (F_A - V_A)\) is soft g-closed. Therefore, \(k[(F_A - U_A) \cap (F_A - V_A)] \subseteq F_A - G_A\). Consequently, \(G_A \subseteq F_A - k[(F_A - U_A) \cap (F_A - V_A)] = F_A - k[F_A - (U_A \cup V_A)]\). By theorem 3.15, \(U_A \cup V_A\) is soft g-open.

Theorem 3.19. Let \((F_A, k)\) be a soft Čech closure space. If \(U_A\) and \(V_A\) are soft g-open subsets of \((F_A, k)\), then \(U_A \cap V_A\) is soft g-open.

Proof. Let \(G_A\) be a soft closed subset of \((F_A, k)\) such that \(G_A \subseteq U_A \cap V_A\). Then, \(F_A - (U_A \cap V_A) \subseteq F_A - G_A\). Consequently, \((F_A - U_A) \cup (F_A - V_A) \subseteq F_A - G_A\). By theorem 3.5, \((F_A - U_A) \cup (F_A - V_A)\) is soft g-closed. Thus, \(k[(F_A - U_A) \cup (F_A - V_A)] \subseteq F_A - G_A\). Hence, \(G_A \subseteq F_A - k[(F_A - U_A) \cup (F_A - V_A)] = F_A - k[F_A - (U_A \cap V_A)]\). By theorem 3.15, \(U_A \cap V_A\) is soft g-open.

Theorem 3.20. Let \((F_A, k)\) be a soft Čech closure space. If \(U_A\) is a soft g-open subset of \(F_A\), then \(G_A = F_A\) whenever \(G_A\) is soft open and \((F_A - k[F_A - U_A]) \cup (F_A - U_A) \subseteq G_A\).

Proof. Assume that \(U_A\) is soft g-open. Let \(G_A\) be a soft g-open subset of \((F_A, k)\) such that \((F_A - k[F_A - U_A]) \cup (F_A - U_A) \subseteq G_A\). Then \((F_A - G_A) \subseteq F_A - ((F_A - k[F_A - U_A]) \cup (F_A - U_A))\). Therefore, \(F_A - G_A \subseteq k[F_A - U_A] \cap U_A\) or equivalently, \(F_A - G_A \subseteq k[F_A - U_A] - (F_A - U_A)\). But, \(F_A - G_A\) is soft closed and \(F_A - U_A\) is soft g-closed. By theorem 3.11, \(F_A - G_A = \emptyset\). Consequently, \(F_A = G_A\).

Remark 3.21. The converse of the above theorem 3.20 is not true as shown in the following example.

Example 3.22. In example 3.7, take \(U_A = F_{11A}\). Then, \((F_A - k[F_A - U_A]) \cup (F_A - U_A) = F_{13A} \subseteq G_A\), whenever \(G_A\) is soft open. This implies, \(G_A = F_A\), but \(U_A\) is not soft g-open.

Theorem 3.23. Let \((F_A, k)\) be a soft Čech closure space and let \(U_A \subseteq F_A\). If \(U_A\) is a soft g-closed, then \(k[U_A] - U_A\) is soft g-open.

Proof. Suppose that \(U_A\) is soft g-open. Let \(G_A\) be a soft closed subset of \((F_A, k)\) such that \(G_A \subseteq k[U_A] - U_A\). By theorem 3.11, \(G_A = \emptyset\) and hence \(G_A \subseteq F_A - k[F_A - (k[U_A] - U_A)]\). By theorem 3.15, \(k[U_A] - U_A\) is soft g-open.

Remark 3.24. The converse of the above theorem 3.23 is not true as shown in the following example.

Example 3.25. In example 3.7, take \(U_A = F_{2A}\). Then, \(k[U_A] - U_A = F_{5A}\), which is soft g-open. But \(U_A\) is not soft g-closed.
4. \( T_1 \)-soft Čech closure space.

In this section, we introduce \( T_1 \)-soft Čech closure spaces through soft \( g \)-closed sets and investigate some of their properties.

**Definition 4.1.** A soft Čech closure space \((FA, k)\) is said to be a \( T_1 \)-space if every soft \( g \)-closed set is soft closed in \( FA \).

**Theorem 4.2.** Let \((FA, k)\) be a soft Čech closure space, then

1. \( T_1 \)-space \( \Rightarrow \) \( T_0 \)-space.
2. \( T_1 \)-space \( \Rightarrow \) \( T_\frac{1}{2} \)-space.

**Proof.** The proof is obvious. ■

**Theorem 4.3.** Let \((FA, k)\) be a soft Čech closure space. Then \((FA, k)\) is a \( T_1 \)-space if and only if every singleton soft subset of \( FA \) is either soft closed or soft open.

**Proof.** Let \((x, u) \in FA\) and suppose that \(\{(x, u)\}\) is not a soft closed subset of \((FA, k)\). Then \(FA - \{(x, u)\}\) is not a soft open subset of \(FA\). Then the only soft open subset containing \(FA - \{(x, u)\}\) is \(FA\). Hence, \(FA - \{(x, u)\}\) is a soft \( g \)-closed subset of \((FA, k)\). Since, \((FA, k)\) is a \( T_1 \)-space. This implies, \(FA - \{(x, u)\}\) is a soft closed subset of \(FA\). Therefore, \(\{(x, u)\}\) is a soft open subset of \(FA\). Conversely, let \(UA\) be a soft \( g \)-closed subset of \((GA, k^*)\). Suppose that \((x, u) \notin UA\). Then \(\{(x, u)\} \subseteq FA - UA\). This implies, \(UA \subseteq FA - \{(x, u)\}\). If \(\{(x, u)\}\) is a soft open subset of \(FA\), then \(FA - \{(x, u)\}\) is a soft closed subset in \(FA\). Now, \(k[UA] \subseteq k[FA - \{(x, u)\}] = FA - \{(x, u)\}\), thus \((x, u) \notin k[UA]\). If \(\{(x, u)\}\) is a soft closed subset of \(FA\), then \(FA - \{(x, u)\}\) is a soft open subset in \(FA\). Since, \(UA\) is soft \( g \)-closed. This implies \(k[UA] \subseteq FA - \{(x, u)\}\). Therefore, \((x, u) \notin k[UA]\). we have, \(k[UA] \subseteq UA\) for both the cases. Thus, \(k[UA] = UA\). Hence, \(UA\) is soft closed subset of \((FA, k)\). Therefore \((FA, k)\) is \( T_1 \)-space. ■

**Theorem 4.4.** Let \((FA, k)\) be a \( T_\frac{1}{2} \)-soft Čech closure space and if \((GA, k^*)\) be a soft closed subspace of \((FA, k)\), then \((G_A, k^*)\) is also \( T_\frac{1}{2} \)-space.

**Proof.** Let \(UA\) be a soft \( g \)-closed subset of \((GA, k^*)\). Since, \(GA \subseteq FA\), then \(UA\) is also soft \( g \)-closed subset of \((FA, k)\). Since, \((FA, k)\) is \( T_1 \)-space, \(UA\) is a soft closed subset of \(FA\). This implies, \(UA\) is a soft closed subset of \(GA\). Therefore, \((GA, k^*)\) is also \( T_\frac{1}{2} \)-space. ■
5. Conclusion

In this paper, we introduced soft g-closed sets in soft Čech closure spaces which are defined over an initial universe with a fixed set of parameters. Also we investigated the behavior relative to union, intersection and soft subspaces of soft g-closed sets as well as soft g-open sets. Through soft g-closed sets, we introduced a new separation axiom, namely $T_{\frac{1}{2}}$-space which lie between $T_0$ and $T_1$.

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