Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 797–822

http://www.ripublication.com/gjpam.htm

L_p-Convergence of Lagrange interpolation polynomials with regular symmetric exponential-type weights

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Abstract

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to [0, \infty)$ be an even function. We consider the exponential-type weights $w(x) = e^{-Q(x)}, x \in \mathbb{R}$. In this paper we show the convergence of the Lagrange interpolation polynomial $L_n(f)$ with the weight w in L_p, $1 < p < \infty$.

AMS subject classification: 41A05, 41A55.
Keywords: Regular symmetric exponential-type weight, quadrature formula, Lagrange interpolation polynomial.

1. Introduction and Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \to \mathbb{R}^+ := [0, \infty)$ be an even function and $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. Then, we can construct the orthonormal polynomials $p_n(x) = p_n(w^2; x)$ of degree n with respect to $w^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w^2(x)dx = \delta_{mn} \quad \text{(Kronecker's delta)}$$

and

$$p_n(x) = \gamma_n x^n + \dots, \gamma_n > 0.$$

We denote the zeros of $p_n(x)$ by

$$-\infty < x_{n,n} < x_{n-1,n} < \dots < x_{2,n} < x_{1,n} < \infty.$$

We denote the Lagrange interpolation polynomial $L_n(f;x)$ for a continuous function f, which is based at the zeros $\{x_{k,n}\}_{k=1}^{n}$ as follows:

$$L_n(f;x) := \sum_{k=1}^{n} f(x_{k,n}) l_{k,n}(x), \quad l_{k,n}(x) := \frac{p_n(x)}{(x - x_{k,n}) p_n'(x_{k,n})}. \tag{1.1}$$

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \le Cf(y)$ for $0 < x < y$. For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \le f(x) \le C_2 g(x)$ for all x. Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{=1}^{\infty}$ we define $c_n \sim d_n$. We denote the class of polynomials of degree at most n by $\mathcal{P}_n$. Throughout $C, C_1, C_2, \dots$ denote positive constants independent of n, x, t, and polynomials of degree at most n. The same symbol does not necessarily denote the same constant in different occurrences. We are interested in the following subclass of weights from [6].

Definition 1.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ be a continuous even function satisfying the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R}\setminus\{0\}$.

(c)
$$\lim_{x\to\infty} Q(x) = \infty.$$

(d) The function
$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$
is quasi-increasing in $(0, \infty)$, with
$$T(x) \ge \Lambda > 1, \quad x \in \mathbb{R}^+\setminus\{0\}.$$

(e) There exists $C_1 > 0$ such that
$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R}\setminus\{0\}.$$

Then, we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad a.e. \quad x \in \mathbb{R} \backslash J,$$

then we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

Example 1.2.

(1) If $T(x)$ is bounded, then the weight $w = \exp(-Q)$ is called the Freud-type weight. The following example gives a Freud-type weight.

$$Q(x) = |x|^\alpha, \quad \alpha > 1.$$

If $T(x)$ is unbounded, then the weight $w = \exp(-Q)$ is called the Erdös-type weight. The following examples give the Erdös-type weights $w = \exp(-Q)$.

(2) ([3, Theorem 3.1]) For $\alpha > 1$, $\ell = 1, 2, 3, \ldots$.

$$Q(x) = Q_{\ell,\alpha}(x) = \exp_\ell(|x|^\alpha) - \exp_l(0),$$

where

$$\exp_\ell(x) = \exp(\exp(\exp \ldots \exp x) \ldots)(\ell - \text{times}).$$

More generally, we define for $\alpha + u > 1$, $\alpha \geq 0$, $u \geq 0$ and $l \geq 1$,

$$Q_{l,\alpha,u}(x) := |x|^u(\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)),$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$. We note that $Q_{l,0,u}$ gives a Freud-type weight.

(3) ([3]) We define $Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1$, $\alpha > 1$.

Mhaskar [7, §9.2] investigates the Lagrange interpolation and obtains some interesting results with respect to the convergence of the Lagrange interpolation polynomial, and he got the following results with the Freud-type weights $w = \exp(-Q)$. We write $g \in C_0(\mathbb{R})$ if $g(x)$ is continuous on $\mathbb{R}$ and

$$|g(x)| \to 0 \quad \text{as} \quad |x| \to \infty.$$

Theorem 1.3. [7, Corollary 9.2.2] Let $w = \exp(-Q)$ be the Freud-type weight, that is, $T(x)$ be bounded. Let $\beta > 1/2$, $f : \mathbb{R} \to \mathbb{R}$, and

$$(1 + x^2)^{\frac{\beta}{2}} w(x) f(x) \in C_0(\mathbb{R}).$$

Let $L_n(f,x)$ denote the Lagrange interpolation polynomial to f at the zeros of $p_n(w^2,x)$. Then

$$\|wL_n(f)\|_{L_2(\mathbb{R})} \le C\left\|(1+x^2)^{\frac{\beta}{2}}w(x)f(x)\right\|_{L_\infty(\mathbb{R})}, n=1,2,\ldots.$$

In particular,

$$\lim_{n\to\infty}\|w(f-L_n(f))\|_{L_2(\mathbb{R})}=0.$$

Theorem 1.4. [7, Theorem 9.2.3] Let $w=\exp(-Q)$ be the Freud-type weight, and let Q'' be increasing on $(0,\infty)$. Let $1<p<\infty$, $\beta>1/p$. We assume that for some $\gamma, 0<\gamma<\beta-1/p$,

$$a_n^{-\gamma}n^{\left|\frac{1}{p}-\frac{1}{2}\right|}\le C, n=1,2,\ldots.$$

If $p>2$, then we also assume that $\beta\le 1$. Then for every $f:\mathbb{R}\to\mathbb{R}$ such that

$$(1+x^2)^{\frac{\beta}{2}}w(x)f(x)\in C_0(\mathbb{R}),$$

we have

$$\|wL_n(f)\|_{L_p(\mathbb{R})}\le C\left\|(1+x^2)^{\frac{\beta}{2}}w(x)f(x)\right\|_{L_\infty(\mathbb{R})}, \quad n=1,2,\ldots.$$

In §2 we give some results with respect to the Lagrange interpolation polynomial, and then we write the quadrature formulas with the weights $w(x)$, which have been given in [9]. In §3 we prepare some lemmas, and in §4 we will prove the theorems. Then we will use the method of Mhaskar. Now, we introduce useful notations. Mhaskar-Rakhmanov-Saff numbers (MRS) a_x is defined as the positive root of following equation:

$$x=\frac{2}{\pi}\int_0^1\frac{a_xuQ'(a_xu)}{(1-u^2)^{1/2}}du, \quad x>0.$$

Let

$$\delta_x=\{xT(a_x)\}^{-2/3}, \quad x>0.$$

The function $\varphi_u(x)$ is defined as follows:

$$\varphi_u(x)=\begin{cases}\dfrac{a_u}{u}\dfrac{1-\frac{|x|}{a_{2u}}}{\sqrt{1-\frac{|x|}{a_u}+\delta_u}}, & |x|\le a_u;\\ \varphi_u(a_u), & a_u<|x|.\end{cases}$$

For $0<p<\infty$ we define the L_p-Christoffel function $\lambda_{n,p}(w;x)$ with a weight w by

$$\lambda_{n,p}(w;x):=\inf_{P\in\mathcal{P}_{n-1}}\int_{-\infty}^{\infty}|Pw|^p(u)du/|P|^p(x).$$

Especially, the L_2-Christoffel function is defined by

$$\lambda_n(w;x):=\inf_{P\in\mathcal{P}_{n-1}}\int_{-\infty}^{\infty}(Pw)^2(u)du/P^2(x)=\frac{1}{\sum_{k=1}^n p_n(x)^2},$$

and the Christoffel numbers $\lambda_{j,n}$, $j=1,2,\ldots,n$ are defined by $\lambda_{j,n}:=\lambda_n(w;x_{j,n})$, $j=1,2,\ldots,n$ (see [6, (9.14),(9.15)]).

2. Theorems

First, we state some quadrature formulas which have been obtained in [9].

Theorem 2.1. [9, Theorem 2.1] Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$ and $b \in \mathbb{R}$. If w is a Freud-type weight, then we assume $a_n = o(1)n^{2/3}$. Then there exist constants $C_1, C_2 > 0$ such that for integer $n \geq 1$,

$$C_1 \int_{-a_n}^{a_n} (1+x^2)^b dx \leq \sum_{j=1}^{n} \lambda_{j,n} w^{-2}(x_{j,n})(1+x_{j,n}^2)^b \leq C_2 \int_{-a_n}^{a_n} (1+x^2)^b dx.$$

Theorem 2.2. [9, Theorem 2.2] Let $w \in \mathcal{F}(C^2+)$. If w is a Freud-type weight, then we assume $a_n = o(1)n^{2/3}$. Let $\beta > 1/2$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If

$$(1+x^2)^\beta w^2(x)|f(x)| \leq C, \quad x \in \mathbb{R},$$

then there exists $n_0 > 0$ such that for $n \geq n_0$

$$\sum_{k=1}^{n} \lambda_{k,n} \left| f(x_{k,n}) \right| \leq C \left\| (1+x^2)^\beta w^2(x) f(x) \right\|_{L_\infty(\mathbb{R})}.$$

In particular, if

$$(1+x^2)^\beta w^2(x)|f(x)| \in C_0(\mathbb{R}),$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{k,n} f(x_{k,n}) = \int_{-\infty}^{\infty} w^2(x) f(x) dx.$$

We define

$$\Phi(x) := \frac{1}{(1+Q(x))^{2/3} T(x)},$$

and

$$\Phi_n(x) := \max\left\{ \delta_n, 1 - \frac{|x|}{a_n} \right\}.$$

Here we note that for $0 < d \leq |x|$,

$$\Phi(x) \sim \frac{Q^{\frac{1}{3}}(x)}{x Q'(x)}.$$

Moreover, we define

$$\Phi^{(\frac{1}{4}-\frac{1}{p})^+}(x) := \begin{cases} 1, & 0 < p < 4; \\ \Phi^{\frac{1}{4}-\frac{1}{p}}(x), & 4 \leq p \leq \infty. \end{cases}$$

Theorem 2.3. cf.[9, Theorem 2.3] Let $w \in \mathcal{F}_\lambda(C^3+)$, $0 < \lambda < 3/2$ (see Definition 2.5, below), and let $1 < p < \infty$. Let $n \geq 1$ be an integer and let $0 < \alpha < 1$. Then for every $P \in \mathcal{P}_n$,

$$\sum_{|x_{k,n}| \leq \alpha a_n} \lambda_{k,n} w^{-2}(x_{k,n}) \left| T^{-\frac{1}{2}}(x_{k,n}) \Phi^{\frac{1}{4}}(x_{k,n}) w(x_{k,n}) P(x_{k,n}) \right|^p$$
$$\leq \; C \int_{-\infty}^{\infty} \left| \Phi^{\frac{1}{4}}(x) w(x) P(x) \right|^p dx.$$

We will prove it in Section 4. We show the following theorems. Give a continuous function f, we define the nth Lagrange interpolation polynomial (1.1) with degree $n-1$, that is,

$$L_n(f; x_{j,n}) = f(x_{j,n}), \quad j = 1, 2, \ldots, n.$$

Theorem 2.4. Let $w \in \mathcal{F}(C^2+)$. If w is a Freud-type weight, then we assume $a_n = o(1)n^{2/3}$. Let $\beta > 1/2$, and let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$(1 + x^2)^{\frac{\beta}{2}} w(x) f(x) \leq C, x \in \mathbb{R}. \tag{2.1}$$

Then we have

$$\|w L_n(f)\|_{L_2(\mathbb{R})} \leq C \left\| (1 + x^2)^{\frac{\beta}{2}} w(x) f(x) \right\|_{L_\infty(\mathbb{R})}, n = 1, 2, 3, \ldots. \tag{2.2}$$

In particular, if

$$(1 + x^2)^{\frac{\beta}{2}} w(x) f(x) \in C_0(\mathbb{R}), \tag{2.3}$$

then

$$\lim_{n \to \infty} \|w(f - L_n(f))\|_{L_2(\mathbb{R})} = 0. \tag{2.4}$$

Now, we can obtain an analogy of Theorem 2.4 for L_p-norm.

Definition 2.5. Let $w = \exp(-Q) \in \mathcal{F}(C^2)$. Let us assume that $Q \in C^{(3)}(\mathbb{R} \setminus \{0\})$ and

$$\left| \frac{Q'''(x)}{Q''(x)} \right| \leq C \left| \frac{Q''(x)}{Q'(x)} \right|$$

hold for $|x| \geq K_2$, furthermore there exists $0 < \lambda < 3/2$ such that

$$\frac{|Q'(x)|}{Q(x)^\lambda} \leq C. \tag{2.5}$$

Then we write $w \in \mathcal{F}_\lambda(C^3+)$.

Remark 2.6. We can give the examples of weights $w \in \mathcal{F}_\lambda(C^3+)$. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let us define

$$\mu_+ := \limsup_{x \to \infty} \frac{Q''(x)}{Q'(x)} / \frac{Q'(x)}{Q(x)}, \quad \mu_- := \liminf_{x \to \infty} \frac{Q''(x)}{Q'(x)} / \frac{Q'(x)}{Q(x)}.$$

If $\mu_+ = \mu_-$, then we say that the weight w is regular. If $Q \in C^{(3)}(\mathbb{R}\setminus\{0\})$ is a regular weight, then we have $\mu_+ = \mu_- = 1$ (see [8, Corollary 5.5]). So we can show (2.5). All weights in Example 1.2 are regular if $Q \in C^{(3)}(\mathbb{R}\setminus\{0\})$.

Theorem 2.7. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$, and if w is a Freud-type weight, then we assume $a_n = o(1)n^{2/3}$. Let $1 < p \le 2$ and $1/p < \beta$. If

$$\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}wf\right\|_{L_\infty(\mathbb{R})} < \infty, \tag{2.6}$$

then we have for $n = 1, 2, 3, \ldots$

$$\|wL_n(f)\|_{L_p(\mathbb{R})} \le C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}wf\right\|_{L_\infty(\mathbb{R})}. \tag{2.7}$$

In particular, if

$$(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}(x)w(x)f(x) \in C_0(\mathbb{R}),$$

then we have

$$\lim_{n\to\infty}\|w(f-L_n(f))\|_{L_p(\mathbb{R})} = 0. \tag{2.8}$$

We define the degree of the best approximation by

$$E_n(w;f) = \inf_{P\in\mathcal{P}_n}\|w(f-P)\|_{L_\infty(\mathbb{R})}.$$

Theorem 2.8. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$, and if w is Freud-type weight, then we assume $a_n = o(1)n^{2/3}$. We suppose that for n large enough,

$$T(a_n) \le Cn^{1/2} \tag{2.9}$$

and

$$Q\left(\frac{a_n}{4}\right) \ge C(\log Q(a_n))^4. \tag{2.10}$$

Let $2 < p < \infty$ and $\beta > 1/p$. If

$$\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-1}(x)w(x)f(x)\right\|_{L_\infty(\mathbb{R})} < \infty, \tag{2.11}$$

then we have

$$\left\|\Phi^{\frac{3}{4}}wL_n(f)\right\|_{L_p(\mathbb{R})} \le C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}wf\right\|_{L_\infty(\mathbb{R})}. \tag{2.12}$$

In particular, if (2.11) is satisfied, then we have

$$(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}(x)w(x)f(x) \in C_0(\mathbb{R}). \tag{2.13}$$

Therefore, we have

$$\lim_{n\to\infty}\left\|\Phi^{\frac{3}{4}}w(f-L_n(f))\right\|_{L_p(\mathbb{R})} = 0. \tag{2.14}$$

Remark 2.9.

(1) If $w \in \mathcal{F}(C^2+)$ is a regular weight, then we have (2.9). In fact, for any $\eta > 0$ we have

$$T(a_n) \le C_\eta n^\eta,$$

where the constant $C_\eta > 0$ depends on only η (see [9, Corollary 5.5]).

(2) The exponentials (1), (2) in Example 1.2 satisfy the following. For each $\ell = 0, 1, 2, \ldots$ and a given $b > 0$, there exists $n_0 > 0$ such that for $n > n_0$

$$Q_{\ell,\alpha}\left(\frac{a_n}{4}\right) \ge C\left(\log Q_{\ell,\alpha}(a_n)\right)^b. \tag{2.15}$$

In fact, in the case of $\ell = 0$ (the Freud-type case) we can easily show that for some $C > 0$ and n large enough,

$$Q_{0,\alpha}\left(\frac{a_n}{4}\right) \ge C Q_{0,\alpha}(a_n) \ge \left(\log Q_{0,\alpha}(a_n)\right)^b.$$

Now, we suppose (2.15) for some $\ell \ge 0$. Then, for n large enough we see

$$\begin{aligned} Q_{\ell+1,\alpha}\left(\frac{a_n}{4}\right) &= \exp Q_{\ell,\alpha}\left(\frac{a_n}{4}\right) \ge \exp\left(\log Q_{\ell,\alpha}(a_n)\right)^b \\ &= \exp\left(\log\log Q_{\ell+1,\alpha}(a_n)\right)^b \ge \exp\left(\log\left(\log Q_{\ell+1,\alpha}(a_n)\right)^b\right) \\ &= \left(\log Q_{\ell+1,\alpha}(a_n)\right)^b. \end{aligned}$$

So, we have the result inductively.

3. Lemmas

To prove the theorems, we need some lemmas.

Lemma 3.1. [6, Theorem 9.3] Let $w \in \mathcal{F}(C^2)$, and let $0 < p < \infty$.

(a) Uniformly for $n \ge 1$ and $|x| \le a_n$, we have

$$\lambda_{n,p}(w; x) \sim \varphi_n(x) w^p(x).$$

(b) Moreover, uniformly for a constant $C > 0$ and $n \ge 1$,

$$\lambda_{n,p}(w; x) \ge C\varphi_n(x) w^p(x), \quad x \in \mathbb{R}.$$

Lemma 3.2. Let $w \in \mathcal{F}(C^2+)$. Then we have the following.

(a) [6, Lemma 13.9]

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n.$$

(b) [6, Lemma 3.6] Given fixed $0 < \alpha$, we have uniformly for $t > 0$,

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)}.$$

Let $0 < \alpha < 1$. Then there exists $C > 0$ such that uniformly for $t > 0$,

$$T(x)\left(1 - \frac{x}{a_t}\right) \geq C, \quad |x| \leq a_{\alpha t}.$$

(c) [6, Lemma 3.4] Uniformly for $t > 0$,

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \quad \text{and} \quad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}.$$

(d) [6, Lemma 3.7 (3.38)] There exists $0 < \eta \leq 2$ such that uniformly for $n > 0$,

$$T(a_n) \leq Cn^{2-\eta}.$$

Lemma 3.3. [7, Theorem 1.2.2] Let $w \in \mathcal{F}(C^2+)$. Let $n \geq 1$ be an integer. Then for all $P \in \mathcal{P}_{2n-1}$, we have

$$\int_{-\infty}^{\infty} P(x)w^2(x)dx = \sum_{k=1}^{n} \lambda_{k,n} P(x_{k,n}).$$

Lemma 3.4. Let $w \in \mathcal{F}(C^2)$.

(1) ([6, Theorem 1.17]) We have

$$\sup_{x\in\mathbb{R}} |p_n(x)|w(x)|x^2 - a_n^2|^{1/4} \sim 1.$$

(2) We have for $k = 1, 2, \ldots, n$,

$$\frac{\Phi^{\frac{1}{4}}(x_{k,n})}{\left|p_n'(x_{k,n})w(x_{k,n})\right|} \leq C\frac{a_n^{3/2}}{n}.$$

Proof. By [6, Theorem 1.19 (a)] we have

$$\left|p_n'(x_{k,n})w(x_{k,n})\right| \sim \varphi_n^{-1}(x_{k,n})a_n^{-1/2}\left(1 - \frac{|x_{k,n}|}{a_n}\right)^{-1/4}.$$

On the other hand by [5, Lemma 3.4] we have $\Phi(x) \leq C\Phi_n(x)$. Therefore, we have

$$\frac{\Phi^{\frac{1}{4}}(x_{k,n})}{\left|p_n'(x_{k,n})w(x_{k,n})\right|} \leq C\Phi_n^{\frac{1}{4}}(x_{k,n})\varphi_n(x_{k,n})a_n^{1/2}\left(1 - \frac{|x_{k,n}|}{a_n}\right)^{1/4}$$
$$\leq C\max\left\{\delta_n, 1 - \frac{|x_{k,n}|}{a_n}\right\}^{1/4}\left(1 - \frac{|x_{k,n}|}{a_{2n}}\right)\left(1 - \frac{|x_{k,n}|}{a_n}\right)^{-1/4}\frac{a_n^{3/2}}{n} \leq C\frac{a_n^{3/2}}{n}.$$

■

Lemma 3.5. [6, Corollary 13.4 (a), p. 329, (12.20)] For $k = 1, 2, \ldots, n$ uniformly, we have

$$x_{k,n} - x_{k+1,n} \sim \varphi_n(x_{k,n})$$

and

$$\varphi_n(x_{k,n}) \sim \varphi_n(x_{k+1,n}).$$

Lemma 3.6. [4, Lemma 3.4 (d)]

(1) Let $0 < \alpha < 1$, $|x_{k,n}| \leq a_{\alpha n}$. Then we have

$$w(x_{k,n}) \sim w(x_{k+1,n}).$$

(2) ([6, Theorem 13.3 (13.9)]) For uniformly $1 \leq j \leq n-1$ and $x \in [x_{j+1,n}, x_{j,n}]$,

$$l_{j+1,n}(x)w(x)w^{-1}(x_{j+1,n}) + l_{j,n}(x)w(x)w^{-1}(x_{j,n}) \sim 1.$$

Lemma 3.7. cf. [2, Theorem 2.7] Let $w \in \mathcal{F}(C^2+)$ and $0 < p \leq \infty$. Then uniformly $n \geq 2$,

$$\left\| \Phi^{(\frac{1}{4}-\frac{1}{p})^+} p_n w \right\|_{L_p(\mathbb{R})} \leq C a_n^{1/p-1/2} \begin{cases} 1, & 0 < p < 4 \text{ or } p = \infty; \\ \log(1+n), & 4 \leq p, \end{cases}$$

where $x^+ = 0$ if $x \leq 0$, and $x^+ = x$ if $x > 0$.

Proof. In [2, Theorem 2.7] we only exchange Φ_n with Φ (in fact, we have $\Phi(x) \leq C\Phi_n(x)$) by [5, Lemma 3.4]. ■

Let $fw \in L_p(\mathbb{R})$. The Fourier-type series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k(w^2, f) p_k(w^2, x), \; a_k(w^2, f) := \int_{-\infty}^{\infty} f(t) p_k(w^2, t) w^2(t) dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f, x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k(w^2, f) p_k(w^2, x).$$

The partial sum $s_n(f)$ admits the representation

$$s_n(f)(x) = \sum_{j=0}^{n-1} a_j p_j(x) = \int_{-\infty}^{\infty} f(t) K_n(x, t) w^2(t) dt,$$

where

$$K_n(x,t) := \sum_{j=0}^{n-1} p_j(x)p_j(t).$$

The Christoffel-Darboux formula

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}$$

is well-known (see [6, p.326 (12.7)]).

Lemma 3.8. [7, Lemma 9.2.6] Let $1 < p < \infty$ and $g \in L_p(\mathbb{R})$. For the Hilbert transform

$$H(g,x) := \lim_{\varepsilon\to 0+}\int_{|x-t|\geq\varepsilon}\frac{g(t)}{x-t}dt, \quad x \in \mathbb{R},$$

we have

$$\|H(g)\|_{L_p(\mathbb{R})} \leq C\,\|g\|_{L_p(\mathbb{R})},$$

where $C > 0$ is a constant depending upon p only.

Lemma 3.9. Let $w \in \mathcal{F}(C^2+)$. Let $1 < p < \infty$ and $n \geq 1$ be an integer. Let $h \in L_{p,w}(\mathbb{R})$ and $h(t) = 0$ for almost all t with $|t| \geq a_n/2$. Then

$$\left\|\Phi^{\frac{1}{4}} w s_n(h)\right\|_{L_p(\mathbb{R})} \leq C\,\|wh\|_{L_p(\mathbb{R})},\, n = 1, 2, 3, \ldots.$$

Proof. By the Christoffel-Darboux formula we have, noting $h(t) = 0$ for almost all t with $|t| \geq a_n/2$,

$$\begin{aligned} s_n(h,x) &= \frac{\gamma_{n-1}}{\gamma_n}\int_{|t|\leq a_n/2}\frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}h(t)w^2(t)dt \\ &= \frac{\gamma_{n-1}}{\gamma_n}\left[p_n(x)H(p_{n-1}hw^2,x) - p_{n-1}(x)H(p_nhw^2,x)\right], \end{aligned}$$

where $H(g,x)$ is the Hilbert transform defined in Lemma 3.8. By Lemma 3.7 with $p = \infty$ and $\gamma_{n-1}/\gamma_n \sim a_n$ (see Lemma 3.2 (a)) we have

$$\begin{aligned} &\Phi^{\frac{1}{4}}(x)w(x)|s_n(h,x)| \\ &\quad\leq Ca_n\Phi^{\frac{1}{4}}(x)w(x)[|p_n(x)||H(p_{n-1}hw^2,x)| + |p_{n-1}(x)||H(p_nhw^2,x)|] \\ &\quad\leq Ca_n^{1/2}[|H(p_{n-1}hw^2,x)| + |H(p_nhw^2,x)|]. \end{aligned}$$

Therefore, from Lemma 3.4 (1) and Lemma 3.8 we have

$$\begin{aligned} &\left\|\Phi^{\frac{1}{4}}(x)w(x)s_n(h,x)\right\|_{L_p(\mathbb{R})} \\ &\quad\leq Ca_n^{1/2}\left\{\left\|H(p_{n-1}hw^2)\right\|_{L_p(\mathbb{R})} + \left\|H(p_nhw^2)\right\|_{L_p(\mathbb{R})}\right\} \\ &\quad\leq Ca_n^{1/2}\left\{\left\|p_{n-1}hw^2\right\|_{L_p(|t|\leq a_n/2)} + \left\|p_nhw^2\right\|_{L_p(|t|\leq a_n/2)}\right\} \end{aligned}$$

$$\leq Ca_n^{1/2}\left\{\|p_{n-1}w\|_{L_\infty(|t|\leq a_n/2)} + \|p_nw\|_{L_\infty(|t|\leq a_n/2)}\right\}\|wh\|_{L_p(\mathbb{R})}$$
$$\leq C\|wh\|_{L_p(\mathbb{R})}.$$

■

Lemma 3.10. [9, Lemma 3.6] Let $w \in \mathcal{F}(C^2+)$, $P \in \mathcal{P}_n$ and let $1 \leq p, q \leq \infty$. Then for $q \leq p$,

$$\|wP\|_{L_q(\mathbb{R})} \leq Ca_n^{\frac{1}{q}-\frac{1}{p}}\|wP\|_{L_p(\mathbb{R})},$$

and for $p < q$,

$$\left\|\frac{1}{\sqrt{T}}wP\right\|_{L_q(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^{\frac{1}{p}-\frac{1}{q}}\|wP\|_{L_p(\mathbb{R})}.$$

If it is true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x|\to\infty} f(x)w(x) = 0,$$

there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty}\|(f - P_n)w\|_{L_\infty(\mathbb{R})} = 0,$$

then we say that w is a positive answer to Bernstein's problem.

Lemma 3.11.

(a) Let $w = \exp(-Q) \in \mathcal{F}(C^2)$ and $\gamma \geq 0$. There exists a weight w_γ such that w_γ is a positive answer to Bernstein's problem, and

$$(1 + x^2)^\gamma w(x) \sim w_\gamma(x).$$

(b) Let $w \in \mathcal{F}_\lambda(C^3+)$, $0 < \lambda < 3/2$, and let $\alpha, \beta \in \mathbb{R}$. Then we can construct a new weight $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}(C^2+)$ such that

$$T(x)^\mu(1 + x^2)^\nu(1 + Q(x))^\alpha(1 + |Q'(x)|)^\beta w(x) \sim w_{\mu,\nu,\alpha,\beta}(x)$$

holds on $\mathbb{R}$.

(c) We put the same assumption in (b) above. Let us define MRS-number for the weight $w_{\mu,\nu,\alpha,\beta} = \exp(-Q_{\mu,\nu,\alpha,\beta})$ by $a_n(Q_{\mu,\nu,\alpha,\beta})$, further we define the function in Definition 1.1 (d) for the weight $w_{\mu,\nu,\alpha,\beta} = \exp(-Q_{\mu,\nu,\alpha,\beta})$ by $T_{\mu,\nu,\alpha,\beta}$. Then there exist $c, C > 0$ such that

$$a_{cn}(Q_{\mu,\nu,\alpha,\beta}) \leq a_n(Q) := a_n \leq a_{Cn}(Q_{\mu,\nu,\alpha,\beta})$$

and

$$T_{\mu,\nu,\alpha,\beta}(x) \sim T(x), \quad x \in \mathbb{R}.$$

Proof. (a) follows from ([9, Lemma 3.9]). Both (b) and (c) follow from [8, Theorem 4.2 and (4.11)]. ■

Lemma 3.12. [4, Lemma 4.1] Let $w \in \mathcal{F}(C^2+)$. Let $0 < \alpha < 1/4$ and

$$\sum_n(x) := \sum_{|x_{k,n}| \geq a_{\alpha n}} |l_{k,n}(x)| w^{-1}(x_{k,n}).$$

Then we have for $|x| \leq a_{\alpha n/2}$ and $|x| \geq a_{2n}$,

$$\sum_n(x) w(x) \leq C.$$

Moreover, for $a_{\alpha n/2} \leq |x| \leq a_{2n}$,

$$\sum_n(x) w(x) \leq C \left(\log(1+n) + a_n^{1/2} |p_n(x) w(x)| T^{-1/4}(a_n)\right).$$

Lemma 3.13. [6, Theorem 1.9] Let $1 \leq p \leq \infty$. For $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\|wP\|_{L_p(\mathbb{R})} \leq C \|wP\|_{L_p(|x| \leq a_n)}.$$

Let $T_k(x)$ be the Chebyshev polynomial of degree k, and then we define

$$K_n^*(x,t) := \begin{cases} \dfrac{1}{2}, & \text{if } n = 1; \\ \dfrac{1}{2} + \displaystyle\sum_{k=1}^{n-1} T_k(x) T_k(t), & \text{if } n = 1, 2, \ldots. \end{cases}$$

Then we have the following.

Lemma 3.14. [7, Lemma 9.2.4 and (9.2.20)] Let $n \geq 1$ be an integer.

(a) We have

$$\frac{n}{20} \leq K_n^*(x,x) \leq n, \quad |x| \leq 1.$$

(b) Let $p > 1$, and $\{y_k\} \subset [-1, 1]$ be a system of points satisfying $|y_k - y_{k+1}| \sim 1/n$ for $k = 1, 2, \ldots, n-1$. Then for $u \in [-1, 1]$,

$$\sum_{k=1}^{n} \left|K_n^*(y_k, u)\right|^p \leq C n^p.$$

4. Proofs of Theorems

Through this section we will use the Mhaskar method of [7, §9.2].

Proof [Proof of Theorem 2.3]. First, from Lemma 3.11 (b), (c) there exists the weight $w^* \in \mathcal{F}(C^2+)$ such that

$$\Phi^{1/4}(x)w(x) \sim w^*(x),$$

further, for the MRS-number a_n^* and $T^*(x)$ with respect to w^*, we see

$$a_{cn}^* \le a_n \le a_{Cn}^*, \quad T^*(x) \sim T(x).$$

Let $R \in \mathcal{P}_m$, $m \ge 1$. From the second inequality in Lemma 3.10 with $q = \infty$ we see

$$\begin{aligned}\left|\frac{1}{\sqrt{T(x)}}\Phi^{1/4}(x)w(x)R(x)\right|^p &\le C\left|\frac{1}{\sqrt{T^*(x)}}w^*(x)R(x)\right|^p \\ &\le C\frac{m}{a_m}\int_{-\infty}^{\infty}\left|w^*(t)R(t)\right|^p dt.\end{aligned}$$

So, using Lemma 3.13 and Lemma 3.11 (c), we have

$$\left|\frac{1}{\sqrt{T(x)}}\Phi^{1/4}(x)w(x)R(x)\right|^p \le C\frac{m}{a_m}\int_{|t|\le a_m}\left|\Phi^{1/4}(t)w(t)R(t)\right|^p dt. \tag{4.1}$$

Now we take $l \ge 1$, $s(> 1/p)$ integers, and let $m := (l+s)n$. We set

$$S(x,t) := K_n^{*s}\left(\frac{x}{a_m}, \frac{t}{a_m}\right),$$

and we apply (4.1) with $R(t) = P(t)S(x,t) \in \mathcal{P}_m$. Since we see $a_m = a_{(l+s)n} \sim a_n$, we have

$$\left|\frac{1}{\sqrt{T(x)}}\Phi^{1/4}(x)w(x)P(x)S(x,x)\right|^p \le C\frac{m}{a_m}\int_{|t|\le a_{rn}}\left|\Phi^{1/4}(t)w(t)P(t)S(x,t)\right|^p dt,$$

where $r > 1$. Therefore, we have for every $x_{k,n}$,

$$\begin{aligned}&\left|\frac{1}{\sqrt{T(x_{k,n})}}\Phi^{1/4}(x_{k,n})w(x_{k,n})P(x_{k,n})S(x_{k,n},x_{k,n})\right|^p \\ &\le C\frac{m}{a_m}\int_{|t|\le a_{rn}}\left|\Phi^{1/4}(t)w(t)P(t)S(x_{k,n},t)\right|^p dt.\end{aligned}$$

Hence, by Lemma 3.14 (a) we have

$$\begin{aligned}&\left|\frac{1}{\sqrt{T(x_{k,n})}}\Phi^{1/4}(x_{k,n})w(x_{k,n})P(x_{k,n})\right|^p \\ &\le C\frac{m}{a_m}\frac{1}{n^{sp}}\int_{|t|\le a_{rn}}\left|\Phi^{1/4}(t)w(t)P(t)S(x_{k,n},t)\right|^p dt.\end{aligned}$$

Using Lemma 3.1 (a) with $p = 2$, we see that if $|x_{k,n}| \le \alpha a_n$, $0 < \alpha < 1$, then

$$\lambda_{k,n} w^{-2}(x_{k,n}) \sim \varphi_n(x_{k,n}) \sim \frac{a_n}{n} \frac{1 - \frac{|x_{k,n}|}{a_{2n}}}{\sqrt{1 - \frac{|x_{k,n}|}{a_n}}} \le C \frac{a_n}{n}.$$

Hence, we have

$$\begin{aligned}
&\sum_{|x_{k,n}| \le \alpha a_n} \lambda_{k,n} w^{-2}(x_{k,n}) \left| T^{-\frac{1}{2}}(x_{k,n}) \Phi^{1/4}(x_{k,n}) w(x_{k,n}) P(x_{k,n}) \right|^p \\
&\le C \frac{a_n}{n} \sum_{|x_{k,n}| \le \alpha a_n} \left| T^{-\frac{1}{2}}(x_{k,n}) \Phi^{1/4}(x_{k,n}) w(x_{k,n}) P(x_{k,n}) \right|^p \\
&\le C \frac{1}{n^{sp}} \sum_{|x_{k,n}| \le \alpha a_n} \int_{|t| \le a_{rn}} \left| \Phi^{1/4}(t) w(t) P(t) S(x_{k,n}, t) \right|^p dt \\
&\le C \int_{|t| \le a_{rn}} \left\{ \left| \Phi^{1/4}(t) w(t) P(t) \right|^p \frac{1}{n^{sp}} \sum_{|x_{k,n}| \le \alpha a_n} \left| K_n^* \left(\frac{x_{k,n}}{a_{rn}}, \frac{t}{a_{rn}} \right) \right|^{sp} \right\} dt \\
&\le C \int_{|t| \le a_{rn}} \left| \Phi^{1/4}(t) w(t) P(t) \right|^p dt \le C \int_{-\infty}^{\infty} \left| \Phi^{1/4}(t) w(t) P(t) \right|^p dt.
\end{aligned}$$

Here we used Lemma 3.14 (b). Consequently, we have the result. ■

Proof [Proof of Theorem 2.4]. By Lemma 3.3 and Theorem 2.2, we have

$$\begin{aligned}
\|w L_n(f)\|_{L_2(\mathbb{R})}^2 &= \sum_{k=1}^{n} \lambda_{k,n} \left\{ L_n(f; x_{k,n}) \right\}^2 = \sum_{k=1}^{n} \lambda_{k,n} f^2(x_{k,n}) \\
&\le C \left\| (1 + x^2)^{\beta} w^2(x) f^2(x) \right\|_{L_\infty(\mathbb{R})} = C \left\| (1 + x^2)^{\frac{\beta}{2}} w(x) f(x) \right\|_{L_\infty(\mathbb{R})}^2 .
\end{aligned}$$

So, we obtain (2.2). We need to show (2.4). By (2.3) and Lemma 3.11 (a) we can select $\{P_{n-1}\}_{n=1}^{\infty} \subset \mathcal{P}_{n-1}$ such that

$$\lim_{n \to \infty} \left\| (1 + x^2)^{\frac{\beta}{2}} w(x) (f(x) - P_{n-1}(x)) \right\|_{L_\infty(\mathbb{R})} = 0. \tag{4.2}$$

Hence, $f(x) - P_{n-1}(x)$ satisfies (2.1) uniformly with respect to n. Therefore, by (2.2) we see

$$\|w(L_n(f - P_{n-1}))\|_{L_2(\mathbb{R})} \le C \left\| (1 + x^2)^{\frac{\beta}{2}} w(x) (f(x) - P_{n-1}(x)) \right\|_{L_\infty(\mathbb{R})} . \tag{4.3}$$

Now, we see

$$\|w(f - L_n(f))\|_{L_2(\mathbb{R})} \le \|w(f - P_{n-1})\|_{L_2(\mathbb{R})} + \|w(L_n(f - P_{n-1}))\|_{L_2(\mathbb{R})} . \tag{4.4}$$

Here, we see

$$\begin{aligned} \|w\,(f - P_{n-1})\|^2_{L_2(\mathbb{R})} &= \int_{-\infty}^{\infty} |w(x)\,(f(x) - P_{n-1}(x))|^2\,dx \\ &\le C \left\| \left(1+x^2\right)^{\frac{\beta}{2}} w\,(f - P_{n-1}) \right\|^2_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} (1+x^2)^{-\beta} dx \\ &\le C \left\| \left(1+x^2\right)^{\frac{\beta}{2}} w\,(f - P_{n-1}) \right\|^2_{L_\infty(\mathbb{R})}. \end{aligned} \tag{4.5}$$

From (4.2), (4.3), (4.4) and (4.5) we have (2.4). ■

Proof [Proof of Theorem 2.7]. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$, and let $1 < p \le 2$. We may suppose that

$$\left\| (1+x^2)^{\frac{\beta}{2}} T^{\frac{1}{2}}(x) \Phi^{-\frac{1}{4}}(x) w(x) f(x) \right\|_{L_\infty(\mathbb{R})} = 1.$$

We write

$$f_1 := \begin{cases} f(x), & |x| \le a_n/4; \\ 0, & a_n/4 < |x| \end{cases} \quad \text{and} \quad f_2 := f - f_1.$$

Then we have

$$\begin{aligned} \|wL_n(f)\|_{L_p(\mathbb{R})} \le\ & \|wL_n(f_1)\|_{L_p(|x| \le a_n/2)} + \|wL_n(f_1)\|_{L_p(a_n/2 \le |x|)} \\ & + \|wL_n(f_2)\|_{L_p(\mathbb{R})} =: I_1 + I_2 + I_3. \end{aligned} \tag{4.6}$$

Let $p' := p/(p-1)$, and let n be fixed. If we consider the function g satisfying $\|wg\|_{L_{p'}([-a_n/2, a_n/2])} = 1$, then by the duality principle [7, A.1.1] we have

$$\|wL_n(f_1)\|_{L_p(|x| \le a_n/2)} = \sup_{\|wg\|_{L_{p'}([-a_n/2, a_n/2])} = 1} \int_{-a_n/2}^{a_n/2} w^2(x) L_n(f_1)(x) g(x) dx.$$

For a function h satisfying $\|wh\|_{L_{p'}(\mathbb{R})} = 1$ and $h(x) = 0$ for $|x| \ge a_n/2$, we will estimate

$$\int_{-\infty}^{\infty} w^2(x) L_n(f_1)(x) h(x) dx.$$

Since we know

$$\int_{-\infty}^{\infty} w^2(x) L_n(f_1)(x)\,(h(x) - s_n(h, x))\,dx = 0, \tag{4.7}$$

(in fact, let $L_n(f_1) =: \sum_{j=0}^{n-1} c_j p_j$, and let $s_n(h, x) =: \sum_{k=0}^{n-1} a_k p_k$, then we easily see that (4.7) holds), from the quadrature formula (Lemma 3.3) and (4.7) we have

$$\begin{aligned} \int_{-\infty}^{\infty} w^2(x) L_n(f_1)(x) h(x) dx &= \int_{-\infty}^{\infty} w^2(x) L_n(f_1)(x) s_n(h, x) dx \\ = \sum_{k=1}^{n} \lambda_{k,n} L_n(f_1, x_{k,n}) s_n(h, x_{k,n}) &= \sum_{k=1}^{n} \lambda_{k,n} f_1(x_{k,n}) s_n(h, x_{k,n}). \end{aligned}$$

Then, using our assumption (2.6) and Hölder's inequality, we have

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty} w^2(x)L_n(f_1)(x)h(x)dx\right| \\
&\le C \sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n}(1+x_{k,n}^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}}(x_{k,n})\Phi^{\frac{1}{4}}(x_{k,n})w^{-1}(x_{k,n})|s_n(h,x_{k,n})| \\
&= C \sum_{|x_{k,n}|\le a_n/4} (\lambda_{k,n}w^{-2}(x_{k,n}))^{1/p}(1+x_{k,n}^2)^{-\frac{\beta}{2}} \\
&\times \left(\lambda_{k,n}w^{-2}(x_{k,n})\right)^{1/p'} T^{-\frac{1}{2}}(x_{k,n})\Phi^{\frac{1}{4}}(x_{k,n})w(x_{k,n})|s_n(h,x_{k,n})| \\
&\le C \left(\sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n}w^{-2}(x_{k,n})(1+x_{k,n}^2)^{-\frac{p\beta}{2}}\right)^{1/p} \\
&\times \sum_{|x_{k,n}|\le a_n/4} \left(\lambda_{k,n}w^{-2}(x_{k,n})\left\{T^{-\frac{1}{2}}(x_{k,n})\Phi^{\frac{1}{4}}(x_{k,n})w(x_{k,n})|s_n(h,x_{k,n})|\right\}^{p'}\right)^{1/p'}.
\end{aligned}
$$

Here we note $p\beta > 1$. So, by Theorem 2.1,

$$
\sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n}w^{-2}(x_{k,n})(1+x_{k,n}^2)^{-\frac{p\beta}{2}} \le C\int_{-a_n}^{a_n}(1+t^2)^{-\frac{p\beta}{2}}dt < C. \tag{4.8}
$$

From Theorem 2.3 with p' and Lemma 3.9, we have

$$
\begin{aligned}
&\sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n}w^{-2}(x_{k,n})\left\{T^{-\frac{1}{2}}(x_{k,n})\Phi^{\frac{1}{4}}(x_{k,n})w(x_{k,n})|s_n(h,x_{k,n})|\right\}^{p'} \\
&\le C\left\|\Phi^{\frac{1}{4}}ws_n(h)\right\|_{L_{p'}(\mathbb{R})}^{p'} \le C\|wh\|_{L_{p'}(\mathbb{R})}^{p'} = C.
\end{aligned} \tag{4.9}
$$

Hence, from (4.8) and (4.9) we have

$$
I_1 \le C. \tag{4.10}
$$

To estimate I_2 we use the Lagrange interpolation formula in the form

$$
L_n(f_1,x) = p_n(x)\frac{\gamma_{n-1}}{\gamma_n}\sum_{k=1}^{n}\lambda_{k,n}p_{n-1}(x_{k,n})\frac{f_1(x_{k,n})}{x-x_{k,n}}
$$

(see [7, p.243]). Since $f_1(t) = 0$ if $|t| > a_n/4$, we have for $|x| \ge a_n/2$, using Lemma 3.7

with $p = \infty$,

$$
\begin{aligned}
&|L_n(f_1, x)| \le Ca_n|p_n(x)| \sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n} \left|p_{n-1}(x_{k,n})\right| \left|\frac{f_1(x_{k,n})}{x - x_{k,n}}\right| \\
&\le C|p_n(x)| \sum_{|x_{k,n}|\le a_n/4} \lambda_{k,n}(1 + x_{k,n}^2)^{-\frac{\beta}{2}} w^{-2}(x_{k,n})T^{-\frac{1}{2}}(x_{k,n}) \\
&\times \Phi^{\frac{1}{4}}(x_{k,n})w(x_{k,n})|p_{n-1}(x_{k,n})| \quad \text{(by } |x - x_{kn}| \ge a_n/4 \text{ and (2.6))} \\
&\le Ca_n^{-1/2}|p_n(x)| \sum_{k=1}^{n} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}}(x_{k,n}).
\end{aligned}
$$

Let $1/p + 1/p' = 1$. Using Theorem 2.1 and $\beta > 1/p$, we have

$$
\begin{aligned}
&\sum_{k=1}^{n} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}}(x_{k,n}) \le \sum_{k=1}^{n} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\frac{\beta}{2}} \\
&\le C \left(\sum_{k=1}^{n} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\frac{p\beta}{2}}\right)^{1/p} \left(\sum_{k=1}^{n} \lambda_{k,n} w^{-2}(x_{k,n})\right)^{1/p'} \\
&\le C \left(\int_{-a_n}^{a_n} (1 + t^2)^{-p\beta/2} dt\right)^{1/p} \left(\int_{-a_n}^{a_n} 1dt\right)^{1/p'} \le Ca_n^{1/p'}.
\end{aligned}
$$

Hence, we have

$$
|L_n(f_1, x)| \le Ca_n^{-1/2+1/p'}|p_n(x)|. \tag{4.11}
$$

Therefore, by Lemma 3.7 again, we have

$$
I_2 = \|wL_n(f_1)\|_{L_p(\frac{a_n}{2}\le|x|)} \le Ca_n^{-\frac{1}{2}+\frac{1}{p'}} \|wp_n\|_{L_p(\mathbb{R})} \le Ca_n^{\frac{1}{p}+\frac{1}{p'}-1} = C. \tag{4.12}
$$

We estimate I_3. By Lemma 3.1, Lemma 3.3 and Theorem 2.1,

$$
\begin{aligned}
&\|wL_n(f_2)\|_{L_2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} w^2(t)L_n(f_2)^2(t)dt = \sum_{|x_{k,n}|\ge a_n/4} \lambda_{k,n}|f(x_{k,n})|^2 \\
&\le \sum_{|x_{k,n}|\ge a_n/4} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\beta} T^{-1}(x_{k,n})\Phi(x_{k,n})^{1/2} \\
&\le C \sum_{|x_{k,n}|\ge a_n/4} \lambda_{k,n} w^{-2}(x_{k,n})(1 + x_{k,n}^2)^{-\beta} \\
&\le Ca_n^{-2\beta} \sum_{|x_{k,n}|\ge a_n/4} \varphi_n(x_{k,n}) \le Ca_n^{-2\beta} \int_{-a_n}^{a_n} dt \le Ca_n^{1-2\beta}.
\end{aligned}
$$

Applying the first inequality in Lemma 3.10 with $1 < p \le 2$,

$$
I_3 = \|wL_n(f_2)\|_{L_p(\mathbb{R})} \le Ca_n^{\frac{1}{p}-\frac{1}{2}} \|wL_n(f_2)\|_{L_2(\mathbb{R})} \le Ca_n^{\frac{1}{p}-\frac{1}{2}} a_n^{\frac{1}{2}-\beta} = Ca_n^{\frac{1}{p}-\beta} \le C. \tag{4.13}
$$

Therefore, by (4.6), (4.10) (4.12) and (4.13) we have (2.7). We need to show (2.8). From Lemma 3.11 (b) we see

$$(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}(x)w(x) \sim w^*(x) \in \mathcal{F}(C^2+).$$

Hence, there exist $P_{n-1} \in \mathcal{P}_{n-1}$, $n = 1, 2, 3, \ldots$ such that

$$\left|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}(x)w(x)\{f(x) - P_{n-1}(x)\}\right| \leq 2C, \quad x \in \mathbb{R},$$

that is, (2.6) holds uniformly for n. Hence, from (2.7) we have

$$\left\|\Phi^{\frac{3}{4}}w(L_n(f - P_{n-1}))\right\|_{L_p(\mathbb{R})} \leq C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}w(f - P_{n-1})\right\|_{L_\infty(\mathbb{R})}. \quad (4.14)$$

Now, we have

$$\|w(f - L_n(f))\|_{L_p(\mathbb{R})} \leq \|w(f - P_{n-1})\|_{L_p(\mathbb{R})} + \|w(L_n(f - P_{n-1}))\|_{L_p(\mathbb{R})}. \quad (4.15)$$

Here, we have

$$\begin{aligned} &\|w(f - P_{n-1})\|_{L_p(\mathbb{R})} \\ \leq\ & C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}w(f - P_{n-1})\right\|_{L_\infty(\mathbb{R})}\left\|(1+x^2)^{-\frac{\beta}{2}}T^{-\frac{1}{2}}\Phi^{\frac{1}{4}}\right\|_{L_p(\mathbb{R})} \\ \leq\ & C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}w(f - P_{n-1})\right\|_{L_\infty(\mathbb{R})}. \end{aligned} \quad (4.16)$$

Here, we used $\left\|(1+x^2)^{-\frac{\beta}{2}}T^{-\frac{1}{2}}\Phi^{\frac{1}{4}}\right\|_{L_p(\mathbb{R})} < \infty$. Consequently, from (4.14), (4.15) and (4.16) we have (2.8). ■

Proof[ProofofTheorem 2.8]. Let $w \in \mathcal{F}_\lambda(C^3+)$ with $0 < \lambda < 3/2$, and let $2 < p < \infty$. We may suppose that

$$\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-1}(x)w(x)f(x)\right\|_{L_\infty(\mathbb{R})} = 1.$$

Then for (2.12) we will show

$$\left\|\Phi^{\frac{3}{4}}wL_n(f)\right\|_{L_p(\mathbb{R})} \leq C.$$

We write for some $0 < \alpha < 1/4$

$$f_1 := \begin{cases} f(x), & |x| \leq a_n/4; \\ 0, & |x| > a_n/4, \end{cases} \quad f_2 := \begin{cases} 0, & |x| \leq a_n/4; \\ f(x), & a_n/4 < |x| \leq a_{\alpha n}, \\ 0, & |x| > a_{\alpha n}, \end{cases}$$

and

$$f_3 := f - f_1 - f_2.$$

Then we have

$$\begin{aligned}
&\left\|\Phi^{\frac{3}{4}} w L_n(f)\right\|_{L_p(\mathbb{R})} \le \left\|\Phi^{\frac{3}{4}} w L_n(f_1)\right\|_{L_p(|x|\le a_n/2)} + \left\|\Phi^{\frac{3}{4}} w L_n(f_1)\right\|_{L_p(|x|\ge a_n/2)} \\
&+ \left\|\Phi^{\frac{3}{4}} w L_n(f_2)\right\|_{L_p(|x|\le a_{2\alpha n})} + \left\|\Phi^{\frac{3}{4}} w L_n(f_2)\right\|_{L_p(|x|\ge a_{2\alpha n})} + \left\|\Phi^{\frac{3}{4}} w L_n(f_3)\right\|_{L_p(\mathbb{R})} \\
&=: J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

By (4.10) we see

$$J_1 \le I_1 \le C. \tag{4.17}$$

To estimate J_2 we can use (4.11) because of $\Phi(x) \le \Phi^{\frac{1}{4}}(x)$, that is, by Lemma 3.7

$$\begin{aligned}
J_2 &= \left\|\Phi^{\frac{3}{4}} w L_n(f_1)\right\|_{L_p(|x|\ge a_n/2)} \le C a_n^{-1/2+1/p'} \Phi^{\frac{1}{2}}(a_n/2) \left\|\Phi^{\frac{1}{4}} w p_n\right\|_{L_p(\mathbb{R})} \\
&\le C\Phi^{\frac{1}{2}}(a_n/2) \begin{cases} 1, & 0 < p < 4 \text{ or } p = \infty; \\ \log(1+n), & p \ge 4, \end{cases}
\end{aligned} \tag{4.18}$$

By our assumption (2.10) we see for n large enough

$$\Phi^{\frac{1}{2}}(a_n/2) = \frac{1}{Q^{1/3}(a_n/2)T^{1/2}(a_n/2)} \le \frac{C}{\log Q(a_n)} \le \frac{C}{\log(n+1)}.$$

Therefore, (4.18) means $J_2 \le C$. To estimate J_3 and J_4, we estimate as follows: We note

$$(1+x^2)^{-\beta} \sim a_n^{-\beta}, \quad \frac{a_n}{4} \le |x| \le a_{\alpha n}.$$

By the assumption (2.11) we see

$$\begin{aligned}
&\left|\Phi^{\frac{3}{4}}(x) w(x) L_n(f_2; x)\right| \\
\le\ & a_n^{-\beta} \Phi^{\frac{3}{4}}(x) w(x) \sum_{a_n/4 \le |x_{k,n}| \le a_{\alpha n}} \left| T^{-\frac{1}{2}}(x_{k,n}) \Phi(x_{k,n}) w^{-1}(x_{k,n}) l_{k,n}(x)\right| \\
=:\ & a_n^{-\beta} \Phi^{\frac{3}{4}}(x) w(x) \sum_{a_n/4 \le |x_{k,n}| \le a_{\alpha n}} J(k, x).
\end{aligned} \tag{4.19}$$

To estimate J_3, we assume $|x| \le a_{2\alpha n}$, and let $x \in [x_{m+1,n}, x_{m,n}]$. Then, by Lemma 3.6 (2) we see

$$l_{m+1,n}(x) w(x) w^{-1}(x_{m+1,n}) + l_{m,n}(x) w(x) w^{-1}(x_{m,n}) \sim 1.$$

Therefore, we have

$$\begin{aligned}
&\Phi^{\frac{3}{4}}(x) w(x) \left(J(m+1, x) + J(m, x)\right) \\
\le\ & C\Phi^{\frac{3}{4}}(x) T^{-\frac{1}{2}}(a_n/4) \Phi(a_n/4) \left(l_{m+1,n}(x) w(x) w^{-1}(x_{m+1,n}) + l_{m,n}(x) w(x) w^{-1}(x_{m,n})\right) \\
\le\ & C\Phi^{\frac{3}{4}}(x) T^{-\frac{1}{2}}(a_n/4) \Phi(a_n/4).
\end{aligned} \tag{4.20}$$

Suppose $k \neq m, m+1$. Then for $|x| \leq a_{\alpha n}$ and $a_n/4 \leq |x_{k,n}| \leq a_{\alpha n}$

$$|x - x_{k,n}| \sim \sum_{\substack{k \leq j \leq m-1 \\ \text{or } m+2 \leq j \leq k}} \varphi_n(x_{j,n}) \geq C|k-m| \frac{a_n}{n} \min\left\{\sqrt{1 - \frac{|x|}{a_n}}, \sqrt{1 - \frac{|x_{k,n}|}{a_n}}\right\}.$$

Hence, with Lemma 3.4(2) we have for $a_n/4 \leq |x_{k,n}| \leq a_{\alpha n}$ and $k \neq m, m+1$

$$\begin{aligned}
&\left|\Phi^{\frac{3}{4}}(x)w(x)J(k,x)\right| = \Phi^{\frac{3}{4}}(x)|p_n(x)w(x)| \left|\frac{T^{-\frac{1}{2}}(x_{k,n})\Phi(x_{k,n})}{(x - x_{k,n})p_n'(x_{k,n})w(x_{k,n})}\right| \\
\leq\ & C\Phi^{\frac{3}{4}}(x)|p_n(x)w(x)| \left|\frac{1}{(x - x_{k,n})\frac{n}{a_n}T^{-\frac{1}{2}}(x_{k,n})\frac{a_n^{3/2}}{n}}\right| \\
\leq\ & C\Phi^{\frac{1}{4}}(x)a_n^{1/2}|p_n(x)w(x)|\Phi^{\frac{3}{4}}(x_{k,n})\frac{1}{|k-m|} \\
& \times \Phi^{\frac{1}{2}}(x)T^{-\frac{1}{2}}(x_{k,n}) \max\left\{\left(1 - \frac{|x|}{a_n}\right)^{-\frac{1}{2}}, \left(1 - \frac{|x_{k,n}|}{a_n}\right)^{-\frac{1}{2}}\right\} \\
\leq\ & C\Phi^{\frac{1}{4}}(x)a_n^{1/2}|p_n(x)w(x)|\Phi^{\frac{3}{4}}(x_{k,n})\frac{1}{|k-m|}.
\end{aligned}$$

Here, we used the following fact: Since $\Phi(x) \leq CT^{-1}(x)$, using Lemma 3.2 (b), we see

$$\Phi^{\frac{1}{2}}(x)T^{-\frac{1}{2}}(x_{k,n}) \max\left\{\left(1 - \frac{|x|}{a_n}\right)^{-\frac{1}{2}}, \left(1 - \frac{|x_{k,n}|}{a_n}\right)^{-\frac{1}{2}}\right\} \leq C.$$

Therefore, we have with (4.20)

$$\begin{aligned}
& \Phi^{\frac{3}{4}}(x)w(x) \sum_{a_n/4 \leq |x_{k,n}| \leq a_{\alpha n}} J(k,x) \\
\leq\ & C\Phi^{\frac{1}{4}}(x)a_n^{1/2}|p_n(x)w(x)| \sum_{\substack{a_n/4 \leq |x_{k,n}| \leq a_{\alpha n} \\ k \neq m, m+1}} \frac{\Phi^{\frac{3}{4}}(x_{k,n})}{|k-m|} + C\Phi^{\frac{3}{4}}(x)T^{-\frac{1}{2}}(a_n/4)\Phi(a_n/4) \\
\leq\ & C\Phi^{\frac{1}{4}}(x)a_n^{1/2}|p_n(x)w(x)|\Phi^{\frac{3}{4}}(a_n/4)\log(1+n) + C\Phi^{\frac{3}{4}}(x)T^{-\frac{1}{2}}(a_n/4)\Phi(a_n/4).
\end{aligned}$$

If we recall (4.19), this implies that

$$\begin{aligned}
J_3 &= \left\|\Phi^{\frac{3}{4}} w L_n(f_2)\right\|_{L_p(|x|\le a_{\alpha n})} \\
&\le C a_n^{1/2-\beta}\left\|\Phi^{\frac{1}{4}} p_n w\right\|_{L_p(|x|\le a_{\alpha n})} \Phi^{\frac{3}{4}}(a_n/4)\log(1+n) \\
&\quad + C a_n^{-\beta}\left\|\Phi^{\frac{3}{4}}\right\|_{L_p(|x|\le a_{\alpha n})} T^{-\frac{1}{2}}(a_n/4)\Phi(a_n/4) \\
&\le C\Phi^{\frac{3}{4}}(a_n/4)\log(1+n) a_n^{1/p-\beta}\begin{cases} 1, & 0<p<4 \text{ or } p=\infty; \\ \log(1+n), & 4\le p, \end{cases} \qquad (4.21)\\
&\quad + C a_n^{-\beta}\left\|(1+|x|)^{-1/2}\right\|_{L_p(|x|\le a_{\alpha n})} \quad (\text{by } \Phi^{3/4}(x)\le (1+|x|)^{-1/2}) \\
&\le C a_n^{1/p-\beta}\frac{(\log(1+n))^2}{Q^{\frac{1}{2}}(a_n/4)} + C a_n^{-\beta+1/p-1/2} \le C
\end{aligned}$$

(note (2.10) and Lemma 3.2 (c)).
Next, we consider the estimate of J_4. Let $a_n/4 \le |x_{k,n}| \le a_{\alpha n}$ and $|x| \ge a_{2\alpha n}$. First we have

$$|x - x_{k,n}| \ge C(a_{2\alpha n} - a_{\alpha n}) \sim \frac{a_n}{T(a_n)}. \qquad (4.22)$$

We see

$$\left|\Phi^{\frac{3}{4}}(x) w(x) J(k,x)\right| = \Phi^{\frac{3}{4}}(x)|p_n(x)w(x)|\left|\frac{T^{-\frac{1}{2}}(x_{k,n})\Phi(x_{k,n})}{(x-x_{k,n})p_n'(x_{k,n})w(x_{k,n})}\right|.$$

By Lemma 3.4 (2), (2.9), (2.10), (4.22) and Lemma 3.2(c), we have

$$\begin{aligned}
&\Phi^{\frac{1}{2}}(x)\left|\frac{T^{-\frac{1}{2}}(x_{k,n})\Phi(x_{k,n})}{(x-x_{k,n})p_n'(x_{k,n})w(x_{k,n})}\right| \le C\frac{a_n^{3/2}}{n}\frac{\Phi^{\frac{1}{2}}(a_n)T(a_n)}{a_n} T^{-\frac{1}{2}}(x_{k,n})\Phi^{\frac{3}{4}}(x_{k,n}) \\
&\le C a_n^{1/2}\frac{T(a_n)}{n}\frac{1}{\left(Q^{2/3}(a_n)T(a_n)\right)^{1/2}}\frac{T^{-\frac{1}{2}}(a_n/4)}{\left(Q^{2/3}(a_n/4)T(a_n/4)\right)^{3/4}} \\
&\le C\frac{a_n^{1/2}}{n}\frac{T^{2/3}(a_n)}{n^{1/3}}\frac{1}{\log^2(1+n)} \le C\frac{a_n^{1/2}}{n}\frac{1}{\log^2(1+n)}.
\end{aligned}$$

From this fact, we have

$$\begin{aligned}
&\Phi^{\frac{3}{4}}(x)w(x)\sum_{a_n/4\le|x_{k,n}|\le a_{\alpha n}} J(k,x) \\
&\le C\Phi^{\frac{1}{4}}(x) a_n^{1/2}|p_n(x)w(x)|\sum_{a_n/4\le|x_{k,n}|\le a_{\alpha n}}\frac{1}{n}\frac{1}{\log^2(1+n)} \\
&\le C\Phi^{\frac{1}{4}}(x) a_n^{1/2}|p_n(x)w(x)|\frac{1}{\log^2(1+n)}.
\end{aligned}$$

Therefore, from Lemma 3.7, and noting (4.19), we have

$$\begin{aligned} J_4 &= \left\|\Phi^{\frac{3}{4}} w L_n(f_2)\right\|_{L_p(|x|\geq a_{\alpha n})} \leq C a_n^{1/2-\beta} \left\|\Phi^{\frac{1}{4}} p_n w\right\|_{L_p(\mathbb{R})} \frac{1}{\log^2(1+n)} \\ &\leq C a_n^{1/p-\beta} \log(1+n) \frac{1}{\log^2(1+n)} \leq C. \end{aligned} \tag{4.23}$$

Finally, we estimate J_5. Using Lemma 3.4 (1), (2) we have

$$\begin{aligned} &\left|\Phi^{\frac{3}{4}}(x) w(x) L_n(f_3; x)\right| = \left|\Phi^{\frac{3}{4}}(x) w(x) \sum_{|x_{k,n}|\geq a_{\alpha n}} f(x_{k,n}) l_{k,n}(x)\right| \\ \leq\ & C\Phi^{\frac{3}{4}}(x) w(x) \sum_{|x_{k,n}|\geq a_{\alpha n}} (1+x_{k,n}^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}}(x_{k,n}) \Phi(x_{k,n}) w^{-1}(x_{k,n}) |l_{k,n}(x)| \\ \leq\ & C a_n^{-\beta} T^{-\frac{1}{2}}(a_{\alpha n}) \Phi^{\frac{3}{4}}(a_{\alpha n}) \Phi^{\frac{1}{2}}(x) \sum_{|x_{k,n}|\geq a_{\alpha n}} \left|\frac{\Phi^{\frac{1}{4}}(x) p_n(x) w(x) \Phi^{\frac{1}{4}}(x_{k,n})}{(x-x_{k,n}) p_n'(x_{k,n}) w(x_{k,n})}\right| \\ \leq\ & C a_n^{-\beta} T^{-\frac{1}{2}}(a_{\alpha n}) \Phi^{\frac{3}{4}}(a_{\alpha n}) \Phi^{\frac{1}{2}}(x) \sum_{|x_{k,n}|\geq a_{\alpha n}} \frac{a_n}{n} \left|\frac{1}{(x-x_{k,n})}\right|. \end{aligned} \tag{4.24}$$

Let $|x| \leq a_{\alpha n/2}$. From $|x - x_{k,n}| \geq a_n/T(a_n)$ and Lemma 3.2 (c) we have

$$\begin{aligned} \left|\Phi^{\frac{3}{4}}(x) w(x) L_n(f_3; x)\right| &\leq C a_n^{-\beta} T^{\frac{1}{2}}(a_n) \Phi^{\frac{3}{4}}(a_{\alpha n}) \Phi^{\frac{1}{2}}(x) \\ &\leq C a_n^{-\beta} T^{\frac{1}{2}}(a_n) Q^{-\frac{1}{2}}(a_{\alpha n}) \Phi^{\frac{1}{2}}(x) \leq C a_n^{-\beta} n^{-\frac{1}{2}} \Phi^{\frac{1}{2}}(x). \end{aligned}$$

Hence, we have for $|x| \leq a_{\alpha n/2}$,

$$\begin{aligned} &\left\|\Phi^{\frac{3}{4}} w L_n(f_3)\right\|_{L_p(|x|\leq a_{\alpha n/2})} \\ \leq\ & C a_n^{-\beta} n^{-\frac{1}{2}} \left\|\Phi^{\frac{1}{2}}\right\|_{L_p(|x|\leq a_{\alpha n/2})} \leq C a_n^{-\beta} n^{-\frac{1}{2}} \left\|\frac{1}{1+Q^{1/3}(x)}\right\|_{L_p(|x|\leq a_{\alpha n/2})} \\ \leq\ & C a_n^{-\beta} n^{-\frac{1}{2}} \left\|\frac{1}{(1+|x|)^{1/3}}\right\|_{L_p(|x|\leq a_{\alpha n/2})} \leq C. \end{aligned} \tag{4.25}$$

For $|x| \geq a_{2n}$, noting Lemma 3.2 (b), (c), we see

$$\begin{aligned} &\left|\Phi^{\frac{3}{4}}(x) w(x) L_n(f_3; x)\right| \leq C a_n^{1-\beta} T^{-\frac{1}{2}}(a_n) \Phi^{\frac{3}{4}}(a_n) Q^{-1/3}(x) |x|^{-1} \frac{1}{1-a_n/a_{2n}} \\ \leq\ & C a_n^{1-\beta} T^{-\frac{1}{2}}(a_n) T^{-\frac{3}{4}}(a_n) Q^{-1/2}(a_n) x^{-2} T(a_n) \\ \leq\ & C a_n^{1-\beta} T^{-\frac{1}{4}}(a_n) \frac{T^{\frac{1}{4}}(a_n)}{n^{\frac{1}{2}}} x^{-2} \leq C a_n^{1-\beta} \frac{1}{n^{\frac{1}{2}} a_n^{\frac{1}{2}}} x^{-\frac{3}{2}} (\because |x| \geq a_n) \leq C a_n^{-\beta} x^{-\frac{3}{2}}. \end{aligned}$$

Therefore, we have

$$\left\|\Phi^{\frac{3}{4}} w L_n(f_3)\right\|_{L_p(|x|\geq a_{2n})} \leq C a_n^{1/p-\beta} \leq C. \tag{4.26}$$

Hence we may consider the integral on $[a_{\alpha n/2}, a_{2n}]$. Using Lemma 3.12 and (4.24), we have for $a_{\alpha n/2} \le |x| \le a_{2n}$,

$$\begin{aligned}
&\left|\Phi^{\frac{3}{4}}(x)w(x)L_n(f_3;x)\right| \\
\le\ & Ca_n^{-\beta}T^{-\frac{1}{2}}(a_{\alpha n})\Phi(a_{\alpha n})\Phi^{\frac{3}{4}}(x)w(x)\sum_{|x_{k,n}|\geqslant a_{\alpha n/2}} w^{-1}(x_{k,n})|l_{k,n}(x)| \\
\le\ & Ca_n^{-\beta}T^{-\frac{1}{2}}(a_{\alpha n})\Phi(a_{\alpha n})\Phi^{\frac{3}{4}}(x)\left(\log(1+n)+a_n^{1/2}|p_n(x)w(x)|\,T^{-1/4}(a_n)\right) \\
\le\ & Ca_n^{-\beta}n^{-\frac{2}{3}}\Phi^{\frac{3}{4}}(x)\left(\log(1+n)+a_n^{1/2}|p_n(x)w(x)|\right).
\end{aligned}$$

Hence, we have by Lemma 3.7 and our assumption,

$$\begin{aligned}
&\left\|\Phi^{\frac{3}{4}}wL_n(f_3)\right\|_{L_p(a_{\alpha n/2}\le|x|\le a_{2n})} \\
\le\ & Ca_n^{-\beta}n^{-\frac{2}{3}}\log(1+n)\left\|\Phi^{\frac{3}{4}}\right\|_{L_p(a_{\alpha n/2}\le|x|\le a_{2n})} + Ca_n^{1/2-\beta}n^{-\frac{2}{3}}\left\|\Phi^{\frac{3}{4}}p_nw\right\|_{L_p(a_{\alpha n/2}\le|x|\le a_{2n})} \\
\le\ & Ca_n^{-\beta}n^{-\frac{2}{3}}\log(1+n)\left\|(1+|x|)^{-1/2}\right\|_{L_p(a_{\alpha n/2}\le|x|\le a_{2n})} + Ca_n^{\frac{1}{p}-\beta}n^{-\frac{2}{3}}\log(1+n) \\
\le\ & C\left\{a_n^{-\beta+1/p-1/2}n^{-2/3}\log(1+n)+a_n^{1/p-\beta}n^{-2/3}\log(1+n)\right\}\le C.
\end{aligned} \tag{4.27}$$

From (4.25), (4.26) and (4.27), we have

$$J_5 = \left\|\Phi^{\frac{3}{4}}wL_n(f_3)\right\|_{L_p(\mathbb{R})} \le C.$$

Therefore we have from (4.17), (4.18), (4.21), (4.23) and (4) we have (2.12).

We need to show (2.14). From Lemma 3.11 (b) we see

$$(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}(x)\Phi^{-\frac{1}{4}}(x)w(x) \sim w^{**} \in \mathcal{F}(C^2+).$$

Hence, noting (2.13), there exist $P_{n-1} \in \mathcal{P}_{n-1}$, $n = 1, 2, 3, \ldots$ such that

$$\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}w(f-P_{n-1})\right\|_{L_\infty(\mathbb{R})} \le 2E_{n-1}(w^{**};f) \to 0 \text{ as } n\to\infty. \tag{4.28}$$

So we see that $f(x) - P_{n-1}(x)$ satisfies (2.11) uniformly for n. Hence, by (2.12) we have

$$\left\|\Phi^{\frac{3}{4}}w(L_n(f-P_{n-1}))\right\|_{L_p(\mathbb{R})} \le C\left\|(1+x^2)^{\frac{\beta}{2}}T^{\frac{1}{2}}\Phi^{-\frac{1}{4}}w(f-P_{n-1})\right\|_{L_\infty(\mathbb{R})}. \tag{4.29}$$

We see

$$\begin{aligned}
\left\|\Phi^{\frac{3}{4}}w(f-L_n(f))\right\|_{L_p(\mathbb{R})} &\le \left\|\Phi^{\frac{3}{4}}w(f-P_{n-1})\right\|_{L_p(\mathbb{R})} + \left\|\Phi^{\frac{3}{4}}w(L_n(f-P_{n-1}))\right\|_{L_p(\mathbb{R})} \\
&=: I_1 + I_2.
\end{aligned} \tag{4.30}$$

We estimate I_1 and I_2. For I_2 we have (4.29);

$$I_2 = \left\| \Phi^{\frac{3}{4}} w(L_n(f - P_{n-1})) \right\|_{L_p(\mathbb{R})} \le C \left\| (1+x^2)^{\frac{\beta}{2}} T^{\frac{1}{2}} \Phi^{-\frac{1}{4}} w(f - P_{n-1}) \right\|_{L_\infty(\mathbb{R})}. \tag{4.31}$$

For I_1 we see

$$\begin{aligned} I_1 &= \left\| \Phi^{\frac{3}{4}} w(f - P_{n-1}) \right\|_{L_p(\mathbb{R})} \\ &\le C \left\| (1+x^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}} \Phi \right\|_{L_p(\mathbb{R})} \left\| (1+x^2)^{\frac{\beta}{2}} T^{\frac{1}{2}} \Phi^{-\frac{1}{4}} w(f - P_{n-1}) \right\|_{L_\infty(\mathbb{R})}. \end{aligned}$$

Since $\left\| (1+x^2)^{-\frac{\beta}{2}} T^{-\frac{1}{2}} \Phi \right\|_{L_p(\mathbb{R})} < \infty$ we have

$$I_1 \le C \left\| (1+x^2)^{\frac{\beta}{2}} T^{\frac{1}{2}} \Phi^{-\frac{1}{4}} w(f - P_{n-1}) \right\|_{L_\infty(\mathbb{R})}. \tag{4.32}$$

Consequently, from (4.28), (4.30), (4.31) and (4.32) we have (2.14), in fact, it follows from

$$\left\| (1+x^2)^{\frac{\beta}{2}} T^{\frac{1}{2}} \Phi^{-\frac{1}{4}} w(f - P_{n-1}) \right\|_{L_\infty(\mathbb{R})} \le C E_{n-1}(w^{**}; f) \to 0 \text{ as } n \to \infty.$$

■

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