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Modeling of Two-Dimensional Supercritical Flow

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Abstract

A derivation of equation system of two-dimensional supercritical flow motion for longitudinal slope is given in this work. It is based on classical equations of flow motion in physical stream plane. A free-flow mode is taken for investigation assuming that resistance forces can be neglected. The flow spreads freely in a wide discharge channel downstream the rectangular pipe.

Keywords: non-pressure rectangular pipe, wide discharge channel with longitudinal slope, supercritical flow, irrotational flow, free spreading of flow.

Introduction

Inspection of the road water disposal facilities indicates their poor operational reliability caused by washout and decay of the facilities understructure. That is why it is necessary to upgrade the mathematical model of calculating the parameters of supercritical flow in the water sluice facilities at free-flow and low-pressure flow modes downstream the culvert in wide discharge channel [1].

Low adequacy of models used in the present-day methods causes the reduction of facilities operating life, early operation failures and the decay of the discharge channel understructure in water sluice facilities of watering and irrigation schemes and also the decay of bottom discharges at water-storage basins. Comparison of experimental and calculated profiles of boundary streamlines of the flow free spreading downstream rectangular pipes that was defined according to the most popular and available methods by I.A. Sherenkov [2] and G.A. Lilitskiy [3] has specified that the results of these methods are not always accurate enough for applied calculations.

A real three-dimensional flow freely spreading in a wide discharge channel could be quite adequately described by two-dimensional equations. These equations are possible if we neglect by projections of velocities and accelerations of water flow motion perpendicular to the surface of a discharge channel. The problem of defining the depths, velocities and geometric pattern of the two-dimensional outflow in a physical stream plane reduces to a system of two quasilinear equations in quotient derivatives. Today this problem has no solution.

Rearrangement of two-dimensional problem from physical stream plane into velocity hodograph plane helps to change the nonlinear system of differential equations in quotient derivatives onto linear homogeneous system that offers analytical solution for the problem of defining depths, velocities and geometric pattern of the flow.

In order to increase the operational reliability of water sluice facilities it is necessary to raise the adequacy of simulative (calculated) and real (experimental) parameters of the flow freely spreading downstream the rectangular culvert in the wide discharge channel.

Objective of the research is to establish the equations of two-dimensional supercritical flow motion provided it spreads freely in a flat mild slope and also to define limits of irrotationall flow spreading, stream velocities and depths.

General conclusions of the research:

- system of equations of irrotational supercritical flow motion in the velocity hodograph plane for a mild slope was established in the research;
- basic system of motion equations for horizontal slope was established in particular;
- boundary problem of defining the flow parameters and its geometry pattern in any point of the stream profile was solved by a simplified method;
- better adequacy of the achieved solution in comparison with previous models was established.

Methodology

A. Basic equations of two-dimensional supercritical flow motion

In works [4,5] the equations of two-dimensional open water flows are deducted from the equations by L. Eiler supplemented by the additive components that take into account forces of flow resistance. The velocity and acceleration perpendicular constituents are considered to be equal to zero and all inertial factors containing these

constituents as multiplier are also zeroed. All resistance forces that include this constituent are neglected.
Then the equations are averaged according to the depth and it results in equations of two-dimensional flow model.
System of dynamic two-dimensional steady flows in orthographic Cartesian coordinates has the following form [4]:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = -g \dfrac{\partial}{\partial x}(z_0 + h) - T_x; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} = -g \dfrac{\partial}{\partial x}(z_0 + h) - T_y, \end{cases} \tag{1}$$

where u_x, u_y – are velocity vector projections at axes $0x$ and $0y$ with average depth: $0x$ – is X-axis of the flow; $0y$ – is axis supplementing the axis $0x$ to the right-handed coordinate system; g – is free-fall acceleration; z_0 – is a slope surface point; h – local flow depth; T_x, T_y – resistant forces constituents, related to the fluid mass unit.
In equations (1) the stream is supposed to be perpendicular to the vertical axis $0z$ provided the slope is horizontal.

Equation of the flow continuity is the following:

$$\frac{\partial}{\partial x}(hu_x) + \frac{\partial}{\partial y}(hu_y) = 0. \tag{2}$$

In total the system of equations of the flow motion and continuity forms the following system of the flow model equations:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = -g \dfrac{\partial}{\partial x}(z_0 + h) - T_x; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} = -g \dfrac{\partial}{\partial x}(z_0 + h) - T_y; \\ \dfrac{\partial}{\partial x}(hu_x) + \dfrac{\partial}{\partial y}(hu_y) = 0, \end{cases} \tag{3}$$

this is a closed system of three equations in quotient derivatives relatively to three unknowns u_x, u_y, h.
System of equations (3) is a system of highly nonlinear equations of mathematical physics [6,7]. Absence of analytical solution to this system of equations causes some difficulties for studying such models by means of mathematical experiment.
Study of the flow motion equations usually starts with the easiest case:

- resistance forces are being neglected: $T_x = T_y = 0$;
- movement is considered to be circulation-free;
- flow slope is considered to be horizontal: $z_0 = 0$.

Despite the significant idealization of the stream flows described by this work and by existing research experience [2,4], this model has wide practical application in hydraulic engineering and road construction.
For circulation-free flux:

$$\Omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0 \tag{4}$$

There is the potential function φ, that: $u_x = \frac{\partial \varphi}{\partial x}$; $u_y = \frac{\partial \varphi}{\partial y}$. (5)

Then system of equations (3) is reduced to one equation relating to potential function:

$$\frac{\partial^2 \varphi}{\partial x^2}\left[c^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2\right] - 2\frac{\partial \varphi}{\partial x}\cdot\frac{\partial \varphi}{\partial y}\cdot\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2}\cdot\left[c^2 - \left(\frac{\partial \varphi}{\partial y}\right)^2\right] = 0, \tag{6}$$

where $c = \sqrt{gh}$.

Many researchers studying the flow hydraulics mention this equation: [4,5,8,9,10,11,12,13]. Bernulli integral for two-dimensional flows is used alongside with the equation (6):

$$H = h + \frac{u^2}{2g} = \text{const}. \tag{7}$$

Equation of the flow continuity (2) gives opportunity to claim that alongside with the potential stream function [14] there is the function $\psi = \psi(x, y)$, meeting the following conditions:

$$hu_x = \frac{\partial \psi}{\partial y};\ hu_y = -\frac{\partial \psi}{\partial x}.$$

Streamline is a line where velocity vector from every point is tangential to this line:

$$\frac{dx}{u_x} = \frac{dy}{u_y}.$$

In this case the line $\psi = \text{const}$ is a streamline.

The equation similar to the equation (6) is correct for the stream function:

$$\frac{\partial^2 \psi}{\partial x^2}\left[1 - \frac{1}{c^2 h^2}\left(\frac{\partial \psi}{\partial y}\right)^2\right] + \frac{\partial^2 \Psi}{\partial y^2}\left[1 - \frac{1}{c^2 h^2}\left(\frac{\partial \psi}{\partial x}\right)^2\right] + \frac{2}{c^2 h^2}\frac{\partial \psi}{\partial x}\cdot\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial x \partial y} = 0. \tag{8}$$

System of equations (3) or equations (6)-(8) are initial for solving different applied problems of the stream of two-dimensional open water flows.

The equations are given in this section in order to study and understand the existing methods of calculating two-dimensional flows.

In this work the water flow motion equation is based on rearrangement of equations (6), (8) into special form by change to a plane of independent coordinates (u, θ), i.e. by change to velocity hodograph plane, where functions ψ, φ are:

$$\psi = \psi(\tau, \theta);\ \varphi = \varphi(\tau, \theta),$$

where $\tau = \frac{V^2}{2gH_0}$; $V^2 = u_x^2 + u_y^2$ – is the velocity modulus square.

Results

A. Rearrangement of the equation system of two-dimensional supercritical flow motion regarding the slope

The derivation of the equation system of two-dimensional supercritical irrotational flow motion in velocity hodograph plane provided the slope is even and horizontal was given in the works [14,15].

An objective of this work is to derive the similar system of motion equations for a flat mild slope.

Take system of the flow motion equation in a physical plane as initial one [4]:

$$\begin{cases} u_x \dfrac{\partial u_x}{\partial x} + u_y \dfrac{\partial u_x}{\partial y} = g \sin \mu - g \cos \mu \left(\dfrac{\partial z_0}{\partial x} + \dfrac{\partial h}{\partial x} \right) - T_x; \\ u_x \dfrac{\partial u_y}{\partial x} + u_y \dfrac{\partial u_y}{\partial y} = -g \cos \mu \left(\dfrac{\partial z_0}{\partial y} + \dfrac{\partial h}{\partial y} \right) - T_x, \end{cases} \tag{9}$$

where:

$u_x, u_y -$ is a projection of a local velocity vector;

$g -$ is gravity acceleration;

$\mu -$ is longitudinal slope angel (pic. 1);

$z_0 -$ is a stream bottom coordinate;

$h -$ is local flow depth;

$T_x, T_y -$ are constituents of resistance forces referred to the fluid mass unit.

Hydrodynamic flow thrust H could be expressed as:

$$H = t + \frac{u^2}{2g} = a - x \sin \mu + (z_0 + h) \cos \mu + \frac{u_x^2 + u_y^2}{2g}. \tag{10}$$

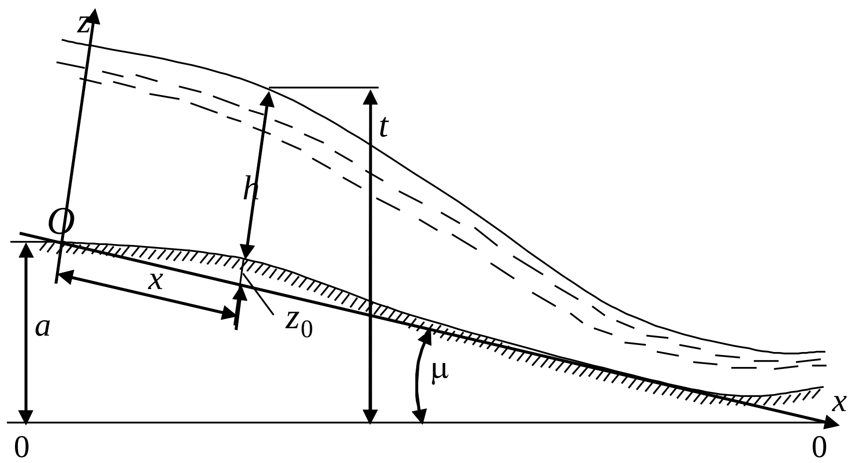

Figure 1: Choice of coordinate system for flows with steep slopes

If slope is flat then $z_0 = 0$, and using the equation (9), we will get [16,17]:

$$\begin{cases} g\dfrac{\partial H}{\partial x} = -T_x + u_y \cdot \Omega; \\ g\dfrac{\partial H}{\partial y} = -T_y + u_x \cdot \Omega, \end{cases} \tag{11}$$

where

$$\Omega = \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}.$$

For irrotational flow if $\Omega = 0$ and there are no forces resisting the flow $T_x = T_x = 0$ then the system (11) implies that $\frac{\partial H}{\partial x} = 0, \frac{\partial H}{\partial y} = 0$ and it follows that

$$H = a - x\sin\mu + h\cos\mu + \frac{u_x^2 + u_y^2}{2g} = const \tag{12}$$

The equation of the flow continuity takes the following form:

$$\frac{\partial(hu_x)}{\partial x} + \frac{\partial(hu_y)}{\partial y} = 0, \tag{13}$$

Where the depth h is calculated along the normal line to the channel bottom.

(13) implies the existence of the stream function $\psi(x, y)$ that ensures meeting the conditions:

$$hu_x = \frac{\partial\psi}{\partial y}; \quad hu_y = -\frac{\partial\psi}{\partial x}. \tag{14}$$

The constant in (12) could be defined from the constraints:

$$x = 0; \quad h = h_0; \quad u = u_0; \quad H = a + h_0\cos\mu + \frac{u_0^2}{2g}; \tag{15}$$

assuming $a = 0$, write (15) in the form:

$$H_0 = h_0\cos\mu + \frac{u_0^2}{2g}. \tag{16}$$

Then the expression for hydrodynamic thrust will have the following form:

$$H = H_0 = -x\sin\mu + h\cos\mu + \frac{u^2}{2g}. \tag{17}$$

Let's introduce the expression

$$H_1(x) = H_0 + x\sin\mu = h\cos\mu + \frac{u^2}{2g}. \tag{18}$$

Let's introduce the flow velocity coefficient τ:

$$\tau = \frac{u^2}{2gH_1(x)}, \tag{19}$$

then

$$u = \tau^{1/2}\sqrt{2gH_1(x)} \text{ and } h = \frac{H_1(x)(1-\tau)}{\cos\mu}. \tag{20}$$

The connection between physical plane of the flux $(x;y)$ and velocity hodograph plane $(\tau;\theta)$ is true for the irrotational flow [14]:

$$dz = \left(d\varphi + i\frac{h_0}{h}d\psi\right)\frac{1}{u}e^{i\theta}, \tag{21}$$

where $z = x + iy$; $\theta -$ is an angle between the velocity vector and axis Ox;

$u -$ is flow local velocity modulus;

$h -$ is local flow depth;

$h_0, u_0 -$ is initial depth and velocity at the output if $x = 0$;

$\varphi -$ is potential function; $\psi -$ is stream function.

Then the following equalities are true:

$$u_x = \frac{\partial\varphi}{\partial x}; \quad u_y = \frac{\partial\varphi}{\partial y}; \quad hu_x = \frac{\partial\psi}{\partial y}; \quad hu_y = -\frac{\partial\psi}{\partial x}. \tag{22}$$

By analogy to the method by S.A. Chaplygin for the perfect gas in case with the horizontal slope the following system is derived from the equations (21) and (22) [14]

$$\begin{cases} \dfrac{1}{u}\dfrac{\partial\varphi}{\partial u} = \dfrac{d}{du}\left(\dfrac{h_0}{uh}\right)\dfrac{\partial\psi}{\partial\theta}; \\ \dfrac{1}{u}\dfrac{\partial\varphi}{\partial\theta} = \dfrac{h_0}{h}\cdot\dfrac{\partial\psi}{\partial u}. \end{cases} \tag{23}$$

It is an intermediate system for a transition to the velocity hodograph plane. It is easier to go to the parameter τ in this system. For this purpose we need to determine the derivatives $\dfrac{\partial\psi}{\partial u}$ and $\dfrac{d}{du}\left(\dfrac{h_0}{uh}\right)$.

First let's determine the derivative $\dfrac{\partial u}{\partial\psi}$ from (20)

$$\frac{\partial u}{\partial\psi} = \frac{\partial\left(\tau^{1/2}\sqrt{2gH_1(x)}\right)}{\partial\psi} =$$

$$\frac{1}{2\tau^{1/2}}\sqrt{2gH_1(x)}\frac{\partial\tau}{\partial\psi} \tag{24}$$

$$+\frac{\tau^{1/2}}{\sqrt{2gH_1(x)}}\cdot g\sin\mu\frac{\partial x}{\partial\psi}.$$

The forth equality (22) implies that

$$\frac{\partial\psi}{\partial x} = -hu\sin\theta, \tag{25}$$

Or with regard to (20)

$$\frac{\partial\psi}{\partial x} = -\frac{H_1(x)(1-\tau)}{\cos\mu}\tau^{1/2}\sqrt{2gH_1(x)}\sin\theta. \tag{26}$$

Using (26), we rearrange the equality (24) to the form:

$$\frac{\partial u}{\partial\psi} = \frac{\sqrt{2gH_1(x)}}{2\tau^{1/2}}\frac{\partial\tau}{\partial\psi} - \frac{g\sin\mu}{\sqrt{2gH_1(x)}}\cdot\frac{\cos\mu}{H_1(x)(1-\tau)\sin\theta\sqrt{2gH_1(x)}}. \tag{27}$$

In a particular case if $\mu=0, H=H_0$ we have $\frac{\partial u}{\partial\psi} = \frac{\sqrt{2gH_0}}{2\tau^{1/2}}\cdot\frac{\partial\tau}{\partial\psi}$ and the second equation of the system (23) changes to the following form:

$$\frac{\partial\varphi}{\partial\theta} = \frac{2h_0\tau}{H_0(1-\tau)}\cdot\frac{\partial\psi}{\partial\tau},$$

i.e. it coincides completely to the second equation of the system, which is correct for the horizontal slope [14]: $\begin{cases}\frac{\partial\varphi}{\partial\tau} = -\frac{h_0}{2H_0}\frac{1-3\tau}{\tau(1-\tau)^2}\frac{\partial\psi}{\partial\theta}; \\ \frac{\partial\varphi}{\partial\theta} = 2\frac{h_0}{H_0}\frac{\tau}{1-\tau}\frac{\partial\psi}{\partial\tau}.\end{cases}$

Then from the equation (27) find the derivative $\frac{\partial\psi}{\partial u}$

$$\frac{\partial\psi}{\partial u} = \frac{1}{\frac{\sqrt{2gH_1(x)}}{2\tau^{1/2}}\frac{\partial\tau}{\partial\psi} - \frac{g\sin\mu}{\sqrt{2gH_1(x)}}\cdot\frac{\cos\mu}{H_1(x)(1-\tau)\sin\theta\sqrt{2gH_1(x)}}}. \tag{28}$$

Rearrange the right part of the equality (28) to the form

$$\frac{\partial\psi}{\partial u} = \frac{\tau^{1/2}\cdot 2H_1^2(x)(1-\tau)\cdot\sin\theta\cdot\frac{\partial\psi}{\partial\tau}}{\sqrt{2gH_1(x)}H_1^2(x)(1-\tau)\cdot\sin\theta - \tau^{1/2}\sin\mu\cdot\cos\mu\cdot\frac{\partial\psi}{\partial\tau}}.$$

Then we rearrange the second equation of the system (23) taking into account (20)

$$\frac{\partial\varphi}{\partial\theta} = \frac{h_0\cdot\tau\cdot 2H_1(x)\sqrt{2gH_1(x)}\cdot\sin\theta\cdot\cos\mu\cdot\frac{\partial\psi}{\partial\tau}}{\sqrt{2gH_1(x)}H_1^2(x)(1-\tau)\cdot\sin\theta - \tau^{1/2}\sin\mu\cdot\cos\mu\cdot\frac{\partial\psi}{\partial\tau}}. \tag{29}$$

Considering (18), we find the derivative $\frac{d}{du}\left(\frac{1}{uh}\right)$:

$$\frac{d}{du}\left(\frac{1}{uh}\right)=\frac{d}{du}\left(\frac{\cos\mu}{u\left(H_0+x\sin\mu-\frac{u^2}{2g}\right)}\right)=$$

$$-\frac{\cos\mu\cdot\left(H_0+x\sin\mu-\frac{3u^2}{2g}+u\cdot\sin\mu\frac{dx}{du}\right)}{u^2\left(H_0+x\sin\mu-\frac{u^2}{2g}\right)^2}. \tag{30}$$

For rearrangement of (30) we find the derivative $\frac{du}{dx}$ with regard to (18)

$$\frac{du}{dx}=\frac{\sqrt{2gH_1(x)}}{2\tau^{1/2}}\frac{d\tau}{dx}+\frac{\tau^{1/2}\sin\mu}{\sqrt{2gH_1(x)}}. \tag{31}$$

Then find the derivative $\frac{d\tau}{dx}$

$$\frac{d\tau}{dx}=\frac{\partial\tau}{\partial\psi}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\tau}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial x}. \tag{32}$$

Rearrange the equality (32) with regard to (20) and (22)

$$\frac{d\tau}{dx}=-\frac{H_1(x)(1-\tau)\tau^{1/2}\sqrt{2gH_1(x)}\sin\theta}{\cos\mu\cdot\frac{\partial\psi}{\partial\tau}}$$

$$+\frac{\tau^{1/2}\sqrt{2gH_1(x)}\cos\theta}{\frac{\partial\varphi}{\partial\tau}}. \tag{33}$$

Taking into account (33) the equality (31) has the following form

$$\frac{du}{dx}=$$

$$g\frac{\tau^{1/2}\sin\mu\cos\mu\frac{\partial\varphi}{\partial\tau}\frac{\partial\psi}{\partial\tau}+\sqrt{2gH_1(x)}H_1^2(x)\cdot\left(\cos\mu\cos\theta\frac{\partial\psi}{\partial\tau}-(1-\tau)\sin\theta\cdot\frac{\partial\varphi}{\partial\tau}\right)+}{\sqrt{2gH_1(x)}\cos\mu\frac{\partial\psi}{\partial\tau}\cdot\frac{\partial\varphi}{\partial\tau}}. \tag{34}$$

Form (34) we derive

$$\frac{dx}{du}=$$

$$\frac{g^{-1}\cdot\sqrt{2gH_1(x)}\cos\mu\frac{\partial\psi}{\partial\tau}\cdot\frac{\partial\varphi}{\partial\tau}}{\tau^{1/2}\sin\mu\cos\mu\frac{\partial\varphi}{\partial\tau}\frac{\partial\psi}{\partial\tau}+\sqrt{2gH_1(x)}H_1^2(x)\cdot\left(\cos\mu\cos\theta\frac{\partial\psi}{\partial\tau}-(1-\tau)\sin\theta\cdot\frac{\partial\varphi}{\partial\tau}\right)}. \quad (35)$$

Put the expression for $\frac{dx}{du}$ into (30), taking into account (24), after the reduction we get

$$\frac{d}{du}\left(\frac{1}{uh}\right)=-\frac{\cos\mu\cdot\left(H_0+x\sin\mu-3\tau\cdot H_1(x)\right)}{2g\tau H_1(x)\cdot\left(H_0+x\sin\mu-\tau\cdot H_1(x)\right)^2}-$$

$$\frac{\cos\mu\cdot\sin\mu}{\tau^{1/2}\sqrt{2gH_1(x)}\left(H_0+x\sin\mu-\tau\cdot H_1(x)\right)^2}\times \quad 36$$

$$\times\frac{g^{-1}\cdot\sqrt{2gH_1(x)}\cos\mu\frac{\partial\psi}{\partial\tau}\cdot\frac{\partial\varphi}{\partial\tau}}{\tau^{1/2}\sin\mu\cos\mu\frac{\partial\varphi}{\partial\tau}\frac{\partial\psi}{\partial\tau}+\sqrt{2gH_1(x)}H_1^2(x)\cdot\left(\cos\mu\cos\theta\frac{\partial\psi}{\partial\tau}-(1-\tau)\sin\theta\cdot\frac{\partial\varphi}{\partial\tau}\right)}.$$

Then in order to determine the derivative $\frac{\partial\varphi}{\partial u}$ in the system (23) we find $\frac{\partial u}{\partial\varphi}$

$$\frac{\partial u}{\partial\varphi}=\frac{\partial\left(\tau^{1/2}\sqrt{2gH_1(x)}\right)}{\partial\varphi}=\frac{1}{2\tau^{1/2}}\sqrt{2gH_1(x)}\frac{\partial\tau}{\partial\varphi}+\frac{\tau^{1/2}}{\sqrt{2gH_1(x)}}\cdot g\sin\mu\frac{\partial x}{\partial\varphi}. \quad (37)$$

(22) implies that

$$\frac{\partial\varphi}{\partial x}=u\cos\theta, \quad (38)$$

Or with regard to (24)

$$\frac{\partial\varphi}{\partial x}=\tau^{1/2}\sqrt{2gH_1(x)}\cos\theta. \quad (39)$$

Using (39), we rearrange the equality (37) to the form:

$$\frac{\partial u}{\partial\varphi}=\frac{\sqrt{2gH_1(x)}}{2\tau^{1/2}\frac{\partial\varphi}{\partial\tau}}-\frac{\sin\mu}{2H_1(x)\cos\theta}. \quad (40)$$

Form (40) we derive $\frac{\partial \varphi}{\partial u}$

$$\frac{\partial \varphi}{\partial u}=\frac{2\tau^{1/2}H_1(x)\cos\theta\frac{\partial \varphi}{\partial \tau}}{\sqrt{2gH_1(x)}H_1(x)\cos\theta+\tau^{1/2}\sin\mu\frac{\partial \varphi}{\partial \tau}}. \tag{41}$$

Taking into account (24), we rearrange the left part of the first equation (23)

$$\frac{1}{u}\frac{\partial \varphi}{\partial u}=\frac{\sqrt{2H_1(x)}\cos\theta\frac{\partial \varphi}{\partial \tau}}{\left(\sqrt{2gH_1(x)}H_1(x)\cos\theta+\tau^{1/2}\sin\mu\frac{\partial \varphi}{\partial \tau}\right)\sqrt{g}}. \tag{42}$$

Taking into account formulas (36) and (42) the first equation of the system (23) will have the following form:

$$\frac{\partial \varphi}{\partial \tau}=-\frac{h_0\left(\sqrt{2gH_1(x)}H_1(x)\cos\theta+\tau^{1/2}\sin\mu\frac{\partial \varphi}{\partial \tau}\right)\sqrt{g}}{\sqrt{2H_1(x)}\cos\theta}$$
$$\left\{\frac{\cos\mu\cdot\left(H_0+x\sin\mu-3\tau\cdot H_1(x)\right)}{2g\tau H_1(x)\cdot\left(H_0+x\sin\mu-\tau\cdot H_1(x)\right)^2}+\right. \tag{43}$$
$$\left.+\frac{g^{-1}\tau^{-1/2}\left(H_0+x\sin\mu-\tau\cdot H_1(x)\right)^{-2}\cdot\cos^2\mu\cdot\sin\mu\cdot\frac{\partial \psi}{\partial \tau}\cdot\frac{\partial \varphi}{\partial \tau}}{\left[\tau^{1/2}\sin\mu\cos\mu\frac{\partial \varphi}{\partial \tau}\frac{\partial \psi}{\partial \tau}+\sqrt{2gH_1(x)}H_1^2(x)\cdot\left(\cos\mu\cos\theta\frac{\partial \psi}{\partial \tau}-(1-\tau)\sin\theta\cdot\frac{\partial \varphi}{\partial \tau}\right)\right]}\right\}$$
$$\frac{\partial \psi}{\partial \theta}.$$

Equations (29) and (43) make equation system of two-dimensional supercritical flow motion considering the slope

$$\begin{cases} \dfrac{\partial\varphi}{\partial\tau} = -\dfrac{h_0\left(\sqrt{2gH_1(x)}H_1(x)\cos\theta + \tau^{1/2}\sin\mu\dfrac{\partial\varphi}{\partial\tau}\right)\sqrt{g}}{\sqrt{2H_1(x)}\cos\theta} \\ \left\{\dfrac{\cos\mu\cdot\left(H_0 + x\sin\mu - 3\tau\cdot H_1(x)\right)}{2g\tau H_1(x)\cdot\left(H_0 + x\sin\mu - \tau\cdot H_1(x)\right)^2} + \right. \\ \left. + \dfrac{g^{-1}\tau^{-1/2}\left(H_0 + x\sin\mu - \tau\cdot H_1(x)\right)^{-2}\cdot\cos^2\mu\cdot\sin\mu\cdot\dfrac{\partial\psi}{\partial\tau}\cdot\dfrac{\partial\varphi}{\partial\tau}}{\left[\tau^{1/2}\sin\mu\cos\mu\dfrac{\partial\varphi}{\partial\tau}\dfrac{\partial\psi}{\partial\tau} + \sqrt{2gH_1(x)}H_1^2(x)\cdot\left(\cos\mu\cos\theta\dfrac{\partial\psi}{\partial\tau} - (1-\tau)\sin\theta\cdot\dfrac{\partial\varphi}{\partial\tau}\right)\right]}\right\} \\ \dfrac{\partial\psi}{\partial\theta}; \\ \dfrac{\partial\varphi}{\partial\theta} = \dfrac{h_0\cdot\tau\cdot 2H_1(x)\sqrt{2gH_1(x)}\cdot\sin\theta\cdot\cos\mu\cdot\dfrac{\partial\psi}{\partial\tau}}{\sqrt{2gH_1(x)}H_1^2(x)(1-\tau)\cdot\sin\theta - \tau^{1/2}\sin\mu\cdot\cos\mu\cdot\dfrac{\partial\psi}{\partial\tau}}. \end{cases} \quad (44)$$

If downstream is horizontal then the system (44) coincides completely with the system derived before that was true for a flat horizontal slope [14].

Equations (44) are constitutive equations of supercritical flow motion with regard to the slope in the velocity hodograph plane. Further from the system (44) we deduce the equations for a mild slope.

Conclusions: there was deduced a system of equations for a supercritical irrotational flow motion with regard to a longitudinal slope in the velocity hodograph plane. The axisymmetric supercritical flow could be viewed as an irrotational one provided the slope is longitudinal and even.

B. A problem of free spreading of supercritical irrotational flow in a wide horizontal slope

If at the free-flow mode a supercritical irrotational flow spreads freely from a rectangular pipe in a wide horizontal slope then the following spreading pattern (see Figure 2) is true. The boundary problem is solved by determining the constant A, which is

$$A = \frac{V_0 b}{2\sin\theta_{max}},$$

if the limit flow spreading angle is known θ_{max}.

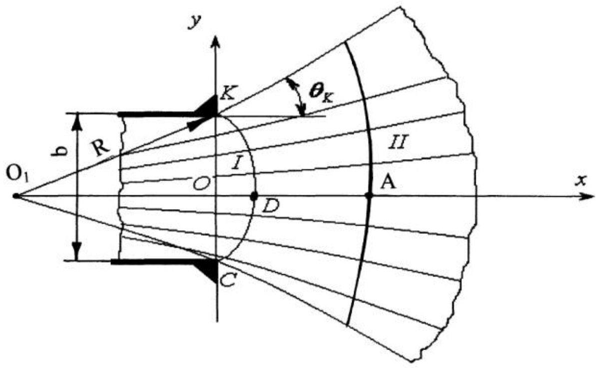

Figure 2: A pattern of free spreading downstream the water pipe *I* – three-dimensional flow; *II* –two-dimensional flow

We consider the equations of supercritical irrotational flow motion for a horizontal and even discharge channel.

$$\begin{cases} \dfrac{\partial \varphi}{\partial \tau} = \dfrac{h_0}{2H_0} \cdot \dfrac{3\tau - 1}{\tau(1-\tau)^2} \cdot \dfrac{\partial \psi}{\partial \theta}; \\ \dfrac{\partial \varphi}{\partial \theta} = 2\dfrac{h_0}{H_0} \cdot \dfrac{\tau}{1-\tau} \cdot \dfrac{\partial \psi}{\partial \tau}. \end{cases} \quad (45)$$

As given in the work [14] the system of equations (45) has a wide range of regular solutions. However, according to the experience of mathematical modeling of free flow spreading, we can take the following structure as a standard problem solution in the velocity hodograph plane that will satisfy the system of equations (45):

$$\begin{cases} \psi = \dfrac{A}{\tau^{1/2}} \sin\theta; \\ \varphi = A \cdot \dfrac{h_0}{H_0} \cdot \dfrac{\cos\theta}{\tau^{1/2}(1-\tau)}. \end{cases} \quad (46)$$

If the project task has a solution in the form of (46), then boundary task has the only solution in defining the parameters if $\tau = 1$.

Relations between the parameters $\theta\, u\, \tau$ arise from the equations (46) along the boundary streamline:

$$\frac{\sin\theta}{\tau^{1/2}} = \sin\theta_{\max} \quad (47)$$

and along the equipotential line passing through a point *A* (see pic. 3) on the flow symmetry axis: $\dfrac{\cos\theta}{\tau^{1/2}(1-\tau)} = \dfrac{1}{\tau_A^{1/2}(1-\tau_A)}$, (48)

where $\theta_{\max}$ – is an angle of flow spreading at infinity; τ_A – is the value of the parameter τ in the point *A*.

The angle $\theta_{\max}$ is determined by common formulas [15]:

First let's show that the parameter τ_D (see pic. 3) coincides to the parameter τ_0.
For that purpose we make the expressions for the flow discharge on the curve KD as:

$$Q_{KD} = \int_0^{\theta_K} V(\theta)\cdot h(\theta)\cdot R\, d\theta, \tag{50}$$

where θ – is the current inclination angle of velocity vector and of the axis Ox along the points of the initial equipotential curve.
Using the ratio (48) for the curve KD

$$\frac{\cos\theta}{\tau^{1/2}(1-\tau)} = \frac{1}{\tau_D^{1/2}(1-\tau_D)}, \tag{51}$$

we determine

$$V(\theta)h(\theta) = H_0\sqrt{2gH_0}\cdot\tau^{1/2}(1-\tau). \tag{52}$$

Determine radius – R from the triangle O_1OK:

$$R = \frac{b}{2\sin\theta_k}. \tag{53}$$

Taking into account equalities (51)-(53) from (50) we deduce:

$$Q_{KD} = h_D V_D\cdot\frac{b}{2} = \frac{Q}{2} = \frac{h_0 V_0 b}{2}, \tag{54}$$

where Q – is the complete flow discharge from the culvert.
From the equality (54) we deduce that:

$$\tau_D = \tau_0. \tag{55}$$

At the Figure 2 we see that:

$$x_D = \frac{b(1-\cos\theta_k)}{2\sin\theta_k}. \tag{56}$$

Parameters θ_k, τ_k are determined by the combined solution of the system:

$$\begin{cases} \dfrac{\sin\theta_K}{\tau_k^{1/2}} = \sin\theta_{\max}; \\ \dfrac{\cos\theta_K}{\tau_k^{1/2}(1-\tau_K)} = \dfrac{1}{\tau_0^{1/2}(1-\tau_0)}. \end{cases} \tag{57}$$

Thus, coupling of flows I and II is characterized by the following parameters: $R, \theta_K, \tau_K, \tau_D, x_D$.

C. *Definition of the flow parameters along its longitudinal axis*

Let's derive formula determining velocities and depths of the flow along its longitudinal symmetry axis. For that purpose we use the equation of the connection between the flux and the velocity hodograph plane:

$$dx + i\,dy = (d\varphi + i\frac{h_0}{h}\cdot d\psi)\frac{1}{V}e^{i\theta}. \tag{58}$$

Putting along the streamline $d\psi \equiv 0$, and taking into account that $\theta \equiv 0$ on the flow symmetry axis with regard to (46) we derive the following differential equation form (58) that connect $dx, d\tau$:

$$dx = \frac{Ah_0}{2H_0\sqrt{2gH_0}} \cdot \frac{3\tau - 1}{\tau^2(1-\tau)^2} \cdot d\tau. \quad (59)$$

Integrating the equation (59) and taking into account the condition $x = x_D ; \tau = \tau_0$, we deduce the following dependence:

$$x = x_D + \frac{Ah_0}{2H_0\sqrt{2gH_0}}\left[\frac{1+\tau}{\tau(1-\tau)} - \ln\frac{1-\tau}{\tau} - \frac{1+\tau_0}{\tau_0(1-\tau_0)} + \ln\frac{1-\tau_0}{\tau_0}\right]. \quad (60)$$

We can see that function $x = x(\tau)$ in the equation (60) – is steadily increasing at the rise of τ from τ_0 to 1.

If we know the parameter τ we can determine local velocity and depth of the flow by formulas:

$$V = \tau^{1/2}\sqrt{2gH_0};$$
$$h = H_0(1-\tau). \quad (61)$$

If we know the parameter τ then we can determine the distance x to the concerned point by the equation (60). If the distance x is given, then the parameter τ is determined from (60) with the help of applied programs set.

D. Model to determine parameters of the boundary streamline in the problem of free spreading water flow

Engineering of amelioration hydraulic structures, water bodies, flood irrigation and also of road drainage facilities requires the determination of water flow at its free and low-pressure spreading outside the culverts to the wide horizontal discharge channel. For that purpose we need to know local depth and velocities of the flow and also the geometry of boundary streamlines. If we know the equation of the boundary streamline we can define the distance to the section of the flow complete spread and then the angle that determine lines of cross hydraulic jumps. The equation of the boundary streamline also helps to determine the angle of the flow spread.

We need the stated parameters in order to choose the understructure of the channel downstream and to set the fluid flow dissipater thus reducing the washing of a discharge channel.

The pattern of supercritical flow spread into the wide horizontal slope is given at the Figure 3.

Downstream the rectangular pipe with the width b the flow has a three-dimensional nature with parameters $V_0, h_{0.}$, where V_0– is the flow velocity module; h_0 - is the flow depth downstream the pipe.

In order to determine the parameters of boundary streamlines we need to choose the original equation system of supercritical flow movement in the velocity hodograph plane [15]. We consider flow to be stable, open with longitudinal symmetry axis.

For supercritical flows τ satisfies the following inequality:

$$\frac{1}{3} < \tau \le 1. \qquad (62)$$

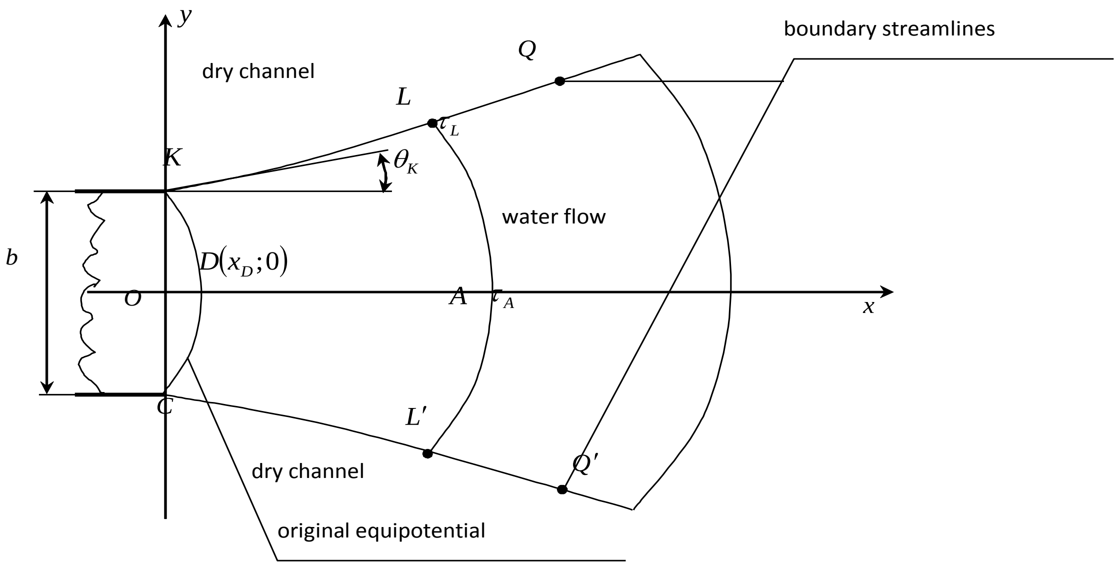

Figure 3: Supercritical flow spread pattern

System of equations (45) is a system of two equations from mathematical physics that is linear relatively to particular derivatives of the function φ, ψ: $\frac{\partial\varphi}{\partial\tau}, \frac{\partial\varphi}{\partial\theta}, \frac{\partial\psi}{\partial\tau}, \frac{\partial\psi}{\partial\theta}$.
According to the work [15], this system has a set of regular solutions. In order to simplify the model we choose the following solution from the set:

$$\begin{cases} \psi = \dfrac{A}{\tau^{1/2}} \sin\theta; \\ \varphi = A \dfrac{h_0}{H_0} \dfrac{\cos\theta}{\tau^{1/2}(1-\tau)}, \end{cases} \qquad (63)$$

Solution (63) determines main qualitative and quantitative features of solving the problem of boundary streamline. Function $\psi = \frac{\sin\theta}{\tau^{1/2}}$ could stay constant along the boundary streamline: $\psi = const$, because the function $\sin\theta$ – rises if the angle θ increases from zero to $\frac{\pi}{2}$; and function $\tau^{1/2}$ – also rises if τ increases from $\frac{1}{3}$ to 1.
According to numerical calculations if we add components to (63) that correspond to the set of solutions distributed at $\sin n\theta, \cos n\theta$, and the relative error is less than 1%, then it does not change numerical ratios in the boundary streamline geometry.
According to the expressions (63) the following dependence between the flow parameters θ and τ will take place along the boundary streamline:

$$\frac{\sin\theta}{\tau^{1/2}} = \sin\theta_{max}. \tag{64}$$

We need the boundary conditions for boundary streamline downstream the pipe, i.e. in the point K to be satisfied:

$$\frac{\sin\theta_{\kappa}}{\tau_{\kappa}^{1/2}} = \sin\theta_{max}. \tag{65}$$

If we take the velocity coefficient at the point D the intercept of original equipotential and flow symmetry axis as τ_D, then according to the second equation of the system (63) we need the following condition to be satisfied along the stated equipotential:

$$\frac{1}{\tau_D^{1/2}(1-\tau_D)} = \frac{\cos\theta_{\kappa}}{\tau_{\kappa}^{1/2}(1-\tau_{\kappa})}. \tag{66}$$

If we know the parameter τ_D then flow parameters in the point K the boundary streamline are determined from the system of trigonometric equations:

$$\begin{cases} \dfrac{\sin\theta_{\kappa}}{\tau_{\kappa}^{1/2}} = \sin\theta_{\infty}; \\ \dfrac{1}{\tau_D^{1/2}(1-\tau_D)} = \dfrac{\cos\theta_{\kappa}}{\tau_{\kappa}^{1/2}(1-\tau_{\kappa})}. \end{cases} \tag{67}$$

If we eliminate the angle θ_K from the equation (67) then we deduce the cubic equation to determine the parameter τ_K:

$$\frac{\tau_{\kappa}(1-\tau_{\kappa})^2}{\tau_D(1-\tau_D)^2} + \tau_{\kappa}\sin^2\theta_{\infty} = 1. \tag{68}$$

The unknown root should satisfy the condition:

$$\tau_0 \le \tau_K < 1. \tag{69}$$

The angle θ_K is acute that is why we use the first equation of the system (67) in order to determine it:

$$\theta_K = \arcsin\left[\tau_K^{1/2}\sin\theta_{\infty}\right]. \tag{70}$$

The velocity parameter at the point D coincides to the flow velocity parameter at the point O. This means that in this model the depths and velocities of OD segment points coincide, i.e. in three-dimensional flow there is a sector where fluid particles move under inertia. According to numerical calculations the value of this sector depends on flow parameters downstream the pipe h_0, V_0 and on the pipe width b. Thus we define the flow parameters in points K and D.

In order to define the boundary streamline coordinates we need to take the equation of the connection between the flux physical plane and the velocity hodograph plane:

$$dx + idy = (d\varphi + i\frac{h_0}{h}d\psi)\frac{1}{V}e^{i\theta}. \tag{71}$$

Taking into account expressions for φ and ψ we deduce the following system of differential equations from (45) for the stream function and the potential function, it will be true along the boundary streamline:

$$\begin{cases} dx = -\dfrac{A\cos\theta \cdot h_0}{\sqrt{2gH_0} \cdot H_0 \cdot \tau^{1/2}} \left[\dfrac{1-3\tau}{2\tau(1-\tau)^2} \cdot \dfrac{\cos\theta}{\tau^{1/2}} d\tau + \dfrac{\tau}{1-\tau} \cdot \dfrac{\sin\theta}{\tau^{3/2}} d\theta \right]; \\ dy = -\dfrac{A\sin\theta \cdot h_0}{\sqrt{2gH_0} \cdot H_0 \cdot \tau^{1/2}} \left[\dfrac{1-3\tau}{2\tau(1-\tau)^2} \cdot \dfrac{\cos\theta}{\tau^{1/2}} d\tau + \dfrac{\tau}{1-\tau} \cdot \dfrac{\sin\theta}{\tau^{3/2}} d\theta \right]. \end{cases} \quad (72)$$

The dependence (65) is true along the boundary streamline. We define the expression for $cos\theta$ from the main trigonometric identity and formula (65):

$$cos\theta = \sqrt{1-\tau\sin^2\theta_{\max}}\,. \quad (73)$$

Along the boundary streamline $d\psi = 0$, then the first equation of the system (45) implies the differential constraint between the flow parameters:

$$cos\theta\, d\theta = \frac{1}{2}\sin\theta \frac{d\tau}{\tau}. \quad (74)$$

Taking into account the dependence (65) we rearrange the equation (74) in the following form:

$$cos\theta\, d\theta = \frac{1}{2}\frac{d\tau}{\tau^{1/2}}\sin\theta_{\max}\,. \quad (75)$$

Then we rearrange the system of equations (72) taking into account (65), (73), (74):

$$\begin{cases} dx = \dfrac{Ah_0}{2H_0\sqrt{2gH_0}} \left[\dfrac{3\tau-1}{\tau^2(1-\tau)^2} - \dfrac{2\sin^2\theta_{\max}}{(1-\tau)^2} \right] d\tau; \\ dy = \dfrac{Ah_0 \sin\theta_{\max}}{H_0\sqrt{2gH_0}}\, d\left[\dfrac{\cos\theta}{\tau^{1/2}(1-\tau)} \right]. \end{cases} \quad (76)$$

Integrating the system (76) we deduce the following parametric equations of the boundary streamline:

$$\begin{cases} x = \dfrac{Ah_0}{2H_0\sqrt{2gH_0}} \left[\dfrac{1+\tau}{\tau(1-\tau)} - \dfrac{2\sin^2\theta_{\max}}{(1-\tau)} - \ln\dfrac{1-\tau}{\tau} - \dfrac{1+\tau_K}{\tau_K(1-\tau_K)} + \ln\dfrac{1-\tau_K}{\tau_K} + \dfrac{2\sin^2\theta_{\max}}{1-\tau_K} \right]; \\ y = \dfrac{b}{2} + \dfrac{Ah_0 \sin\theta_{\max}}{H_0\sqrt{2gH_0}} \left[\dfrac{\cos\theta}{\tau^{1/2}(1-\tau)} - \dfrac{\cos\theta_K}{\tau_K^{1/2}(1-\tau_K)} \right]. \end{cases} \quad (77)$$

Thus, using formulas (77) we determine the coordinates of arbitrary point L at the boundary streamline with parameters τ and θ, therefore we determine the geometry of the boundary streamline. We use the following system of equations in order to determine the flow parameters in the point L at the boundary streamline:

$$\begin{cases} \dfrac{\sin\theta_L}{\tau_L^{1/2}} = \sin\theta_{\max}; \\ \dfrac{\cos\theta_L}{\tau_L^{1/2}(1-\tau_L)} = \dfrac{1}{\tau_A^{1/2}(1-\tau_A)}. \end{cases} \quad (78)$$

Note that depending on the abscissa x_A the following formula of velocity coefficient distribution is true for a streamline of a zero flow at the symmetry axis:

$$x_A = x_D + \frac{V_0 \cdot h_0 \cdot b}{4H_0\sqrt{2gH_0}\sin\theta_\infty} \left\{ \frac{1+\tau_A}{\tau_A(1-\tau_A)} - \ln\frac{1-\tau_A}{\tau_A} - \frac{1+\tau_0}{\tau_0(1-\tau_0)} + \ln\frac{1-\tau_0}{\tau_0} \right\}. \quad (79)$$

Thus we determine the value τ_A from (79) using the set x_A and then we determine parameters θ_L and τ_L from the system (78) and coordinates of the point L: x_L, y_L from the system (77). Note that if we know the parameters in the point L: θ_L and τ_L, then we can determine the value of flow velocity and depth at the point L of the boundary streamline

$$\begin{cases} V_L = \tau_L^{1/2}\sqrt{2gH_0}; \\ h_L = H_0(1-\tau_L). \end{cases} \quad (80)$$

The model given in this work ensures that concordance of model and experimental parameters within the limits of downstream width $\beta \le 7$ is more accurate than in any other methods.

Despite the simplified character of the model (forces of the flow resistance occurring during it's spreading over the horizontal concrete slope), it could be used for engineering the hydraulic structures which have free flow spreading downstream the culverts.

E. Model to determine parameters of arbitrary streamline in the problem of free water flow spread

The solution of the problem of determining the parameters of arbitrary streamline of the flow spreading freely downstream the culverts is given in this section.

We take the equation system from mathematical physics (45) that describe the flow motion in the velocity hodograph plane as an initial system of equations of supercritical flow motion. The pattern of flow spread in the physical plane is given at the pic. 4.

First we determine coordinates of the point T at the initial equipotential provided we know the flow coefficient of a streamline passing this point.

The boundary streamline cuts 50% of the total water rate in the culvert from the flow symmetry axis Ox, it is $0{,}5 \cdot Q$.

The streamline passing the point *T*, cuts the following flow rate from the symmetry axis:

$$Q_T = 0.5 \cdot Q \cdot K_T \text{, here } K_T = \frac{\sin\theta_{\max}^T}{\sin\theta_{\max}}. \tag{81}$$

Thus, if the flow coefficient is given $0 \le K_T \le 1$, then we have a single value of a streamline. From the equation (81) we can define the angle $\theta_{\max}^T$:

$$\theta_{\max}^T = \arcsin(K_T \sin\theta_{\max}). \tag{82}$$

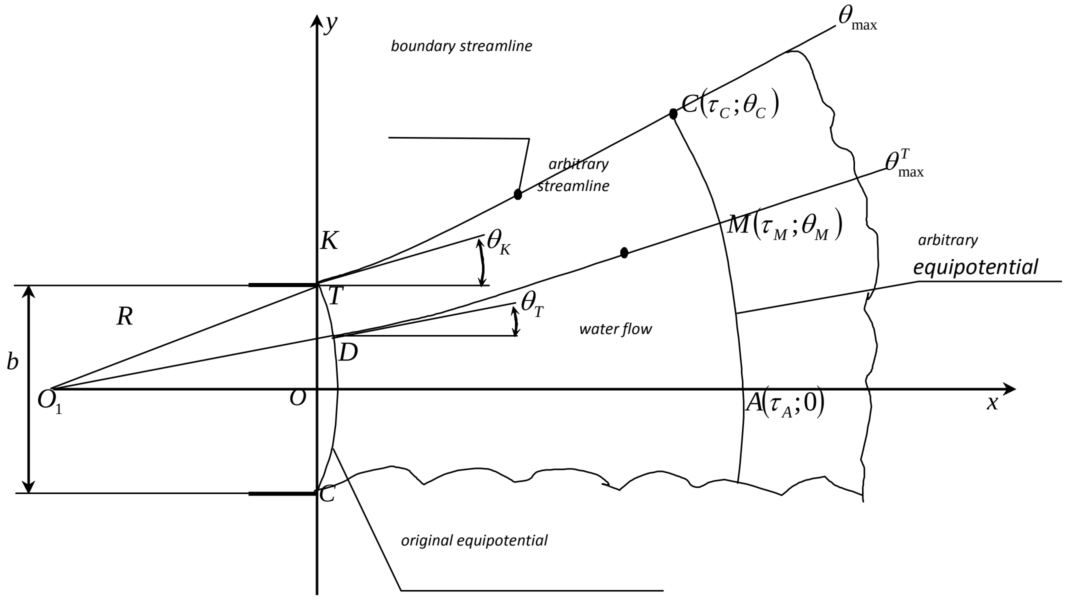

Figure 4: Flow spread pattern

Parameters τ_T, θ_T at the point *T* are determined by solving the system of equations:

$$\begin{cases} \dfrac{\sin\theta_T}{\tau_T^{1/2}} = \sin\theta_{\max}^T; \\ \dfrac{1}{\tau_0^{1/2}(1-\tau_0)} = \dfrac{\cos\theta_T}{\tau_T^{1/2}(1-\tau_T)}. \end{cases} \tag{83}$$

Here we determine the solution of the system (83), using the condition:

$$\tau_0 < \tau_T < \tau_K. \tag{84}$$

We determine the coordinates of the point *T* from the geometry of original equipotential and arbitrary streamline:

$$\begin{cases} x_T = R\cos\theta_T - R\cos\theta_K, \\ y_T = R\sin\theta_T. \end{cases} \tag{85}$$

Note that this streamline will also pass the point $A_1(0, K_T \cdot \frac{b}{2})$.

We specify the stabilization of the arbitrary equipotential passing the point *A* by the depth $h_A = \delta \cdot h_0$, where 0 < δ < 1.

We determine the square of the velocity coefficient by the depth h_A:

$$\tau_A = 1 - \frac{h_A}{H_0} \tag{86}$$

and also the flow velocity at the point *A*:

$$V_A = \tau_A^{1/2}\sqrt{2gH_0}. \tag{87}$$

Point *M* is at the intercept of the streamline that passes the point *T* and equipotential that passes the point *A*, then according to the theory [15]; it's coordinates satisfy the following system of trigonometric equations:

$$\begin{cases} \dfrac{\cos\theta_M}{\tau_M^{1/2}(1-\tau_M)} = \dfrac{1}{\tau_A^{1/2}(1-\tau_A)}; \\ \dfrac{\sin\theta_M}{\tau_M^{1/2}} = \sin\theta_{\max}^T, \end{cases} \tag{88}$$

where $\sin\theta_{\max}^T = \dfrac{\sin\theta_T}{\tau_T^{1/2}}$.

Then using the certain parameters θ_M, τ_M and the certain coordinates of the point *T* we can determine the coordinates of the point *M* from the following system of differential equations for a streamline [15]:

$$\begin{cases} dx = \dfrac{A\cos\theta}{\tau^{1/2}\sqrt{2gH_0}}\left\{-\dfrac{h_0}{H_0}\cdot\dfrac{1-3\tau}{2\tau(1-\tau)^2}\cdot\dfrac{\cos\theta}{\tau^{1/2}}\cdot d\tau - \dfrac{h_0}{H_0}\cdot\dfrac{\tau}{1-\tau}\cdot\dfrac{\sin\theta}{\tau^{3/2}}\cdot d\theta\right\}; \\ dy = \dfrac{\sin\theta}{\tau^{1/2}\sqrt{2gH_0}}\cdot d\varphi. \end{cases} \tag{89}$$

For the flow in the whole $A = \dfrac{V_0 b}{2\sin\theta_{\max}}$.

Using direct interaction we derive from the second equation of the system (89):

$$\begin{aligned} y_M &= y_T + \frac{A\sin\theta_{\max}^T}{\sqrt{2gH_0}}\left\{\frac{\cos\theta_M}{\tau_M^{1/2}(1-\tau_M)} - \frac{\cos\theta_T}{\tau_T^{1/2}(1-\tau_T)}\right\}\frac{h_0}{H_0} = \\ &= y_T + \frac{A\sin\theta_{\max}^T}{\sqrt{2gH_0}}\left\{\frac{1}{\tau_A^{1/2}(1-\tau_A)} - \frac{1}{\tau_0^{1/2}(1-\tau_0)}\right\}\frac{h_0}{H_0}. \end{aligned} \tag{90}$$

Using the equation of the link between parameters θ, τ along the streamline that passes the point *M*:

$$\frac{\sin\theta}{\tau^{1/2}} = \sin^{T}_{\max}.$$

And rearranging the first equation of the system (90), we get the equation:

$$dx = \frac{A}{\sqrt{2gH_0}} \cdot \frac{h_0}{2H_0} \left\{ \frac{(3\tau-1)(1-\tau\cdot K)}{\tau^2(1-\tau)^2} d\tau - \frac{K\, d\tau}{(1-\tau)\tau} \right\}, \tag{91}$$

where $K = \sin^2 \theta^{T}_{\max}$.

Then after the integration of the equation (91), we determine the abscissa of the point *M* by formula:

$$x_M = x_T + \frac{A}{\sqrt{2gH_0}} \cdot \frac{h_0}{2H_0} \left\{ I_1 - KI_2 - KI_3 \right\} \Bigg|_{\tau_T}^{\tau_M}, \tag{92}$$

and: $I_1 = \int \frac{(3\tau-1)d\tau}{\tau^2(1-\tau)^2} = \frac{1+\tau}{\tau(1-\tau)} + \ln\frac{\tau}{1-\tau}$; $I_2 = \int \frac{3\tau-1}{\tau(1-\tau)^2} \cdot d\tau = \frac{2}{1-\tau} + \ln\frac{1-\tau}{\tau}$;

$I_3 = \ln\frac{\tau}{1-\tau} = \int \frac{d\tau}{\tau(1-\tau)}$.

Thus, the results of this work give opportunity for hydraulic structure engineers to determine parameters of the flow freely spreading downstream the culvert leaving out friction forces. A number of further works will be devoted to the consideration of friction forces using the method based on velocity hodograph plane. This method was used to calculate the model flow parameters and to check their adequacy in regard to experimental parameters.

Method based on the velocity hodograph plane helps to decompose the calculation of the whole flow parameters into the calculation of parameters of the separate flow elements and to derive practical analytical formulas and functional dependences between the flow parameters for the boundary streamline, symmetry axis, arbitrary streamline, arbitrary equipotential. Thus, it is possible to use substitutions in certain methods of calculating hydraulic structure elements by substituting separate formulas to new and more accurate ones.

Here is the comparison of experimental [2,3] and model data according to the suggested method.

Experiment No. 1.

$h_0 = 9.27 \cdot 10^{-2} m; b = 0.16 m;$

$Q = 2.19 \cdot 10^{-2} \frac{m^3}{s};$

$V_0 = 1.47654 \frac{m}{s}; F_0 = 2.397.$

$\beta = \frac{B}{b}$; $B -$ is flow width; $F_0 = \frac{V_0^2}{gh_0} -$ is Froude number downstream the pipe.

According to this comparison we can make a conclusion that close to the pipe discharge and before the relative flow widening $\beta = 5 \div 7$, adequacy of the distinguished parameters is:

- sufficient for application at hydraulic construction;
- exceeds the degree of convergence of the flow parameters according to existing methods [2,5].

Results of this simplified model will be further used for consideration of friction forces and as a basis for the development of numerical methods of calculations of supercritical freely spreading flows with regard to friction forces from the discharge channel.

Qualitative features of spreading the flow by this simplified model were described in details in works [18,19,20]. Mathematical modeling in particular helped to determine a flow section where fluid particles move by inertia without changing velocities and depths. This flow section is quite little and located right behind the outlet of a culvert.

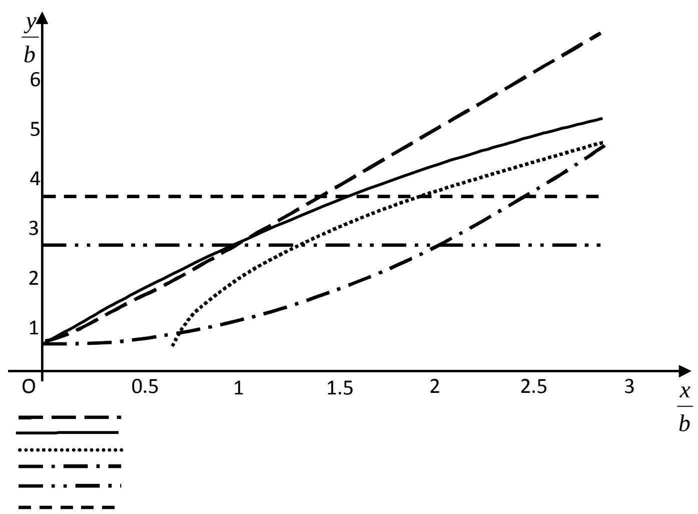

Figure 5: Diagrams of boundary flow line according to different authors and to the experiment Theoretical result; Experimental data; According to G.A. Lilitsky; According to I.A. Sherenkov; $\beta = 5$; $\beta = 7$.

Discussion

The transition from physical plane of the stream to velocity hodograph plane helps to find analytical solutions for potential function and stream function and also to determine basic flow parameters for two-dimensional problem of supercritical flow.

Derivation of equations of irrotational supercritical flow motion in the velocity hodograph plane with longitudinal slope gives opportunity to find analytical solution for potential function and stream function in a more general case. Thus, we can determine depth and velocity of the flow in a stream and also geometry of the free flow spreading.

The derived equations of irrotational supercritical flow motion provided the slope is longitudinal are generalization of equations of irrotational supercritical flow motion in the velocity hodograph plane with horizontal slope that authors derived before. These equations coincide to the first ones if the slope is zero.
Offered solutions of boundary streamlines, depths and velocities of the flow are more adequate than values of flow parameters derived by famous authors I.A. Sherenkov and G.A Lilitsky. Mismatch of boundary streamline geometry and flow parameters to an experiment do not exceed 7% at the flow widening $\beta = 5 \div 7$.

Conclusion

Advancement of calculation methods of two-dimensional supercritical flow depends on a number of questions that include studying the features of differential and integral equations describing the flow spreading, finding solutions for these equations, studying the geometry and kinematics of the flow, defining its hydraulic features. Calculation methods need to be advanced because there are contradictions in methods by different authors.
There was worked out a fundamentally new approach to solving hydraulic problems that are connected to two-dimensional supercritical steady irrotational flows due to existing streamline and potential function. For the first time there was suggested an analytical method which allowed converting equations of quotient derivatives for potential of velocities and stream functions in problem of two-dimensional supercritical flows into the system of two linear homogeneous differential equations.
While setting the boundary tasks we used method of differentiation onto a problem of velocity potential and a problem of stream function with their further combination and finding solution for the whole two-dimensional problem. Model of two-dimensional flow is taken as a basis for the general form of dependences defined in problems. The adequacy of calculated and real parameters depends on secondary factors, on modeling of equipotentials and transformation of classical algorithms for solving problems.
The offered method helped to solve the problem of free spreading flow downstream the rectangular culvert. Introduction of the developed method for calculation of two-dimensional flow in downstream of hydraulic structures has economic impact. It increases the accuracy of calculations of the hydraulic structures downstream and therefore increases the reliability of work and decreases material content of projected hydraulic structures.
Further researches are expected to comprise the solution for the two-dimensional plan task of the movement of the open two-dimensional supercritical flow for a mild slope. There will be also discussed the solution for two-dimensional stationary task for standard hydraulic structures: road drains, minor bridges, outlet works with high cills. The solution of stationary task of free supercritical flow spreading will help afterwards to solve different two-dimensional plan tasks of flow spreading in the downstream of hydraulic structures taking into account the resistance forces and the slope. The following task to be solved is a plan task of free spreading of two-dimensional supercritical flow downstream the circular pipes due to the fact that

circular pipes are more frequently used in hydraulic construction. The solution of stationary task of open two-dimensional supercritical flow spreading is expected to become a basis for the solution of a non-stationary task of the supercritical flow movement in a wide streambed.

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