Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 761-781

http://www.ripublication.com

Analytical Approach to Exact Solutions for Stochastic Fractional Hirota-Satsuma Coupled KdV Equations

Hossam A. Ghany[1], H. E. Hussain[1] and M. Al Qurashi[1]
[1] *Department of Mathematics, Faculty of Science, Taif University, Taif, Saudi Arabia.*

Abstract

In this paper we will construct exact travelling wave solutions of nonlinear stochastic fractional partial differential equations, by using modified fractional sub-equation method. The main idea of this method is to take full advantage of the fractional Riccati equation, which has many exact solutions. Moreover, white noise functional solutions are obtained for the Wick-type stochastic fractional Hirota-Satsuma coupled KdV equations via Hermite transform and white noise analysis. These solutions include stochastic exponential decay, soliton and periodic wave solutions.

Keywords: Hirota-Satsuma coupled KdV equations; Fractional calculus; Riccati equation; White noise; Hermite transform.

Introduction

This paper is devoted to consider the generalized fractional Hirota-Satsuma coupled KdV equations given by

$$\begin{cases} D_t^{\alpha} u - p(t) D_x^{3\alpha} u - q(t) u D_x^{\alpha} u - r(t) D_x^{\alpha}(vw) = 0, \\ D_t^{\alpha} v + D_x^{3\alpha} v - \varepsilon\, u D_x^{\alpha} v = 0, \\ D_t^{\alpha} w + D_x^{3\alpha} w - \varepsilon\, u D_x^{\alpha} w = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (-\infty,\infty) \times (0,\infty)$, $\alpha \in (0,1]$, $\varepsilon \in R$, $D_t^{\alpha} u$ and $D_x^{\alpha} u$ are the modified Riemann-Liouville derivatives and $p(t), q(t)$ and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. As pointed in [8, 21, 31] the interaction of two long waves with different dispersion relations can be described by Eq.(1.1). Moreover, if the problem is considered in random environment, we can get random fractional Hirota-Satsuma coupled KdV equations. In order to give the exact solutions of this random model, we only consider it in white noise environment, that is, we will study the following Wick-type stochastic fractional Hirota-Satsuma coupled KdV equations

$$\begin{cases} D_t^{\alpha} U - P(t)\Diamond D_x^{3\alpha} U - Q(t)\Diamond U\Diamond D_x^{\alpha} U - R(t)\Diamond D_x^{\alpha}(V\Diamond W) = 0, \\ D_t^{\alpha} V + D_x^{3\alpha} V - \varepsilon\, U\Diamond D_x^{\alpha} V = 0, \\ D_t^{\alpha} W + D_x^{3\alpha} W - \varepsilon\, U\Diamond D_x^{\alpha} W = 0 \end{cases} \tag{1.2}$$

where ``$\Diamond$" is the Wick product on the Kondratiev distribution space $(S)_{-1}$ and $P(t), Q(t)$ and $R(t)$ are $(S)_{-1}$ – valued functions [22]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology and material science can be better described by fractional partial differential equations FPDEs [26, 28, 29]. Consequently, considerable attention has been given to the solution of the FPDEs. There are many methods for calculating the approximate solutions for nonlinear FPDEs such as the variational iterations method [32], Adomian decomposition method [2, 3], the homotopy perturbation method [17, 18] and the Exp-function method [25, 34, 35, 36, 39]. The exact solutions for nonlinear FPDEs are still under study until now. Li and He [24] introduced complex transform for reducing FPDEs into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus. Also, He [19] introduced a new method to look for exact travelling wave solutions of nonlinear FPDEs. This method is called fractional sub-equation method, and it is based on the homogeneous balance principle [38] and Jumarie’s Riemann-Liouville derivatives [23]. It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in random environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. Wadati [33] first answered the interesting question, “How does external noise affect the motion of solitons?" and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. The cauchy problems associated with stochastic PDEs was discussed by many authors, e.g., de Bouard and Debussche [4,5], Debussche and Printems [6,7], Printems [30] and Ghany and Hyder [12]. On the basis of white noise functional analysis [22], Ghany et al. [9-16] studied more intensely the white noise functional solutions for some nonlinear stochastic PDEs. The motivation of this paper is to propose a modified fractional sub-equation method to seek new exact travelling wave solutions for the variable coefficients fractional Hirota-Satsuma coupled KdV equations with the modified Riemann-Liouville derivatives . These solutions include exponential decay, soliton and periodic travelling wave solutions. Then, with the help of Hermit transform and white noise analysis, we employ these solutions to find new white noise functional solutions for the stochastic fractional Hirota-Satsuma coupled KdV equations in white noise environment, especially in its Wick version.

The rest of this paper is organized as follows: In Section 2, we recall the definition and some properties of the modified Riemann-Liouville derivative and introduce a modified fractional subequation method. In Section 3, we apply the modified fractional sub-equation method to explore exact travelling wave solutions for Eq.(1.1). In Section 4, we use the Hermite transform to obtain white noise functional solutions for Eq.(1.2). In Section 5, we give illustrative examples for the investigated model. The last section is devoted to summary and discussion.

Preliminaries

Suppose that $S(R^d)$ and $S'(R^d)$ are the Hida test function space and the Hida distribution space on R^d, respectively. Let $h_n(x)$ be Hermite polynomials and put

$$\zeta_n = e^{-x^2} h_n(\sqrt{2}x)/((n-1)!\pi)^{\frac{1}{2}}, \quad n1. \tag{2.1}$$

then, the collection $\{\zeta_n\}_{n1}$ constitutes an orthogonal basis for $L_2(R)$. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ denote d-dimensional multi-indices with $\alpha_1, \alpha_2, ..., \alpha_d \in N$. The family of tensor products

$$\zeta_\alpha := \zeta_{(\alpha_1, \alpha_2, ..., \alpha_d)} = \zeta_{\alpha_1} \otimes \zeta_{\alpha_2} \otimes ... \otimes \zeta_{\alpha_d} \tag{2.2}$$

forms an orthogonal basis for $L_2(R^d)$. Suppose that $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, ..., \alpha_d^{(i)})$ is the i-th multi-index number in some fixed ordering of all d-dimensional multi-indices α. We can, and will, assume that this ordering has the property that

$$i < j \Rightarrow \alpha_1^{(i)} + \alpha_2^{(i)} + ... + \alpha_d^{(i)} < \alpha_1^{(j)} + \alpha_2^{(j)} + ... + \alpha_d^{(j)} \tag{2.3}$$

i.e., the $\{\alpha^{(j)}\}_{j=1}^{\infty}$ occurs in an increasing order. Now define

$$\eta_i := \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes ... \otimes \zeta_{\alpha_d^{(i)}} \quad . \tag{2.4}$$

We need to consider multi-indices of arbitrary length. For simplification of notation, we regard multi-indices as elements of the space $(N_0^N)_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ with elements $\alpha_i \in N_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. We write $J = (N_0^N)_c$, for $\alpha \in J$. Define

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(<\omega, \eta_i>), \quad \omega = (\omega_1, \omega_2, ..., \omega_d) \in S'(R^d) \tag{2.5}$$

For a fixed $n \in N$ and for all $k \in N$, suppose the space $(S)_1^n$ consists of those $f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in \bigoplus_{k=1}^{n} L_2(\mu)$ with $c_\alpha \in R^n$ such that

$$\| f \|_{1,k}^2 = \sum_\alpha c_\alpha^2 (\alpha!)^2 (2N)^{k\alpha} < \infty \tag{2.6}$$

where, $c_\alpha^2 = |c_\alpha|^2 = \sum_{k=1}^n (c_\alpha^{(k)})^2$ if $c_\alpha = (c_\alpha^{(1)}, c_\alpha^{(2)}, ..., c_\alpha^{(n)}) \in R^n$ and μ is the white noise measure on $(S'(R), B(S'(R)))$, $\alpha! = \prod_{k=1}^\infty \alpha_k!$ and $(2N)^\alpha = \prod_j (2j)^{\alpha_j}$ for $\alpha \in J$. The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_\alpha b_\alpha H_\alpha(\omega)$ with $b_\alpha \in R^n$ such that $\| f \|_{-1,-q} = \sum_\alpha b_\alpha^2 (2N)^{-q\alpha} < \infty$ for some $q \in N$. The family of seminorms $\| f \|_{1,k}, k \in N$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$< F, f > = \sum_\alpha (b_\alpha, c_\alpha) \alpha! \tag{2.7}$$

where (b_α, c_α) is the inner product in R^n.

The Wick product $f \lozenge F$ of two elements $f = \sum_\alpha a_\alpha H_\alpha, F = \sum_\beta b_\beta H_\beta \in (S)_{-1}^n$ with $a_\alpha, b_\beta \in R^n$, is defined by

$$f \lozenge F = \sum_{\alpha,\beta} (a_\alpha, b_\beta) H_{\alpha+\beta} \tag{2.8}$$

The spaces $(S)_1^n, (S)_{-1}^n, S(R^d)$ and $S'(R^d)$ are closed under Wick products. For $F = \sum_\alpha b_\alpha H_\alpha \in (S)_{-1}^n$, with $b_\alpha \in R^n$, the Hermite transformation of F, is defined by

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_\alpha b_\alpha z^\alpha \in C^N \tag{2.9}$$

where $z = (z_1, z_2, ...) \in C^N$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$, if $\alpha \in J$, where $z_j^0 = 1$. For $F, G \in (S)_{-1}^n$ we have

$$\widetilde{F \lozenge G}(z) = \tilde{F}(z).\tilde{G}(z) \tag{2.10}$$

for all z such that $\tilde{F}(z)$ and $\tilde{G}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of C^N defined by $(z_1^1, z_2^1, ..., z_n^1).(z_1^2, z_2^2, ..., z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$. Let $X = \sum_\alpha a_\alpha H_\alpha$, then the vector $c_0 = \tilde{X}(0) \in R^N$ is called the generalized expectation of X which denoted by $E(X)$. Suppose that $g : U \to C^M$ is an analytic function, where U is a neighborhood of $E(X)$. Assume that the Taylor series of g around $E(X)$ have coefficients in R^M. Then the Wick version $g^\lozenge(X) = \mathcal{H}^{-1}(g \circ \tilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z)=\sum a_{\alpha}(z-E(X))^{\alpha}$, with $a_{\alpha}\in R^{M}$, then $g^{\Diamond}(z)=\sum a_{\alpha}(z-E(X))^{\Diamond\alpha}\in(S)_{-1}^{M}$. Suppose that modelling consideration leads us to consider an stochastic fractional PDE as follows:

$$A(t,x,\partial_{t^{\alpha}},\Delta_{x^{\beta}},U,\omega)=0 \tag{2.11}$$

where A is some given function $U=U(t,x,\omega)$ is an unknown (generalized) stochastic process, and the operators $\partial_{t^{\alpha}}=\frac{\partial^{\alpha}}{\partial t^{\alpha}},\Delta_{x^{\beta}}=(\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}},\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}},\ldots,\frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}})$ when $x=(x_1,x_2,\ldots,x_d),\beta=(\beta_1,\beta_2,\ldots,\beta_d)\in R^{d}$. Firstly, we interpret all products as Wick products and all functions as their Wick versions. Wick version of Eq. (2.11) is written as follows:

$$A^{\Diamond}(t,x,\partial_{t^{\alpha}},\Delta_{x^{\beta}},U,\omega)=0 \tag{2.12}$$

Secondly, we take the Hermite transformation of Eq. (2.12), which turns Wick products into ordinary products (between complex numbers), so the equation takes the form

$$\tilde{A}(t,x,\partial_{t^{\alpha}},\Delta_{x^{\beta}},\tilde{U},z_1,z_2,\ldots)=0 \tag{2.13}$$

where $\tilde{U}=\mathrm{H}(U)$ is the Hermite transformation of U and $z_1,z_2,\ldots$ are complex numbers.

Definition 1: A measurable function

$$u:R^{d}\rightarrow(S)_{-1}^{N}$$

is called $(S)_{-1}^{N}$-process. The partial derivative $\frac{\partial u}{\partial x_k}$ of an $(S)_{-1}^{N}$ u is defined by

$$\frac{\partial u}{\partial x_k}(x_1,\ldots,x_d)=lim_{\Delta x_k\rightarrow0}\frac{u(x_1,\ldots x_k+\Delta x_k,x_d)-u(x_1,\ldots,x_d)}{\Delta x_k}$$

provided the limit exists in $(S)_{-1}^{N}$. Let u be a continuous $(\mathrm{S})_{-1}-$process, and let $h>0$ denote a constant discretization span. Define the forward operator $FW_{x_k}(h)$, by

$$FW_{x_k}(h)u(x):=u(x_1,\ldots,x_k+h,x_{k+1},\ldots,x_d). \tag{2.14}$$

Then for $0<\alpha\leq1$, the $\alpha-$order fractional difference of u is defined by the expression

$$\Delta_{x_k}^{\alpha}u(x):=(FW_{x_k}(h)-1)^{\alpha}.u(x)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\alpha}{j}u(x_1,\ldots,x_k+(\alpha-j)h,x_{k+1},\ldots,x_d), \tag{2.15}$$

and its $\alpha-$order fractional derivative is given by

$$D_{x_k}^{\alpha}u(x)=\lim_{h\downarrow0}\frac{\Delta_{x_k}^{\alpha}u(x)}{h^{\alpha}}. \tag{2.16}$$

provided the limit exists in $(\mathrm{S})_{-1}$. In terms of the Hermite transform the limit on the right-hand side of (2.14) exists if and only if there exists an element $Y\in(\mathrm{S})_{-1}$ such that

$\frac{1}{h^{\alpha}}\Delta^{\alpha}_{x_k}\widetilde{u}(x,z) \to \widetilde{Y}(z)$ pointwise boundedly (uniformly) in $K_{\sigma}(\delta)$ for some $\sigma < \infty, \delta > 0$, where

$$K_{\sigma}(\delta) = \{z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\mu \neq 0} |z^{\mu}|^2 (2\mathbb{N})^{\sigma\mu} < \delta\}$$

If this is the case, then Y is denoted by $D^{\alpha}_{x_k} u(x)$.

Let us denote by $L^1(a,b;(\mathcal{S})_{-1})$ the space of all strongly integrable $(\mathcal{S})_{-1}-$processes on $[a,b]$, then for $X \in L^1(a,b;(\mathcal{S})_{-1})$ we can set the $\alpha-$order Riemann-Liouville fractional integral operator and the modified Riemann-Liouville fractional derivative as follows:

Definition 2: The $\alpha-$order Riemann-Liouville fractional integral operator of X is defined as

$$J^{\alpha}X(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}X(\tau)d\tau,$$

$$\text{for } \alpha > 0, t \in [a,b] \text{ and } J^0 X(t) := X(t). \tag{2.17}$$

When we apply Hermite transform to solve stochastic differential equations the following observation is important. Assume that the $(\mathcal{S})_{-1}-$process $X(t,\omega)$ has an $\alpha-$order fractional derivative and

$$D^{\alpha}_t X(t,\omega) = F(t,\omega) \text{ in } (\mathcal{S})_{-1}, \tag{2.18}$$

this equivalent to saying that

$$\lim_{h \downarrow 0} \frac{\Delta^{\alpha}_{x_k}\widetilde{X}(t,z)}{h^{\alpha}} = \widetilde{F}(t,z) \tag{2.19}$$

uniformly for $z \in K_{\sigma}(\delta)$ for some $\sigma < \infty, \delta > 0$. For this it is clearly necessary that

$$D^{\alpha}_t \widetilde{X}(t,z) = \widetilde{F}(t,z) \quad \text{forall } z \in K_{\sigma}(\delta), \tag{2.20}$$

but apparently not sufficient, because we also need that the pointwise convergence is bounded for $z \in K_{\sigma}(\delta)$. The following result is sufficient for our purposes.

Lemma 1: *Suppose* $X(t,\omega)$ *and* $F(t,\omega)$ *are* $(\mathcal{S})_{-1}-$processes *such that*

- $D^{\alpha}_t \widetilde{X}(t,z) = \widetilde{F}(t,z)$ *for each* $(t,z) \in (a,b) \times K_{\sigma}(\delta)$ *and that*
- $\widetilde{F}(t,z)$ *is a bounded function for* $(t,z) \in (a,b) \times K_{\sigma}(\delta)$ *and continuous with respect to* $t \in (a,b)$ *for each* $z \in K_{\sigma}(\delta)$. *Then* $X(t,\omega)$ *has an* $\alpha-$order *fractional derivative and for each* $t \in (a,b)$

$$D^{\alpha}_t X(t,\omega) = F(t,\omega) \quad \text{in } (\mathcal{S})_{-1}. \tag{2.21}$$

Proof. According to the fractional counterpart of the mean value theorem [?], we have

$$\frac{1}{h^{\alpha}}\Delta_t^{\alpha}\tilde{X}(t,z)=\frac{\Gamma(1+\alpha)}{h^{\alpha}}(\tilde{X}(t+h,z)-\tilde{X}(t,z))=\tilde{F}(t+\theta h,z), \tag{2.22}$$

for some $\theta\in[0,1]$ and for each $z\in K_{\sigma}(\delta)$. So if the hypotheses (i), (ii) hold, then

$$\lim_{h\downarrow 0}\frac{\Delta_t^{\alpha}\tilde{X}(t,z)}{h^{\alpha}}=\tilde{F}(t,z) \tag{2.23}$$

pointwise boundedly for $z\in K_{\sigma}(\delta)$. Taking Hermite transform of (2.15) and using [2, Lemma 2.8.5], we get the following result

Lemma 2: *Let* $X(t)$ *be an* $(\mathcal{S})_{-1}-$process. *Suppose there exist* $\sigma<\infty$, $\delta>0$ *such that*

$$\sup\{\tilde{X}(t,z):t\in[a,b],\,z\in K_{\sigma}(\delta)\}<\infty \tag{2.24}$$

and $\tilde{X}(t,z)$ is a continuous function with respect to $t\in[a,b]$ for each $z\in K_{\sigma}(\delta)$. Then the $\alpha-$order Riemann-Liouville fractional integral operator of $X(t)$ exists and

$$J^{\alpha}\tilde{X}(t)(z)=J^{\alpha}\tilde{X}(t,z),\quad \text{for}\,\alpha\geq 0,\,t\in[a,b],\,z\in K_{\sigma}(\delta). \tag{2.25}$$

In the case of higher order derivatives we have the following result

Lemma 3: *Suppose there exist an open interval* I, *real numbers* σ,δ *and a function* $u:I\times K_{\sigma}(\delta)\to\mathbb{C}$ *such that*

$$D_x^{2\alpha}u(x,z)=\tilde{F}(x,z),\ \text{for}\ (x,z)\in I\times K_{\sigma}(\delta) \tag{2.26}$$

where $F(x)\in(\mathcal{S})_{-1}$ for all $x\in I$. Suppose $D_x^{2\alpha}u$ is bounded for $(x,z)\in I\times K_{\sigma}(\delta)$ and continuous with respect to $x\in I$ for each $z\in K_{\sigma}(\delta)$. Then there exists $U(x)\in(\mathcal{S})_{-1}$ such that

$$D_x^{2\alpha}U(x)=F(x),\ \text{for}\ x\in I. \tag{2.27}$$

Proof. By the fractional counterpart of the mean value theorem again, we have

$$\frac{1}{h^{2\alpha}}\Delta_x^{2\alpha}u(x,z)=\frac{\Gamma^2(1+\alpha)}{h^{2\alpha}}(u(x+2h,z)-2u(x+h,z)+u(x,z))=\tilde{F}(x+\theta h,z) \tag{2.28}$$

for some $\theta\in[0,1]$ and for each $z\in K_{\sigma}(\delta)$. So if (2.14) and the assumptions on $D_x^{2\alpha}u$ hold, then

$$\lim_{h\downarrow 0}\frac{\Delta_x^{2\alpha}\tilde{u}(t,z)}{h^{2\alpha}}=\tilde{F}(x,z) \tag{2.29}$$

pointwise boundedly for $z \in K_\sigma(\delta)$. According to [2, Lemma 2.8.5], we can apply the inverse Hermite transform to Eq.(2.17) and get

$$D_x^{2\alpha} U(x) = F(x) \text{ in } (\mathrm{S})_{-1} \text{ and for all } x \in I, \tag{2.30}$$

where $u(x,z) = \tilde{U}(x)(z)$ for all $(x,z) \in I \times K_\sigma(\delta)$ +

More generally , we can apply the argument of Lemma 2.1 repeatedly and get the following result

Theorem 2.4. *Suppose* $u(x,t,z)$ *is a solution (in the usual strong, pointwise sense) of the equation*

$$\tilde{\Omega}(x,t,D_t^\alpha, D_{x_1}^\alpha, \ldots, D_{x_d}^\alpha, u, z) = 0 \tag{2.31}$$

for (x,t) *in some bounded open set* $G \subset \mathrm{R}^d \times \mathrm{R}_+$, *and for all* $z \in K_\sigma(\delta)$, *for some* σ, δ. *Moreover, suppose that* $u(x,t,z)$ *and all its partial fractional derivatives, which are involved in (2.19), are (uniformly) bounded for* $(x,t,z) \in G \times K_\sigma(\delta)$, *continuous with respect to* $(x,t) \in G$ *for each* $z \in K_\sigma(\delta)$ *and analytic with respect to* $z \in K_\sigma(\delta)$, *for all* $(x,t) \in G$. *Then there exists* $U(x,t) \in (\mathrm{S})_{-1}$ *such that* $u(x,t,z) = \tilde{U}(t,x)(z)$ *for all* $(t,x,z) \in G \times K_\sigma(\delta)$ *and* $U(x,t)$ *solves (in the strong sense) the equation*

$$\Omega^\diamond(t,x,D_t^\alpha, D_{x_1}^\alpha, \ldots, D_{x_d}^\alpha, U, \omega) = 0 \text{ in } (\mathrm{S})_{-1}. \tag{2.32}$$

There are different definitions for fractional derivatives, for more details see [28]. In our paper we use the modified Riemann-Liouville derivative defined by Jumarie [23]

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^x (x-y)^{-\alpha-1}[f(y)-f(0)]dy, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-y)^{-\alpha}[f(y)-f(0)]dy, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, & n \le \alpha < n+1, \; n \in \mathrm{N} \end{cases} \tag{2.33}$$

which has merits over the original one, for example, the α − order derivative of a constant is zero. Some properties of the modified Riemann-Liouville derivative were summarized in [23], three useful formulas of them are

$$\begin{cases} D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ D_x^\alpha (u(x)v(x)) = u(x) D_x^\alpha v(x) + v(x) D_x^\alpha u(x), \\ D_x^\alpha [f(u(x)] = \dfrac{df}{du} D_x^\alpha u(x) = \left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{2.34}$$

Now, we outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear FPDE, say, in two variables

$$F(u,u_x,u_t,D_x^{\alpha}u,D_t^{\alpha}u,...)=0,\quad 0<\alpha\le 1 \tag{2.35}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u=u(x,t)$ explicitly, we first introduce the following transformation

$$u=u(\xi),\qquad \xi=\xi(x,t) \tag{2.36}$$

which converts Eq.(2.3) into a fractional ordinary differential equation

$$G(u,u',u'',D_{\xi}^{\alpha}u,D_{\xi}^{2\alpha}u,...)=0. \tag{2.37}$$

Next we introduce a new variable $Y=Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_{\xi}^{\alpha}Y=h_0+h_1Y+h_2Y^2,\quad 0<\alpha\le 1, \tag{2.38}$$

where h_0, h_1 and h_2 are arbitrary constants. Eq.(2.6) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha=1$ Eq.(2.6) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feed back gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [27, 38]. Then we propose the following series expansion as a solution of Eq.(2.3)

$$u(x,t)=u(\xi)=\sum_{k=0}^{n}a_k(x,t)Y^k(\xi)+\sum_{k=1}^{n}b_k(x,t)Y^{-k}(\xi), \tag{2.39}$$

where $a_k(k=0,1,...,n), b_k(k=1,...,n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in equation Eq.(2.5). Inserting Eq.(2.7) into Eq.(2.5) and using Eq.(2.6) will give an algebraic equation in powers of Y . Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k and b_k With the aid of Mathematica, we can determine a_k and b_k . According to the recent paper by Zhang et al. [38], we can deduce the following set of solutions of Eq.(2.6).

$$\begin{cases} Y_1(\xi) = E_\alpha(\xi) - 1, & h_0 = h_1 = 1,\, h_2 = 0, \\ Y_2(\xi) = \coth_\alpha(\xi) \pm \operatorname{csch}_\alpha(\xi),\, Y_3(\xi) = \tanh_\alpha(\xi) \pm i\,\operatorname{sech}_\alpha(\xi), & h_0 = -h_2 = \frac{1}{2},\, h_1 = 0, \\ Y_4(\xi) = \frac{1}{2}\tan_\alpha(2\xi),\, Y_5(\xi) = \frac{1}{2}\cot_\alpha(2\xi), & h_0 = \frac{1}{4}h_2 = 1,\, h_1 = 0, \end{cases} \tag{2.40}$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)}, \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)}, \operatorname{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)}, \operatorname{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2}, \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2}, \tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)},$$

$$\cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)}, \sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i}, \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

defined by the Mittag-Leffler function $E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+j\alpha)}$. For more details about the generalized exponential, hyperbolic and trigonometric functions see [29].

Exact Travelling Wave Solutions for Eq.(1.1)

In this section, we apply Hermite transform, white noise analysis, and modified fractional sub-equation method to explore exact travelling wave solutions for Eq.(1.1). Taking the Hermite transform of Eq.(1.2), we get the deterministic system

$$\begin{cases} D_t^\alpha \tilde{U}(x,t,z) = \tilde{P}(t,z) D_x^{3\alpha} \tilde{U}(x,t,z) + \tilde{Q}(t,z)\tilde{U}(x,t,z) D_x^\alpha \tilde{U}(x,t,z) \\ \qquad + \tilde{R}(t,z) D_x^\alpha (\tilde{V}(x,t,z)\tilde{W}(x,t,z)), \\ D_t^\alpha \tilde{V}(x,t,z) = -D_x^{3\alpha} \tilde{V}(x,t,z) + \varepsilon \tilde{U}(x,t,z) D_x^\alpha \tilde{V}(x,t,z), \\ D_t^\alpha \tilde{W}(x,t,z) = -D_x^{3\alpha} \tilde{W}(x,t,z) + \varepsilon \tilde{U}(x,t,z) D_x^\alpha \tilde{W}(x,t,z), \end{cases} \tag{3.1}$$

where $z = (z_1, z_2, \ldots) \in (\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter. To look for the travelling wave solution of Eq.(3.1), we make the transformations

$$u(x,t,z) := \tilde{U}(x,t,z) = \phi(\xi(x,t,z)),\; v(x,t,z) := \tilde{V}(x,t,z) = \psi(\xi(x,t,z))$$

and

$$w(x,t,z) := \tilde{W}(x,t,z) = \chi(\xi(x,t,z)), \text{with } \xi(x,t,z) = kx + s\int_0^t l(\tau,z)\,d\tau + c,$$

where k, s and c are arbitrary constants which satisfy $ks \neq 0$, $l(t,z)$ is a non zero functions of the indicated variables to be determined. So, Eq.(3.1) can be changing into the form

$$\begin{cases} (sl)^{\alpha} D_{\xi}^{\alpha} \phi = k^{3\alpha} p D_{\xi}^{3\alpha} \phi + k^{\alpha} q \phi D_{\xi}^{\alpha} \phi + k^{\alpha} r D_{\xi}^{\alpha} (\psi \chi), \\ (sl)^{\alpha} D_{\xi}^{\alpha} \psi = -k^{3\alpha} D_{\xi}^{3\alpha} \psi + \varepsilon k^{\alpha} \phi D_{\xi}^{\alpha} \psi, \\ (sl)^{\alpha} D_{\xi}^{\alpha} \chi = -k^{3\alpha} D_{\xi}^{3\alpha} \chi + \varepsilon k^{\alpha} \phi D_{\xi}^{\alpha} \chi, \end{cases} \tag{3.2}$$

where $p(t,z) := \tilde{P}(t,z)$, $q(t,z) := \tilde{Q}(t,z)$ and $r(t,z) := \tilde{R}(t,z)$. Balancing the highest order linear terms and nonlinear terms in Eq.(3.2), gives the following ansatzes:

$$\begin{cases} u(x,t,z) = a_0(t,z) + a_1(t,z) Y(\xi) + a_2(t,z) Y^2(\xi) + b_1(t,z) Y^{-1}(\xi) + b_2(t,z) Y^{-2}(\xi), \\ v(x,t,z) = c_0(t,z) + c_1(t,z) Y(\xi) + c_2(t,z) Y^2(\xi) + d_1(t,z) Y^{-1}(\xi) + d_2(t,z) Y^{-2}(\xi), \\ w(x,t,z) = e_0(t,z) + e_1(t,z) Y(\xi) + e_2(t,z) Y^2(\xi) + f_1(t,z) Y^{-1}(\xi) + f_2(t,z) Y^{-2}(\xi) \end{cases} \tag{3.3}$$

where $Y(\xi)$ satisfies the fractional Riccati equation (2.6). By substituting Eq.(3.3) along with Eq.(2.6) into Eq.(3.2), collect the coefficients of Y^k $(k = -5,-4,...,5)$ and set them to be zero, we will obtain a system of algebraic equations in the unknowns a_k, c_k, e_k $(k = 0,1,2)$, b_k, d_k, f_k $(k = 1,2)$ and l of the form

$$\begin{cases} (sl)^{\alpha} G_i^{\phi} = k^{3\alpha} p K_i^{\phi} + k^{\alpha} q \zeta_i^{\phi} + k^{\alpha} r(\rho_i^1 + \rho_i^2), & i = 0,1,2,3, \\ (sl)^{\alpha} H_i^{\phi} = k^{3\alpha} p L_i^{\phi} + k^{\alpha} q \eta_i^{\phi} + k^{\alpha} r(\lambda_i^1 + \lambda_i^2), & i = 1,2,3, \\ k^{2\alpha} p K_i^{\phi} + q \zeta_i^{\phi} + r(\rho_i^1 + \rho_i^2) = 0, & i = 4,5, \\ k^{2\alpha} p L_i^{\phi} + q \eta_i^{\phi} + r(\lambda_i^1 + \lambda_i^2) = 0, & i = 4,5, \\ (sl)^{\alpha} G_i^{\psi} = -k^{3\alpha} K_i^{\psi} + 3k^{\alpha} \zeta_i^{\psi}, & i = 0,1,2,3, \\ (sl)^{\alpha} H_i^{\psi} = -k^{3\alpha} L_i^{\psi} + 3k^{\alpha} \eta_i^{\psi}, & i = 1,2,3, \\ -k^{2\alpha} K_i^{\psi} + 3\zeta_i^{\psi} = 0, & i = 4,5, \\ -k^{2\alpha} L_i^{\psi} + 3\eta_i^{\psi} = 0, & i = 4,5, \\ (sl)^{\alpha} G_i^{\chi} = -k^{3\alpha} K_i^{\chi} + 3k^{\alpha} \zeta_i^{\chi}, & i = 0,1,2,3, \\ (sl)^{\alpha} H_i^{\chi} = -k^{3\alpha} L_i^{\chi} + 3k^{\alpha} \eta_i^{\chi}, & i = 1,2,3, \\ -k^{2\alpha} K_i^{\eta} + 3\zeta_i^{\chi} = 0, & i = 4,5, \\ -k^{2\alpha} L_i^{\eta} + 3\eta_i^{\chi} = 0, & i = 4,5. \end{cases} \tag{3.4}$$

where, $G_0^\phi = h_0 a_1 - h_2 b_1$, $G_1^\phi = 2h_0 a_2 + h_1 a_1$, $G_2^\phi = 2h_1 a_2 + h_2 a_1$, $G_3^\phi = 2h_2 a_2$,
$G_0^\psi = h_0 c_1 - h_2 d_1$, $G_1^\psi = 2h_0 c_2 + h_1 c_1$, $G_2^\psi = 2h_1 c_2 + h_2 c_1$, $G_3^\psi = 2h_2 c_2$, $G_0^\chi = h_0 e_1 - h_2 f_1$,
$G_1^\chi = 2h_0 e_2 + h_1 e_1$, $G_2^\chi = 2h_1 e_2 + h_2 e_1$, $G_3^\chi = 2h_2 e_2$, $H_1^\phi = -(2h_2 b_2 + h_1 b_1)$,
$H_2^\phi = -(2h_1 b_2 + h_0 b_1)$, $H_3^\phi = -2h_0 b_2$, $H_1^\psi = -(2h_2 d_2 + h_1 d_1)$, $H_2^\psi = -(2h_1 d_2 + h_0 d_1)$,
$H_3^\psi = -2h_0 d_2$ $H_1^\chi = -(2h_2 f_2 + h_1 f_1)$, $H_2^\chi = -(2h_1 f_2 + h_0 f_1)$, $H_3^\chi = -2h_0 f_2$,
$K_0^\phi = h_0(2h_0 G_2^\phi + h_1 G_1^\phi) + h_2(2h_2 H_2^\phi + h_1 H_1^\phi)$,
$K_1^\phi = 2h_0(3h_0 G_3^\phi + 2h_1 G_2^\phi + h_2 G_1^\phi) + h_1(2h_0 G_2^\phi + h_1 G_1^\phi)$,
$K_2^\phi = 3h_0(3h_1 G_3^\phi + 2h_2 G_2^\phi) + 2h_1(3h_0 G_3^\phi + 2h_1 G_2^\phi + h_2 G_1^\phi) + h_2(2h_0 G_2^\phi + h_1 G_1^\phi)$,
$K_3^\phi = 2h_2(3h_0 G_3^\phi + 2h_1 G_2^\phi + h_2 G_1^\phi) + 3h_1(3h_1 G_3^\phi + 2h_2 G_2^\phi) + 12h_0 h_2 G_3^\phi$,
$K_4^\phi = 3h_2(3h_1 G_3^\phi + 2h_2 G_2^\phi) + 12h_1 h_2 G_3^\phi$, $K_5^\phi = 12h_2^2 G_3^\phi$,
$K_0^\psi = h_0(2h_0 G_2^\psi + h_1 G_1^\psi) + h_2(2h_2 H_2^\psi + h_1 H_1^\psi)$,
$K_1^\psi = 2h_0(3h_0 G_3^\psi + 2h_1 G_2^\psi + h_2 G_1^\psi) + h_1(2h_0 G_2^\psi + h_1 G_1^\psi)$,
$K_2^\psi = 3h_0(3h_1 G_3^\psi + 2h_2 G_2^\psi) + 2h_1(3h_0 G_3^\psi + 2h_1 G_2^\psi + h_2 G_1^\psi) + h_2(2h_0 G_2^\psi + h_1 G_1^\psi)$,
$K_3^\psi = 2h_2(3h_0 G_3^\psi + 2h_1 G_2^\psi + h_2 G_1^\psi) + 3h_1(3h_1 G_3^\psi + 2h_2 G_2^\psi) + 12h_0 h_2 G_3^\psi$,
$K_4^\psi = 3h_2(3h_1 G_3^\psi + 2h_2 G_2^\psi) + 12h_1 h_2 G_3^\psi$, $K_5^\psi = 12h_2^2 G_3^\psi$,
$K_0^\chi = h_0(2h_0 G_2^\chi + h_1 G_1^\chi) + h_2(2h_2 H_2^\chi + h_1 H_1^\chi)$,
$K_1^\chi = 2h_0(3h_0 G_3^\chi + 2h_1 G_2^\chi + h_2 G_1^\chi) + h_1(2h_0 G_2^\chi + h_1 G_1^\chi)$,
$K_2^\chi = 3h_0(3h_1 G_3^\chi + 2h_2 G_2^\chi) + 2h_1(3h_0 G_3^\chi + 2h_1 G_2^\chi + h_2 G_1^\chi) + h_2(2h_0 G_2^\chi + h_1 G_1^\chi)$,
$K_3^\chi = 2h_2(3h_0 G_3^\chi + 2h_1 G_2^\chi + h_2 G_1^\chi) + 3h_1(3h_1 G_3^\chi + 2h_2 G_2^\chi) + 12h_0 h_2 G_3^\chi$,
$K_4^\chi = 3h_2(3h_1 G_3^\chi + 2h_2 G_2^\chi) + 12h_1 h_2 G_3^\chi$, $K_5^\chi = 12h_2^2 G_3^\chi$,
$L_1^\phi = 2h_2(3h_2 H_3^\phi + 2h_1 H_2^\phi + h_0 H_1^\phi) + h_1(2h_2 H_2^\phi + h_1 H_1^\phi)$,
$L_2^\phi = 3h_2(3h_1 H_3^\phi + 2h_0 H_2^\phi) + 2h_1(3h_2 H_3^\phi + 2h_1 H_2^\phi + h_0 H_1^\phi) + h_0(2h_2 H_2^\phi + h_1 H_1^\phi)$,
$L_3^\phi = 2h_0(3h_2 H_3^\phi + 2h_1 H_2^\phi + h_0 H_1^\phi) + 3h_1(3h_1 H_3^\phi + 2h_0 H_2^\phi) + 12h_0 h_2 H_3^\phi$,
$L_4^\phi = 3h_0(3h_1 H_3^\phi + 2h_0 H_2^\phi) + 12h_1 h_0 H_3^\phi$, $L_5^\phi = 12h_0^2 H_3^\phi$,
$L_1^\psi = 2h_2(3h_2 H_3^\psi + 2h_1 H_2^\psi + h_0 H_1^\psi) + h_1(2h_2 H_2^\psi + h_1 H_1^\psi)$,
$L_2^\psi = 3h_2(3h_1 H_3^\psi + 2h_0 H_2^\psi) + 2h_1(3h_2 H_3^\psi + 2h_1 H_2^\psi + h_0 H_1^\psi) + h_0(2h_2 H_2^\psi + h_1 H_1^\psi)$,
$L_3^\psi = 2h_0(3h_2 H_3^\psi + 2h_1 H_2^\psi + h_0 H_1^\psi) + 3h_1(3h_1 H_3^\psi + 2h_0 H_2^\psi) + 12h_0 h_2 H_3^\psi$,
$L_4^\psi = 3h_0(3h_1 H_3^\psi + 2h_0 H_2^\psi) + 12h_1 h_0 H_3^\psi$, $L_5^\psi = 12h_0^2 H_3^\psi$,
$L_1^\chi = 2h_2(3h_2 H_3^\chi + 2h_1 H_2^\chi + h_0 H_1^\chi) + h_1(2h_2 H_2^\chi + h_1 H_1^\chi)$,
$L_2^\chi = 3h_2(3h_1 H_3^\chi + 2h_0 H_2^\chi) + 2h_1(3h_2 H_3^\chi + 2h_1 H_2^\chi + h_0 H_1^\chi) + h_0(2h_2 H_2^\chi + h_1 H_1^\chi)$,
$L_3^\chi = 2h_0(3h_2 H_3^\chi + 2h_1 H_2^\chi + h_0 H_1^\chi) + 3h_1(3h_1 H_3^\chi + 2h_0 H_2^\chi) + 12h_0 h_2 H_3^\chi$,
$L_4^\chi = 3h_0(3h_1 H_3^\chi + 2h_0 H_2^\chi) + 12h_1 h_0 H_3^\chi$, $L_5^\chi = 12h_0^2 H_3^\chi$,
$\zeta_0^\phi = a_0 G_0^\phi + a_1 H_1^\phi + a_2 H_2^\phi + b_1 G_1^\phi + b_2 G_2^\phi$, $\zeta_1^\phi = a_0 G_1^\phi + a_1 G_0^\phi + a_2 H_1^\phi + b_1 G_2^\phi + b_2 G_3^\phi$,

$\zeta_2^{\phi} = a_0 G_2^{\phi} + a_1 G_1^{\phi} + a_2 G_0^{\phi} + b_1 G_3^{\phi}$, $\zeta_3^{\phi} = a_0 G_3^{\phi} + a_1 G_2^{\phi} + a_2 G_1^{\phi}$, $\zeta_4^{\phi} = a_1 G_3^{\phi} + a_2 G_2^{\phi}$,
$\zeta_5^{\phi} = a_2 G_3^{\phi}$, $\eta_1^{\phi} = a_0 H_1^{\phi} + a_1 H_2^{\phi} + a_2 H_3^{\phi} + b_1 G_0^{\phi} + b_2 G_1^{\phi}$,
$\eta_2^{\phi} = a_0 H_2^{\phi} + a_1 H_3^{\phi} + b_2 G_0^{\phi} + b_1 H_1^{\phi}$, $\eta_3^{\phi} = a_0 H_3^{\phi} + b_1 H_2^{\phi} + b_2 H_1^{\phi}$, $\eta_4^{\phi} = b_1 H_3^{\phi} + b_2 H_2^{\phi}$,
$\eta_5^{\phi} = b_2 H_3^{\phi}$, $\zeta_0^{\psi} = c_0 G_0^{\phi} + c_1 H_1^{\phi} + c_2 H_2^{\phi} + d_1 G_1^{\phi} + d_2 G_2^{\phi}$,
$\zeta_1^{\psi} = c_0 G_1^{\phi} + c_1 G_0^{\phi} + c_2 H_1^{\phi} + d_1 G_2^{\phi} + d_2 G_3^{\phi}$, $\zeta_2^{\psi} = c_0 G_2^{\phi} + c_1 G_1^{\phi} + c_2 G_0^{\phi} + d_1 G_3^{\phi}$,
$\zeta_3^{\psi} = c_0 G_3^{\phi} + c_1 G_2^{\phi} + c_2 G_1^{\phi}$, $\zeta_4^{\psi} = c_1 G_3^{\phi} + c_2 G_2^{\phi}$, $\zeta_5^{\psi} = c_2 G_3^{\phi}$,
$\eta_1^{\psi} = c_0 H_1^{\phi} + c_1 H_2^{\phi} + c_2 H_3^{\phi} + d_1 G_0^{\phi} + d_2 G_1^{\phi}$, $\eta_2^{\psi} = c_0 H_2^{\phi} + c_1 H_3^{\phi} + d_2 G_0^{\phi} + d_1 H_1^{\phi}$,
$\eta_3^{\psi} = c_0 H_3^{\phi} + d_1 H_2^{\phi} + d_2 H_1^{\phi}$, $\eta_4^{\psi} = d_1 H_3^{\phi} + d_2 H_2^{\phi}$, $\eta_5^{\psi} = d_2 H_3^{\phi}$,
$\zeta_0^{\chi} = e_0 G_0^{\phi} + e_1 H_1^{\phi} + e_2 H_2^{\phi} + f_1 G_1^{\phi} + f_2 G_2^{\phi}$, $\zeta_1^{\chi} = e_0 G_1^{\phi} + e_1 G_0^{\phi} + e_2 H_1^{\phi} + f_1 G_2^{\phi} + f_2 G_3^{\phi}$,
$\zeta_2^{\chi} = e_0 G_2^{\phi} + e_1 G_1^{\phi} + e_2 G_0^{\phi} + f_1 G_3^{\phi}$, $\zeta_3^{\chi} = e_0 G_3^{\phi} + e_1 G_2^{\phi} + e_2 G_1^{\phi}$, $\zeta_4^{\chi} = e_1 G_3^{\phi} + e_2 G_2^{\phi}$,
$\zeta_5^{\chi} = e_2 G_3^{\phi}$, $\eta_1^{\chi} = e_0 H_1^{\phi} + e_1 H_2^{\phi} + e_2 H_3^{\phi} + f_1 G_0^{\phi} + f_2 G_1^{\phi}$,
$\eta_2^{\chi} = e_0 H_2^{\phi} + e_1 H_3^{\phi} + f_2 G_0^{\phi} + f_1 H_1^{\phi}$, $\eta_3^{\chi} = e_0 H_3^{\phi} + f_1 H_2^{\phi} + f_2 H_1^{\phi}$, $\eta_4^{\chi} = f_1 H_3^{\phi} + f_2 H_2^{\phi}$,
$\eta_5^{\chi} = f_2 H_3^{\phi}$, $\rho_0^1 = e_0 G_0^{\psi} + e_1 H_1^{\psi} + e_2 H_2^{\psi} + f_1 G_1^{\psi} + f_2 G_2^{\psi}$,
$\rho_1^1 = e_0 G_1^{\psi} + e_1 G_0^{\psi} + e_2 H_1^{\psi} + f_1 G_2^{\psi} + f_2 G_3^{\psi}$, $\rho_2^1 = e_0 G_2^{\psi} + e_1 G_1^{\psi} + e_2 G_0^{\psi} + f_1 G_3^{\psi}$,
$\rho_3^1 = e_0 G_3^{\psi} + e_1 G_2^{\psi} + e_2 G_1^{\psi}$, $\rho_4^1 = e_1 G_3^{\psi} + e_2 G_2^{\psi}$, $\rho_5^1 = e_2 G_3^{\psi}$,
$\lambda_1^1 = e_0 H_1^{\psi} + e_1 H_2^{\psi} + e_2 H_3^{\psi} + f_1 G_0^{\psi} + f_2 G_1^{\psi}$, $\lambda_2^1 = e_0 H_2^{\psi} + e_1 H_3^{\psi} + f_2 G_0^{\psi} + f_1 H_1^{\psi}$,
$\lambda_3^1 = e_0 H_3^{\psi} + f_1 H_2^{\psi} + f_2 H_1^{\psi}$, $\lambda_4^1 = f_1 H_3^{\psi} + f_2 H_2^{\psi}$, $\lambda_5^1 = f_2 H_3^{\psi}$,
$\rho_0^2 = c_0 G_0^{\chi} + c_1 H_1^{\chi} + c_2 H_2^{\chi} + d_1 G_1^{\chi} + d_2 G_2^{\chi}$, $\rho_1^2 = c_0 G_1^{\chi} + c_1 G_0^{\chi} + c_2 H_1^{\chi} + d_1 G_2^{\chi} + d_2 G_3^{\chi}$,
$\rho_2^2 = c_0 G_2^{\chi} + c_1 G_1^{\chi} + c_2 G_0^{\chi} + d_1 G_3^{\chi}$, $\rho_3^2 = c_0 G_3^{\chi} + c_1 G_2^{\chi} + c_2 G_1^{\chi}$, $\rho_4^2 = c_1 G_3^{\chi} + c_2 G_2^{\chi}$,
$\rho_5^2 = c_2 G_3^{\chi}$, $\lambda_1^2 = c_0 H_1^{\chi} + c_1 H_2^{\chi} + c_2 H_3^{\chi} + d_1 G_0^{\chi} + d_2 G_1^{\chi}$,
$\lambda_2^2 = c_0 H_2^{\chi} + c_1 H_3^{\chi} + d_2 G_0^{\chi} + d_1 H_1^{\chi}$, $\lambda_3^2 = c_0 H_3^{\chi} + d_1 H_2^{\chi} + d_2 H_1^{\chi}$, $\lambda_4^2 = d_1 H_3^{\chi} + d_2 H_2^{\chi}$ and
$\lambda_5^2 = d_2 H_3^{\chi}$.

In the remaining part of this section we investigate and solve our problem for some particular cases for the fractional Riccati equation (2.6) as follows.

Case A. If we set $h_0 = h_1 = 1,\ h_2 = 0$ in Eq.(2.6), and use Mathematica to solve the resulting system, we will obtain the following set of solutions

Set 1:

$$a_1 = a_2 = c_1 = c_2 = e_1 = e_2 = 0,\ a_0 = \frac{pk^{2\alpha}}{q} - \frac{4rp}{9q}\left(\frac{6r}{p} - \frac{21p}{q}\right)^{-1} k^{2\alpha},\ c_0 = e_0 = \frac{pk^{2\alpha}}{3},$$

$$b_1 = \left(\frac{6r}{p} - \frac{21p}{q}\right)k^{2\alpha},\ b_2 = \frac{-6pk^{2\alpha}}{q},\ d_1 = f_1 = \frac{2}{3}k^{2\alpha},\ d_2 = f_2 = 4k^{2\alpha},\ l = \left(\frac{2pk^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}},$$

under condition that $77p^2 - 64qr = 0$. Substituting these values in Eq.(3.3) and using Eq.(2.8), we obtain the following exponential decay wave solution of Eq.(3.1):

$$u_1(x,t,z) = \frac{k^{2\alpha} p(t,z)}{q(t,z)} - \frac{4k^{2\alpha} r(t,z) p^2(t,z)}{54r(t,z)q(t,z) - 189p^2(t,z)} + \frac{k^{2\alpha}[(6r(t,z)q(t,z) - 21p^2(t,z))(Y_1[\xi_1(x,t,z)]) - 6p^2(t,z)]}{p(t,z)q(t,z)(Y_1[\xi_1(x,t,z)])^2}, \tag{3.5}$$

$$v_1(x,t,z) = \frac{k^{2\alpha}}{3} p(t,z) + \frac{2k^{2\alpha}(Y_1[\xi_1(x,t,z)] + 6)}{3(Y_1[\xi_1(x,t,z)])^2}, \tag{3.6}$$

$$w_1(x,t,z) = \frac{k^{2\alpha}}{3} p(t,z) + \frac{2k^{2\alpha}(Y_1[\xi_1(x,t,z)] + 6)}{3(Y_1[\xi_1(x,t,z)])^2}, \tag{3.7}$$

with

$$\xi_1(x,t,z) = kx + (2k^{3\alpha})^{\frac{1}{\alpha}} \int_0^t (p(\tau,z))^{\frac{1}{\alpha}} d\tau + c_1.$$

Case B. If we set $h_0 = -h_2 = \frac{1}{2}$, $h_1 = 0$ in Eq.(2.6), and use Mathematica to solve the resulting system, we will obtain the following sets of solutions

Set2:

$$a_1 = b_1 = c_1 = c_2 = d_1 = e_1 = e_2 = f_1 = 0,\ a_0 = \frac{(2+3p)k^{2\alpha}}{q}, c_0 = -e_0 = \frac{pk^{2\alpha}}{3}, a_2 = 2b_2 = \frac{-3pk^{2\alpha}}{q},$$

$$f_2 = d_2 = k^{2\alpha}, l = \left(\frac{(2+p)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}.$$

Substituting these values in Eq.(3.3) and using Eq.(2.8), we obtain the following soliton wave solutions of Eq.(3.1)

$$u_i(x,t,z) = \frac{(2+3p(t,z))k^{2\alpha}}{q(t,z)} - \frac{3p(t,z)k^{2\alpha}}{2q(t,z)}(2Y_i^2[\xi_2(x,t,z)] + Y_i^{-2}[\xi_2(x,t,z)]), \tag{3.8}$$

$$v_i(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + k^{2\alpha} Y_i^{-2}[\xi_2(x,t,z)], \tag{3.9}$$

$$w_i(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + k^{2\alpha} Y_i^{-2}[\xi_2(x,t,z)], \tag{3.10}$$

with $i = 2,3$ and

$$\xi_2(x,t,z) = kx + k^3 \int_0^t (2 + p(\tau,z))^{\frac{1}{\alpha}} d\tau + c_2.$$

Set3:

$a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = e_1 = e_2 = f_1 = 0,\ a_0 = \frac{(2+4p)k^{2\alpha}}{q},\ c_0 = -e_0 = \frac{pk^{2\alpha}}{3},\ b_2 = \frac{-3pk^{2\alpha}}{2q},$

$f_2 = d_2 = k^2,\ l = \left(\frac{(2+p)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}.$

Substituting these values in Eq.(3.3) and using Eq.(2.8), we obtain the following soliton wave solutions of Eq.(3.1)

$$u_{i+2}(x,t,z) = \frac{(2+4p(t,z))k^{2\alpha}}{q(t,z)} - \frac{3p(t,z)k^{2\alpha}}{2q(t,z)} Y_i^{-2}[\xi_2(x,t,z)], \tag{3.11}$$

$$v_{i+2}(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + k^2 Y_i^{-2}[\xi_2(x,t,z)], \tag{3.12}$$

$$w_{i+2}(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + k^2 Y_i^{-2}[\xi_2(x,t,z)] \tag{3.13}$$

with $i = 2,3$.

Case C. If we set $h_0 = \frac{1}{4}h_2 = 1,\ h_1 = 0$ in Eq.(2.6), and use Mathematica to solve the resulting system, we will obtain the following set of solutions

Set4:

$a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = e_1 = f_1 = f_2 = 0,\ a_0 = \frac{(32+31p)k^{2\alpha}}{q} - \frac{4rk^{2\alpha}}{9},\ a_2 = -\frac{96pk^{2\alpha}}{q},$

$c_0 = -e_0 = \frac{pk^{2\alpha}}{3},\ c_2 = e_2 = 64k^{2\alpha},\ l = \left(\frac{(p-32)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}.$

Substituting these values in Eq.(3.3) and using Eq.(2.8), we obtain the following periodic wave solutions of Eq.(3.1)

$$u_{i+2}(x,t,z) = \frac{(32+31p(t,z))k^{2\alpha}}{q(t,z)} - \frac{4r(t,z)k^{2\alpha}}{9} - \frac{96p(t,z)k^{2\alpha}}{q(t,z)} Y_i^{2}[\xi_3(x,t,z)], \tag{3.14}$$

$$v_{i+2}(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + 64k^{2\alpha} Y_i^{-2}[\xi_3(x,t,z)], \tag{3.15}$$

$$w_{i+2}(x,t,z) = \frac{p(t,z)k^{2\alpha}}{3} + 64k^{2} Y_i^{-2}[\xi_3(x,t,z)] \tag{3.16}$$

with $i = 4,5$ and

$$\xi_3(x,t,z) = kx + k^3 \int_0^t (p(\tau,z)-32)^{\frac{1}{\alpha}} d\tau + c_3.$$

At the end of this section we should remark that, there exists an infinitely number of exact travelling wave solutions for Eq.(1.1); these solutions come from solving the system (3.4) with regard to the fractional Riccati equation (2.6). The above mentioned cases are just to clarify how far our technique is applicable.

White Noise Functional Solutions of Eq.(1.2)

In this section, we use the inverse Hermite transform to obtain white noise functional solutions for Eq.(1.2). The properties of generalized exponential, hyperbolic and trigonometric functions yield that there exists a bounded open set $G \subset \mathbb{R}\times\mathbb{R}_+$, $m<\infty$, $n>0$ such that the solution $\{u(x,t,z), v(x,t,z), w(x,t,z)\}$ of Eq.(3.1) and all its fractional derivatives which are involved in Eq.(3.1) are uniformly bounded for $(x,t,z)\in G\times K_m(n)$, continuous with respect to $(x,t)\in G$ for all $z\in K_m(n)$ and analytic with respect to $z\in K_m(n)$, for all $(x,t)\in G$. From [22], there exist $U(x,t), V(x,t), W(x,t) \in (\mathcal{S})_{-1}$ such that $u(x,t,z) = \tilde{U}(x,t)(z)$, $v(x,t,z)=\tilde{V}(x,t)(z)$ and $w(x,t,z)=\tilde{W}(x,t)(z)$ for all $(x,t,z)\in G\times K_m(n)$ and $\{U(x,t), V(x,t), W(x,t)\}$ solves (in the strong sense in $(\mathcal{S})_{-1}$) Eq.(1.2) in $(\mathcal{S})_{-1}$. Hence, for $P(t)Q(t)R(t)\neq 0$, the white noise functional solutions of Eq.(1.2) can be obtained by applying the inverse Hermite transform to Eqs.(3.5)-(3.16) as follows:

• Stochastic exponential decay wave solution:

$$\begin{aligned} U_1(x,t) &= \frac{k^{2\alpha}P(t)}{Q(t)} - \frac{4k^{2\alpha}R(t)\lozenge P^{\lozenge 2}(t)}{54R(t)\lozenge Q(t)-189P^{\lozenge 2}(t)} \\ &+ \frac{k^{2\alpha}[(6R(t)\lozenge Q(t)-21P^{\lozenge 2}(t))\lozenge(Y_1^{\lozenge}[\Xi_1(x,t)])-6P^{\lozenge 2}(t)]}{P(t)\lozenge Q(t)\lozenge (Y_1^{\lozenge}[\Xi_1(x,t)])^{\lozenge 2}}, \end{aligned} \tag{4.1}$$

$$V_1(x,t) = \frac{k^{2\alpha}}{3}P(t) + \frac{2k^{2\alpha}(Y_1^{\lozenge}[\Xi_1(x,t)]+6)}{3(Y_1^{\lozenge}[\Xi_1(x,t)])^{\lozenge 2}}, \tag{4.2}$$

$$W_1(x,t) = \frac{k^{2\alpha}}{3}P(t) + \frac{2k^{2\alpha}(Y_1^{\lozenge}[\Xi_1(x,t)]+6)}{3(Y_1^{\lozenge}[\Xi_1(x,t)])^{\lozenge 2}}, \tag{4.3}$$

with

$$\Xi_1(x,t,z) = kx + (2k^{3\alpha})^{\frac{1}{\alpha}} \int_0^t (P(\tau))^{\lozenge\frac{1}{\alpha}} d\tau + c_1.$$

• Stochastic soliton wave solutions:

$$U_i(x,t)=\frac{k^{2\alpha}(2+3P(t))}{Q(t,z)}-\frac{3k^{2\alpha}P(t)}{2Q(t)}\Diamond(2Y_i^{\Diamond 2}[\Xi_2(x,t)]+Y_i^{\Diamond(-2)}[\Xi_2(x,t)]), \tag{4.4}$$

$$V_i(x,t)=\frac{k^{2\alpha}P(t)}{3}+k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_2(x,t)], \tag{4.5}$$

$$W_i(x,t)=\frac{k^{2\alpha}P(t)}{3}+k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_2(x,t)], \tag{4.6}$$

$$U_{i+2}(x,t)=\frac{k^{2\alpha}(2+4P(t))}{Q(t)}-\frac{3k^{2\alpha}P(t)}{2Q(t)}\Diamond Y_i^{\Diamond(-2)}[\Xi_2(x,t)], \tag{4.7}$$

$$V_{i+2}(x,t)=\frac{k^{2\alpha}P(t)}{3}+k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_2(x,t)], \tag{4.8}$$

$$W_{i+2}(x,t)=\frac{k^{2\alpha}P(t)}{3}+k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_2(x,t)], \tag{4.9}$$

with $i=2,3$ and

$$\Xi_2(x,t)=kx+k^3\int_0^t(2+P(\tau))^{\Diamond\frac{1}{\alpha}}d\tau+c_2.$$

• Stochastic periodic wave solutions:

$$U_{i+2}(x,t)=\frac{k^{2\alpha}(32+31P(t))}{Q(t)}-\frac{4k^{2\alpha}R(t)}{9}-\frac{96k^{2\alpha}P(t)}{Q(t)}\Diamond Y_i^{\Diamond 2}[\Xi_3(x,t)], \tag{4.10}$$

$$V_{i+2}(x,t)=\frac{k^{2\alpha}P(t)}{3}+64k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_3(x,t)], \tag{4.11}$$

$$W_{i+2}(x,t)=\frac{k^{2\alpha}P(t)}{3}+64k^{2\alpha}Y_i^{\Diamond(-2)}[\Xi_3(x,t)], \tag{4.12}$$

with $i=4,5$ and

$$\Xi_3(x,t)=kx+k^3\int_0^t(P(\tau)-32)^{\Diamond\frac{1}{\alpha}}d\tau+c_3.$$

We observe that for different forms of $P(t), Q(t)$ and $R(t),$ we can get different solutions of Eq.(1.2) from Eqs.(4.1)-(3.12).

Examples

Let $\dot{B}_t$ be the Gaussian white noise, where B_t is Brownian motion. We have the Hermite transform

$$\tilde{B}(t,z) = \sum_{k=1}^{\infty} z_k \int_0^t \pi_k(s)ds \text{. Since } E_\alpha^\lozenge(B_t) = E_\alpha(B_t - t^2/2),$$

we have

$$\tan_\alpha^\lozenge(B_t) = \tan_\alpha(B_t - t^2/2),\ \cot_\alpha^\lozenge(B_t) = \cot_\alpha(B_t - t^2/2),$$
$$\tanh_\alpha^\lozenge(B_t) = \tanh_\alpha(B_t - t^2/2),\ \coth_\alpha^\lozenge(B_t) = \coth_\alpha(B_t - t^2/2),$$
$$\text{sech}_\alpha^\lozenge(B_t) = \text{sech}_\alpha(B_t - t^2/2) \text{ and } \text{csch}_\alpha^\lozenge(B_t) = \text{csch}_\alpha(B_t - t^2/2).$$

Suppose $P(t) = R(t) = \theta_1 Q(t)$ and $Q(t) = (q(t) + \theta_2 \dot{B}_t)^\alpha,$

where θ_1, θ_2 are arbitrary constants and $q(t)$ is integrable or bounded measurable function on R_+. Hence, for $Q(t) \neq 0$ the white noise functional solutions of Eq.(1.2) in non-Wick version are as follows

$$U_8(x,t) = \theta_1 k^{2\alpha} - \frac{4\theta_1 k^{2\alpha} Q(t)}{9(6-21\theta_1)} + k^{2\alpha}[(6-21\theta_1)Y_1[\Xi_1^*(x,t)] - 6\theta_1]Y_1^{-2}[\Xi_1^*(x,t)], \quad (5.1)$$

$$V_8(x,t) = \frac{\theta_1 k^{2\alpha} Q(t)}{3} + \frac{2k^{2\alpha}}{3}(Y_1[\Xi_1^*(x,t)] + 6)Y_1^{-2}[\Xi_1^*(x,t)], \quad (5.2)$$

$$W_8(x,t) = \frac{\theta_1 k^{2\alpha} Q(t)}{3} + \frac{2k^{2\alpha}}{3}(Y_1[\Xi_1^*(x,t)] + 6)Y_1^{-2}[\Xi_1^*(x,t)], \quad (5.3)$$

with

$$\Xi_1^*(x,t) = kx + (2k^3)^{\frac{1}{\alpha}} \theta_1 \left(\int_0^t q(\tau)d\tau - \theta_2 B_t + \frac{\theta_2 t^2}{2} \right) + c_1,$$

$$U_{i+7}(x,t) = \frac{k^{2\alpha}(2 + 3\theta_1 Q(t))}{Q(t)} - \frac{3\theta_1 k^{2\alpha}}{2}(2Y_i^2[\Xi_2^*(x,t)] + Y_i^{-2}[\Xi_2^*(x,t)]), \quad (5.4)$$

$$V_{i+7}(x,t) = \frac{\theta_1 k^{2\alpha} Q(t)}{3} + k^{2\alpha} Y_i^{-2}[\Xi_2^*(x,t)], \quad (5.5)$$

$$W_{i+7}(x,t) = \frac{\theta_1 k^{2\alpha} Q(t)}{3} + k^{2\alpha} Y_i^{-2}[\Xi_2^*(x,t)], \quad (5.6)$$

$$U_{i+9}(x,t)=\frac{k^{2\alpha}(2+4\theta_1 Q(t))}{Q(t)}+\frac{3\theta_1 k^{2\alpha}}{2}Y_i^{-2}[\Xi_2^*(x,t)], \tag{5.7}$$

$$V_{i+9}(x,t)=\frac{k^{2\alpha}\theta_1 Q(t)}{3}-k^{2\alpha}Y_i^{-2}[\Xi_2^*(x,t)], \tag{5.8}$$

$$W_{i+9}(x,t)=\frac{k^{2\alpha}\theta_1 Q(t)}{3}-k^{2\alpha}Y_i^{-2}[\Xi_2^*(x,t)], \tag{5.9}$$

with $i=2,3$ and

$$\Xi_2^*(x,t)=kx+k^3\left(3\int_0^t q(\tau)d\tau-\theta_2 B_t+\frac{\theta_2 t^2}{2}\right)+c_2,$$

$$U_{i+9}(x,t)=\frac{k^{2\alpha}(32+31\theta_1 Q(t))}{Q(t)}+\frac{4k^{2\alpha}\theta_1 Q(t)}{9}-96\theta_1 k^{2\alpha}Y_i^2[\Xi_3^*(x,t)], \tag{5.10}$$

$$V_{i+9}(x,t)=\frac{k^{2\alpha}\theta_1 Q(t)}{3}-64k^{2\alpha}Y_i^{-2}[\Xi_3^*(x,t)], \tag{5.11}$$

$$V_{i+9}(x,t)=\frac{k^{2\alpha}\theta_1 Q(t)}{3}-64k^{2\alpha}Y_i^{-2}[\Xi_3^*(x,t)], \tag{5.12}$$

with $i=4,5$ and

$$\Xi_3^*(x,t)=kx+k^3\left(30\int_0^t q(\tau)d\tau+\theta_2 B_t-\frac{\theta_2 t^2}{2}\right)+c_3.$$

Summary and Discussion

Our first interest in this work is to implement new strategies that give white noise functional solutions of the variable coefficients Wick-type stochastic fractional Hirota-Satsuma coupled KdV equations. The strategies that will be pursued in this work rest mainly on Hermite transform, white noise theory and modified fractional sub-equation method, all of which are employed to find white noise functional solutions of Eq.(1.2). The proposed schemes, as we believe, are entirely new and introduce new solutions in addition to the well-known traditional solutions. The ease of using these methods are showed its power to determine shock or solitary type of solutions. Obviously, the planner which we have proposed in this paper can be also applied to other nonlinear PDEs in mathematical physics such as KdV-Burgers, modified KdV-Burgers, Sawada-Kotera, Zhiber-Shabat and Benjamin-Bona-Mahony equations. Note that, if $\alpha=1$, Eq.(1.2) is reduced to the stochastic generalized Hirota-Satsuma coupled KdV equations. Also, if $\alpha=1$, $p=0.5$ and $r=-q=3$, Eq.(1.1) is reduced to the generalized Hirota-Satsuma coupled KdV equations. Hence, our results can be considered a generalization of the work due to Ghany [9] and Wu [37]. Moreover, there is a unitary mapping between the Gaussian white noise space and the Poisson white noise space, this connection was given by Benth and Gjerde [1]. Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients $P(t), Q(t)$ and $R(t)$ are Poisson white noise functions in Eq.(1.2).

References

[1] E. Benth and J. Gjerde, *Potential Anal.* 8 (1998), 179-193.
[2] V. Daftardar-Gejji and H. Jafari, *Appl. Math. Comput.* 189 (2007), 541-548.
[3] V. Daftardar-Gejji and S. Bhalekar, *Appl. Math. Comput.* 202 (2008), 113-120.
[4] A. de Bouard and A. Debussche, *J. Funct. Anal.* 154 (1998), 215-251.
[5] A. de Bouard and A. Debussche, *J. Funct. Anal.* 169 (1999), 532-558.
[6] A. Debussche and J. Printems, *Physica D* 134 (1999), 200-226.
[7] A. Debussche and J. Printems, *J. Comput. Anal. Appl.* 3 (2001), 183-206.
[8] Z. Z. Ganji, D. D. Ganji and Y. Rostamiyan, *Applied Mathematical Modelling* 33 (2009), 3107-3113.
[9] H. A. Ghany, *Chin. J. Phys.* 49 (2011), 926-940.
[10] H. A. Ghany, *Chin. J. Phys.* (2013), 875.
[11] H. A. Ghany and A. Hyder, *International Review of Physics* 6 (2012), 153-157.
[12] H. A. Ghany and A. Hyder, *J. Comput. Anal. Appl.* 15 (2013), 1332-1343.
[13] H. A. Ghany and A. Hyder, *Kuwait Journal of Science.* 41 (2013).
[14] H. A. Ghany and A. Hyder, Chin. Phys. B, 23 (2014), 060503.
[15] H. A. Ghany and M. S. Mohammed, *Chin. J. Phys.*, 50 (2012), 619-627.
[16] H. A. Ghany, A. S. Okb El Bab, A. M. Zabal and A. Hyder, *Chin. Phys. B* 22 (2013) , 080501.
[17] A. Golbabai and K. Sayevand, *Nonlinear Sci. Lett. A* 1 (2010), 147-154.
[18] A. Golbabai and K. Sayevand, *J. Comput. Math. Appl.* 62 (2011), 1003-1012.
[19] J. H. He, *International Journal of Non-Linear Mechanics* 35 (2000), 37-43.
[20] J. H. He, S. K. Elagan and Z. B. Li, *Phys. Lett. A* 376 (2012), 257-259.
[21] R. Hirota and J. Satsuma, *Physics Letters A* 85 (1981), 407-408.
[22] H. Holden, B. Øsendal, J. Ubøe and T. Zhang, *Stochastic partial differential equations*, Birhkäuser: Basel, 1996.
[23] G. Jumarie, *Comput. Math. Appl.* 51 (2006), 1367-1376.
[24] Z. B. Li and H. J. He, *Math. Comput. Appl.* 15 (2010),970-973.
[25] B. A. Li and M. L. Wang, *Chin. Phys. B* 14 (2005), 1698-1706.

[26] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons: New York, 1993.

[27] Z. Odibat and S. Momani, *Chaos Solitons and Fractals* 36 (2008), 167-174.

[28] K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press: New York, 1974.

[29] I. Podlubny, *Fractional differential equations*, Academic Press: New York, 1999.

[30] J. Printems, *J. Differential Equations* 153 (1999), 338-373.

[31] M. Shateri and D. D. Ganji, *International Journal of Differential Equations* 2010 (2010), 1-11.

[32] N. H. Sweilam, M. M. Khader and R. F. Al-Bar, *Phys. Lett. A* 371 (2007) 26-33

[33] M. Wadati, *J. Phys. Soc. Jpn.* 52 (1983), 2642-2648.

[34] M. L. Wang and Y. B. Zhou, *Phys. Lett. A* 318 (2003), 84-92.

[35] Y. M. Wang, M. L. Wang and J. L. Zhang, *Chin. Phys. B* 12 (2003), 1341-1348.

[36] Y. M. Wang, M. L. Wang, J. L. Zhang, D. F. Ren and Z. D. Fang, *Chin. Phys. B* 12 (2003), 825-830.

[37] Y. T. Wu, X. G. Geng, X. B. Hu and S.M. Zhu, *Phys. Lett. A* 255 (1999), 259-264.

[38] S. Zhang, Q. A. Zong, D. Liu and Q. Gao, *Communications in Fractional Calculus* 1 (2010), 48-52.

[39] Y. B. Zhou, M. L. Wang and Y. M. Wang, *Phys. Lett. A* 308 (2003), 31-36.