Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 507–516

http://www.ripublication.com/gjpam.htm

Secure Restrained Domination in the Join and Corona of Graphs

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Abstract

Let G be a connected simple graph. A restrained dominating set S of the vertex set of G, $V(G)$ is a *secure restrained dominating set* of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a restrained dominating set of G. The minimum cardinality of a secure restrained dominating set of G, denoted by $\gamma_{sr}(G)$, is called the *secure restrained domination number* of G. A secure restrained dominating set of cardinality $\gamma_{sr}(G)$ is called a γ_{sr}-set of G. In [7], Pushpam and Suseendran paper's "Secure Restrained Domination in Graphs" studied few properties of secure restrained domination number of certain classes of graphs and evaluate $\gamma_{sr}(G)$ values for trees, unicyclic graphs, split graphs and generalized Petersen graphs. In this paper, we characterize the secure restrained dominating sets in the join and corona of two graphs and give some important results.

AMS subject classification: 05C69.
Keywords: Dominating set, secure dominating set, restrained dominating set, secure restrained dominating set.

1. Introduction

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [9]. However, it was not until 1977, following an article [2] by Ernie Cockayne and Stephen Hedetniemi, that domination in graphs became an area of study by many researchers. One type of domination parameter is the secure domination in graphs. This

[1]This research is funded by the Office of Research University of San Carlos, Cebu City, Philippines.

was studied and introduced by E.J. Cockayne et al. [1, 3]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. Other type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [6] indirectly as a vertex partitioning problem. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place a few guards as possible. In [4, 5], Enriquez and Canoy, introduced the concepts of secure convex and restrained convex domination in graphs. In [7], Pushpam and Suseendran paper's "Secure Restrained Domination in Graphs" studied few properties of secure restrained domination number of certain classes of graphs and evaluate $\gamma_{sr}(G)$ values for trees, unicyclic graphs, split graphs and generalized Petersen graphs. In this paper, we characterize the secure restrained dominating sets in the join and corona of two graphs and give some important results. For other graph concepts, readers may refer to [8].

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the *vertex-set* of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the *edge-set* of G. The elements of $V(G)$ are called *vertices* and the cardinality $|V(G)|$ of $V(G)$ is the *order* of G. The elements of $E(G)$ are called *edges* and the cardinality $|E(G)|$ of $E(G)$ is the *size* of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \varnothing$, then G is called an empty graph. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The elements of $N_G(v)$ are called *neighbors* of v. The *closed neighborhood* of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = \bigcup_{v \in X} N_G(v)$. The *closed neighborhood* of X in G is the set $N_G[X] = \bigcup_{v \in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ [resp. $N_G(x)$] will be denoted by $N[x]$ [resp. $N(x)$].

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $x \in S$ such that $xv \in E(G)$, i.e., $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. A dominating set S of $V(G)$ is a *secure dominating set* of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the *secure domination number* of G. A secure dominating set of cardinality $\gamma_s(G)$ is called a *γ_s-set* of G. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. A restrained dominating set S of $V(G)$ is a *secure restrained dominating set* of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a restrained dominating set of G. The minimum cardinality of a secure restrained dominating set of G, denoted by $\gamma_{sr}(G)$, is called the *secure restrained domination number* of G. A secure restrained dominating set of cardinality $\gamma_{sr}(G)$ is called a γ_{sr}*-set* of G.

Unless otherwise stated, all graphs in this paper are assumed to be simple and connected.

2. Results

We need the following Lemma and Remark for our first result.

Lemma 2.1. If S is a secure restrained dominating set of a graph G, then S is a secure dominating set of G.

Proof. Suppose that S is a secure restrained dominating set of G. Then S is a restrained dominating set of G, that is, S is a dominating set of G. Now, let $u \in V(G) \setminus S$. Then there exists $v \in S$ such that $uv \in E(G)$ and $S_u = (S \setminus \{v\}) \cup \{u\}$ is a restrained dominating set of G, that is, S_u is a dominating set of G. Hence, S is a secure dominating set of G. ■

The converse of Lemma 2.1 is not necessarily true. For example in $G = P_6 = [v_1, v_2, v_3, v_4, v_5, v_6]$, the set $S = \{v_1, v_3, v_5\}$ is a secure dominating set but not a restrained dominating set of G, that is, S is not a secure restrained dominating set of G.

The *join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge-set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. A nonempty subset S of $V(G)$, where G is any graph, is a *clique* in G if the graph $\langle S \rangle$ induced by S is complete.

Remark 2.2. Let G and H be complete graphs. Then $\gamma_s(G + H) = 1$.

The next result characterized the secure restrained dominating sets in the join of two graphs.

Theorem 2.3. Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$ is a secure restrained dominating set in $G + H$ if and only if one of the following statements holds:

(i) S is a secure dominating set of G and $|S| \geq 2$.

(ii) S is a secure dominating set of H and $|S| \geq 2$.

(iii) $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$ and

(a) S_G is a dominating set of G and S_H is a dominating set of H; or

(b) S_G is dominating set of G and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H; or

(c) S_H is dominating set of H and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G; or

(d) $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H.

(iv) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ $(|S_G| \geq 2)$ and $S_H = \{w\} \subset V(H)$ and $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G.

(v) $S = S_G \cup S_H$ where $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ $(|S_H| \geq 2)$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H.

(vi) $S = S_G \cup S_H$ where $S_G \subseteq V(G)$ $(|S_G| \geq 2)$ and $S_H \subseteq V(H)$ $(|S_H| \geq 2)$.

Proof. Suppose that S is a secure restrained dominating set of $G + H$. Consider the following cases:

Case 1. Suppose that $S \subseteq V(G)$ or $S \subseteq V(H)$.
If $S \subseteq V(G)$, then S is a secure dominating set of G by Lemma 2.1. Now suppose that $|S| = 1$, say $S = \{a\}$. Since S is a secure restrained dominating set of $G + H$, $\{z\}$ is a dominating set in $G + H$ (and hence in H) for every $z \in V(H)$. This implies that H is a complete graph, contrary to our assumption. Thus, $|S| \geq 2$. This shows that statement (i) holds. Similarly, statement (ii) holds if $S \subseteq V(H)$.

Case 2. Suppose that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Then $S = S_G \cup S_H$. Consider the following subcases.

Subcase 1 Suppose that $S_G = \{v\} \subset V(G)$ and $S_H = \{w\} \subset V(H)$. If S_G is a dominating set of G and S_H is a dominating set of H, then we are done with $(iiia)$. Suppose that S_G is a dominating set of G and S_H is not a dominating set of H. Let $x \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since S is a secure restrained dominating set of $G + H$, $\{w, x\}$ is a dominating set in $G + H$ (and hence in H). Since $wx \notin E(H)$, $xy \in E(H)$ for every $y \notin N_H(w)$. This implies that $y \in (V(H) \setminus S_H) \setminus N_H(S_H)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(H) \setminus S_H) \setminus N_H(S_H) \rangle$ induced by $(V(H) \setminus S_H) \setminus N_H(S_H)$ is complete. Hence, $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H. This proves $(iiib)$. Similarly, if S_H is dominating set of H and S_G is not a dominating set of G, then $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This proves $(iiic)$. If S_G is not a dominating set of G and S_H is not a dominating set of H, then $(iiid)$ holds by following similar arguments in $(iiib)$ and $(iiic)$.

Subcase 2 Suppose that $S_G \subseteq V(G)$ $(|S_G| \geq 2)$ and $S_H = \{w\} \subset V(H)$. If S_G is a dominating set of G, then (i) holds. Suppose that S_G is not a dominating set of G. Let $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since S is a secure restrained dominating set of $G + H$, $S_x = (S \setminus \{w\}) \cup \{x\}$ is a dominating set in $G + H$ (and hence in G). Since $vx \notin E(G)$ for every $v \in S_G$, $xy \in E(G)$ for every $y \notin N_G(S_G)$ (otherwise, S_x is not dominating set in $G + H$). This implies that $y \in (V(G) \setminus S_G) \setminus N_G(S_G)$. Since x was arbitrarily chosen, it follows that the subgraph $\langle (V(G) \setminus S_G) \setminus N_G(S_G) \rangle$ induced by $(V(G) \setminus S_G) \setminus N_G(S_G)$

is complete. Hence $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. This proves (iv). Similarly, (v) holds, if $S_G = \{v\} \subset V(G)$ and $S_H \subseteq V(H)$ $(|S_H| \geq 2)$.

Subcase 3 Suppose that $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$. Let $|S_G| \geq 2$. If S_G is a dominating set of G, then (i) holds. Suppose that S_G is not a dominating set of G. If $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G, then (iv) holds. Suppose that $(V(G) \setminus S_G) \setminus N_G(S_G)$ is not a clique in G. If $|S_H| = 1$, say $S_H = \{w\}$, then there exists $x \in (V(G) \setminus S_G) \setminus N_G(S_G)$ such that $S_x = (S \setminus \{w\}) \cup \{x\}$ is not a dominating set of G(and hence in $G + H$). This contradict to our assumption that S is a secure restrained dominating set of $G + H$. Thus, $|S_H| \geq 2$. Similarly, if $|S_H| \geq 2$ and $(V(H) \setminus S_H) \setminus N_H(S_H)$ is not a clique in H, then $|S_G| \geq 2$. This proves (vi).

For the converse, suppose first that statement (i) holds. Since S is a dominating set of G, S is a dominating set of $G + H$. Let $u \in V(G + H) \setminus S$. Then there exists $v \in S$ such that $uv \in E(G + H)$. If $u \in V(G)$ then $uw \in E(G + H)$ for all $w \in V(H) \subset V(G+H) \setminus S$. If $u \in V(H)$, then there exists $w \in V(H) \subset V(G+H) \setminus S$ such that $uw \in E(G + H)$ since H is a connected non-complete graph. Thus, S is a restrained dominating set of $G + H$. Further, since S is a secure dominating set of G, there exists $v \in S \cap N_G(u)$ such that $S_u = (S \setminus \{v\}) \cup \{u\}$ is a dominating set of G (and hence S_u is a dominating set of $G + H$) if $u \in V(G)$. Since $|S| \geq 2$, S_u is a dominating set of $G + H$ if $u \in V(H)$. Thus, S_u is a dominating set of $G + H$. Now, let $x \in V(G + H) \setminus S_u$. Then there exists $y \in S_u$ such that $xy \in E(G + H)$. If $x \in V(G)$, then $xz \in E(G + H)$ for all $z \in V(H) \subset V(G + H) \setminus S_u$. If $x \in V(H)$ then since H is a connected non-complete graph, there exists $(w \in V(H) \setminus S_u) \subset V(G + H) \setminus S_u$ such that $xw \in E(G + H)$. Thus, S_u is a restrained dominating set of $G + H$.

Accordingly, S is a secure restrained dominating set of $G + H$ if (i) holds. Similarly, if (ii) holds, then S is a secure restrained dominating set of $G + H$.

Suppose that $(iiia)$ holds. Then $S = \{v, w\}$ is a dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. Then $vx \in E(G + H)$. If $x \in V(G)$, then $xy \in E(G + H)$ for all $y \in V(H) \setminus S \subset V(G + H) \setminus S$. If $x \in V(H)$, $xz \in E(G + H)$ for all $z \in V(G) \setminus S \subset V(G + H) \setminus S$. Thus, S is a restrained dominating set of $G + H$. Now, let $u \in V(G + H) \setminus S$. Then $uv \in E(G + H)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$. Since $S_H = \{w\}$ is a dominating set in H (and hence in $G+H$), S_u is a dominating set of $G+H$. Let $z \in V(G + H) \setminus S_u$. Then $zw \in E(G + H)$. If $z \in V(G)$, then $zy \in E(G + H)$ for all $y \in V(H) \setminus S_u \subset V(G + H) \setminus S_u$. If $z \in V(H)$, then $zx \in E(G + H)$ for all $x \in V(G) \setminus S_u \subset V(G + H) \setminus S_u$. This implies that S_u is a restrained dominating set of $G + H$.

Accordingly, S is a secure restrained dominating set of $G + H$ if $(iiia)$ holds.

Suppose that $(iiib)$ holds. Since $S_G = \{v\}$ is a dominating set of G (and hence of $G + H$), $S = S_G \cup S_H$ is a dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. Then $xv \in E(G+H)$. If $x \in V(G)$, then $xy \in E(G+H)$ for all $y \in V(H) \setminus S \subset V(G+H) \setminus S$. If $x \in V(H)$, then $xz \in E(G + H)$ for all $z \in V(G) \setminus S \subset V(G + H) \setminus S$. Thus, S is a restrained dominating set of $G + H$. Now, let $u \in V(G + H) \setminus S$. Consider the following two cases.

Case 1 If $u \notin N_H(w)$, then $u \in V(H)$. This implies that $uv \in E(G+H)$ and $S_u = (S \setminus \{v\}) \cup \{u\} = \{w, u\}$. Since $u \notin N_H(w)$, $u \in (V(H) \setminus S_H) \setminus N_H(S_H)$ where $(V(H) \setminus S_H) \setminus N_H(S_H)$ is a clique in H. This implies that S_u is a dominating set in H (and hence in $G+H$). Let $x \in V(G+H) \setminus S_u$. Then there exists $a \in S_u$, say w, such that $xw \in E(G+H)$. If $x \in V(G)$, then $xy \in E(G+H)$ for all $y \in V(H) \setminus S_u \subset V(G+H) \setminus S_u$. If $x \in V(H)$, then $xz \in E(G+H)$ for all $z \in V(G) \setminus S_u \subset V(G+H) \setminus S_u$. Thus, S_u is a restrained dominating set of $G+H$.

Case 2 If $u \in N_H(w)$, then $u \in V(G)$ or $u \in V(H)$. Consider first that $u \in V(G)$. Then $uw \in E(G+H)$ and $S_u = S \setminus \{w\} \cup \{u\} = \{v, u\} \subset V(G)$. Since $\{v\}$ is a dominating set of G (and hence in $G+H$), S_u is a dominating set of $G+H$. Let $x \in V(G+H) \setminus S_u$. Then there exists $b \in S_u$, say v, such that $xv \in E(G+H)$. If $x \in V(G)$, then $xz \in E(G+H)$ for all $z \in V(H) \setminus S_u \subset V(G+H) \setminus S_u$. If $x \in V(H)$, then $xy \in E(G+H)$ for all $y \in V(G) \setminus S_u \subseteq V(G+H) \setminus S_u$ (note that G is connected non-complete graph). Thus, S_u is a restrained dominating set of $G+H$. Similarly, if $u \in V(H)$, then S_u is a restrained dominating set of $G+H$.

Accordingly, S is a secure restrained dominating set of $G+H$ if $(iiib)$ holds. Similarly, S is a secure restrained dominating set of $G+H$ if $(iiic)$ holds.

Suppose that $(iiid)$ holds. Then $vz \in E(G+H)$ for all $z \in V(H)$ and $wx \in E(G+H)$ for all $x \in V(G)$. This implies that $N_{G+H}[S] = N_{G+H}[v] \cup N_{G+H}[w] = V(G+H)$, that is, S is a dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. Then there exists $a \in S$ such that $ax \in E(G+H)$. If $x \in V(G)$, then $xz \in E(G+H)$ for all $z \in V(H) \setminus S \subset V(G+H) \setminus S$. If $x \in V(H)$, then $xy \in E(G+H)$ for all $y \in V(G) \setminus S \subset V(G+H) \setminus S$. Thus, S is a restrained dominating set of $G+H$. Let $u \in (V(G) \setminus S_G) \setminus N_G(S_G)$ where $(V(G) \setminus S_G) \setminus N_G(S_G)$ is a clique in G. Then $S_u = \{v, u\}$ is a dominating set in G (and hence in $G+H$). Let $x \in V(G+H) \setminus S_u$. Then there exists $a \in S_u$, say u, such that $ux \in E(G+H)$. If $x \in V(G)$, then $xz \in E(G+H)$ for all $z \in V(H) \setminus S_u \subset V(G+H) \setminus S_u$. If $x \in V(H)$, then $xy \in E(G+H)$ for all $y \in V(G) \setminus S_u \subset V(G+H) \setminus S_u$. Thus, S_u is a restrained dominating set of $G+H$. Similarly, if $u \in (V(H) \setminus S_H) \setminus N_H(S_H)$ where $(V(H) \setminus S_H) \setminus N_H(S_G)$ is a clique in H. Then $S_u = \{w, u\}$ is a restrained dominating set of $G+H$.

Accordingly, S is a secure restrained dominating set of $G+H$ if $(iiid)$ holds. Similarly, S is a secure restrained dominating set of $G+H$ if any of the following (iv), (v), or (vi) holds. ■

Corollary 2.4. Let G and H be connected non-complete graphs and let $S_G \subset V(G)$ and $S_H \subset V(H)$. Then

$$\gamma_{sr}(G+H) = \begin{cases} 2, \text{ if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma_s(G) = 2 \text{ or } \gamma_s(H) = 2 \\ 3, \text{ if } |S_G| = 2 \text{ and } (V(G) \setminus S_G) \setminus N_G(S_G) \text{ is a clique in G or} \\ \quad |S_H| = 2 \text{ and } (V(H) \setminus S_H) \setminus N_H(S_H) \text{ is a clique in H} \\ 4, \text{ if otherwise.} \end{cases}$$

The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith

copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

Remark 2.5. Let G and H be nontrivial connected graphs. A nonempty subset S of $V(G \circ H)$ is a dominating set of $G \circ H$ if and only if $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v.

The following result characterize the secure restrained dominating sets in the corona of two connected graphs.

Theorem 2.6. Let G and H be nontrivial connected graphs. A nonempty subset S of $V(G \circ H)$ is a secure restrained dominating set of $G \circ H$ if and only if for each $v \in V(G)$, $\langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices and one of the following is satisfied.

(i) $S = V(G)$ and H is complete.

(ii) $S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right)$, where $S_v \cup \{u\}$ is a dominating set of H^v for each $u \in V(H^v) \setminus S_v$.

(iii) $S = \left(\bigcup_{v \in V(G)} S_v \right)$, where S_v is a secure dominating set of H^v.

Proof. Suppose that a nonempty subset S of $V(G \circ H)$ is a secure restrained dominating set of $G \circ H$. Then $\langle V(G \circ H) \setminus S \rangle = \bigcup_{v \in V(G)} \langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices, that is, for each $v \in V(G)$, $\langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices. Since S is a dominating set, in view of Remark 2.5, $V(G) \subseteq S$ or $\bigcup_{v \in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v. Consider the following cases.

Case 1 Suppose that $V(G) \subseteq S$. If $S = V(G)$ and suppose that H is non-complete, then there exist distinct vertices $x, y \in V(H)$ such that $xy \notin E(H)$. This implies that for each $v \in V(G)$, $(S \setminus \{v\}) \cup \{x\}$ is not a dominating set of $G \circ H$ contrary to our assumption that S is a secure restrained dominating set of $G \circ H$. Thus, H is complete. This proves statement (i). If $S \neq V(G)$, then $V(G) \subset S$. Let $x \in S \setminus V(G) = \bigcup_{v \in V(G)} S_v$, where for each $v \in V(G)$, $S_v \subseteq V(H^v)$. This implies that $\bigcup_{v \in V(G)} S_v \subset S$, that is, $V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right) \subseteq S$. Since $x \in S$ implies that $x \in$

$\bigcup_{v\in V(G)} S_v$, it follows that $x \in V(G) \cup \left(\bigcup_{v\in V(G)} S_v\right)$. Thus, $S \subseteq V(G) \cup \left(\bigcup_{v\in V(G)} S_v\right)$, that is, $S = V(G) \cup \left(\bigcup_{v\in V(G)} S_v\right)$, where $S_v \subseteq H^v$. By assumption, S is a secure restrained dominating set of $G \circ H$. Let $y \in V(G \circ H) \setminus S$. Then for each $v \in V(G)$, $(S \setminus \{v\}) \cup \{y\} = V(G) \setminus \{v\} \cup S_v \cup \{y\}$ is a restrained dominating set of $G \circ H$. This implies that for each $v \in V(G)$, $S_v \cup \{y\}$ is a dominating set of H^v. This proves statement (ii).

Case 2 Suppose that $\bigcup_{v\in V(G)} (S_v) \subseteq S$ where for each $v \in V(G)$, S_v is a dominating set of H^v. In case 1, $V(G) \subseteq S$, suppose that $V(G) \not\subseteq S$. Then $S \subseteq \bigcup_{v\in V(G)} V(H^v)$ with S_v is a dominating set of H^v. If for each $v \in V(G)$, S_v is not a secure dominating set of H^v then there exists $u \in V(H^v) \setminus S_v$ such that for every $x \in S_v$, $xu \notin E(H^v)$ and $S'_v = (S_v \setminus \{x\}) \cup \{u\}$ is not a dominating set in H^v. Thus, $S_u = \bigcup_{v\in V(G)} S'_v$ is not a dominating set of $G \circ H$ contrary to our assumption that S is a secure restrained dominating set of $G \circ H$. This implies that for each $v \in V(G)$, S_v is a secure dominating set of H^v. Now, let $x \in S$. Since $x \notin V(G)$, it follows that $x \in \bigcup_{v\in V(G)} V(H^v)$. Suppose that for each $v \in V(G)$, $x \notin \bigcup_{v\in V(G)} S_v$. Then for each $v \in V(G)$, $x \notin S_v$. Since for each $v \in V(G)$, S_v is a secure dominating set of H^v, there exists $y \in S_v \cap N_{H^v}(x)$ such that $S'_v = (S_v \setminus \{y\}) \cup \{x\}$ is a dominating set in H^v for all $v \in V(G)$. This implies that $x \in S_x = \bigcup_{v\in V(G)} S'_v$, that is, $x \notin S$. Thus, $S \subseteq \bigcup_{v\in V(G)} S_v$. Accordingly, $S = \bigcup_{v\in V(G)} S_v$, where S_v is a secure dominating set of H^v. This proves statement (iii).

For the converse, suppose that for each $v \in V(G)$, $\langle V(H^v) \setminus S_v\rangle$ is a subgraph without isolated vertices and statement (i) or (ii) or (iii) holds. First, if statement (i) holds, then $S = V(G)$ is a dominating set of $G \circ H$ by Remark 2.5. Since H is nontrivial connected graph, for each $v \in V(G)$, H^v has no isolated vertices. This implies that $\bigcup_{v\in V(G)} H^v = \langle V(G \circ H) \setminus S\rangle$ is a subgraph without isolated vertices. Thus, S is a retrained dominating set of $V(G \circ H)$. Now, let $u \in V(G \circ H) \setminus S$. Since H is complete, for each $v \in V(G)$, $v \in S \cap N_{G\circ H}(u)$ and $S_u = S \setminus \{v\} \cup \{u\}$ is a dominating set of $G \circ H$. Now, let $y \in V(G \circ H) \setminus S_u$ $(y \neq v)$. Then there exists $z \in S_u$, say $z = u$ such that $yu \in E(G \circ H)$ and $yv \in E(G \circ H)$ for some $v \in V(G \circ H) \setminus S_u$. If $z \neq u$, then $yz \in E(G \circ H)$ and $yw \in E(G \circ H)$ for some $w \in V(G \circ H) \setminus S_u$ (since H is nontrivial complete graph). Thus, S_u is a restrained dominating set of $G \circ H$. Accordingly, S is a secure restrained dominating set of $G \circ H$ if statement (i) holds.

If statement (ii) holds, then $S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v \right)$ is a dominating set of $G \circ H$ by Remark 2.5. Since $\langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices, it follows that $\langle V(G \circ H) \setminus S \rangle = \bigcup_{v \in V(G)} \langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices. Thus, S is a restrained dominating set of $G \circ H$. Let $u \in V(G \circ H) \setminus S$. Then for each $v \in V(G)$,

$$S_u = (S \setminus \{v\}) \cup \{u\} = (V(G) \setminus \{v\}) \cup \left(\bigcup_{v' \in V(G) \setminus \{v\}} S_{v'} \right) \cup (S_v \cup \{u\}).$$

Since for each $v \in V(G)$, $S_v \cup \{u\}$ is a dominating set of $V(H^v) \setminus S_v$, it follows that S_u is a dominating set of $V(G \circ H)$. Now, let $y \in V(G \circ H) \setminus S_u$ $(y \neq v)$. Then there exists $z \in S_u$, say $z = u$ such that $yu \in E(G \circ H)$ and $yv \in E(G \circ H)$ for some $v \in V(G \circ H) \setminus S_u$. If $z \neq u$, then $yz \in E(G \circ H)$ and $yw \in E(G \circ H)$ for some $w \in V(G \circ H) \setminus S_u$ where $w \neq v$ (since H is nontrivial graph). Thus, S_u is a restrained dominating set of $G \circ H$. Accordingly, S is a secure restrained dominating set of $G \circ H$ if statement (ii) holds.

If statement (iii) holds, then $S = \left(\bigcup_{v \in V(G)} S_v \right)$ is a dominating set of $G \circ H$ by Remark 2.5. Since $\langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices, it follows that $\langle V(G \circ H) \setminus S \rangle = V(G) \cup \bigcup_{v \in V(G)} \langle V(H^v) \setminus S_v \rangle$ is a subgraph without isolated vertices. Thus, S is a restrained dominating set of $G \circ H$. Let $u \in V(G \circ H) \setminus S$. Then there exists $x \in S$ such that $xu \in E(G \circ H)$ and $S_u = (S \setminus \{x\}) \cup \{u\} = \bigcup_{v' \in (V(G) \setminus \{v\})} S_{v'} \cup [(S_v \setminus \{x\}) \cup \{u\}]$. If $u \notin V(G)$, for each $v \in V(G)$, $(S_v \setminus \{x\}) \cup \{u\}$ is a dominating set of H^v (note that S_v is a secure dominating set). This implies that S_u is a dominating set of $V(G \circ H)$. If $u \in V(G)$, then for each $v \in V(G)$, $(S_v \setminus \{x\}) \cup \{u\}$ is a dominating set in $v + H^v$. Again S_u is a dominating set of $G \circ H$. Now, let $y \in V(G \circ H) \setminus S_u$. If $u \notin V(G)$, then there exists $z \in S_u$ such that $yz \in E(G \circ H)$ and $yv \in E(G \circ H)$ for some $v \in V(G) \setminus S_u \subseteq V(G + H) \setminus S_u$. If $u \in V(G)$, then $yu \in E(G \circ H)$ and $yw \in E(G \circ H)$ for some $w \in V(H^v) \setminus S_v \subseteq V(G \circ H) \setminus S_u$ (since H is nontrivial graph). Thus, S_u is a restrained dominating set of $G \circ H$. Accordingly, S is a secure restrained dominating set of $G \circ H$ if statement (iii) holds. ■

Corollary 2.7. Let G be connected graph and $H = K_n$ with $n \geq 2$. Then $\gamma_{sr}(G \circ H) = |V(G)|$.

Remark 2.8. Let G be connected graph and $H = \overline{K}_n$. Then $\gamma_{sr}(G \circ H) = |V(G \circ H)|$.

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