Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2016), pp. 473-479

http://www.ripublication.com

On n-power-hyponormal operators

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Abstract

In this paper we introduce the new class of n-power-hyponormal operators acting on a complex Hilbert space H. We give some basic properties of these operators.

Keywords: Operator, n-power-hyponormal, 3-Normal, 4-Normal, idempotent operator.

Introduction.

Throughout this paper, $B(H)$ denotes to the algebra of all bounded linear operators acting on a complex Hilbert space H. An operator T is said to n-normal operator if $T^nT^* = T^*T^n$; normal if $TT^* = T^*T$ (it is clear that a bounded normal operator is n-normal operator for any n); self adjoint if $T^* = T$; positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0$ for all $x \in H$, and seky-normal if $T^2 = T^{*^2}$; projection if $T^2 = T = T^*$. For an operator $T \in H$, if $\|Tx\| = \|x\|$ for all $x \in H$

(or equivalently $T^*T = I$), then T is called an isometry; T is called unitary if $TT^* = T^*T = I$. An operator T on H is called hyponormal if $TT^* \leq T^*T$.

n-power-hyponormal operators

Definition

$T \in B(H)$ is called an n-power-hyponormal operators if $T^nT^* \le T^*T^n$.

It is easy to observe that, this new class includes all normal, all n-normal and all hyponormal operators.

Proposition 1

If S, $T \in B(H)$ are unitarily equivalent and if T is n-power-hyponormal operators then so is S.

Proof

Let T be an n-power-hyponormal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^*$ so $S^n = UT^nU^*$.

We have

$$\begin{aligned} S^nS^* &= UT^nU^*\left(UTU^*\right)^* \\ &= UT^nU^*UTU^* \\ &= UT^nT^*U^* \\ &\le UT^*T^nU^* \\ &= S^*S^n. \end{aligned}$$

Hence, $S^nS^* \le S^*S^n$ that is S is n-power-hyponormal operator.

Proposition 2

Let $T \in B(H)$ be an n-power-hyponormal operator. Then T^* is co-n-power-hyponormal.

Proof

Since T is n-power-hyponormal operator. We have

$$\begin{aligned} T^nT^* \le T^*T^n &\Rightarrow \left(T^nT^*\right)^* \le \left(T^*T^n\right)^* \\ &\Rightarrow \left(T^*\right)^*\left(T^n\right)^* \le \left(T^n\right)^*\left(T^*\right)^* \\ &\Rightarrow T\left(T^*\right)^n \le \left(T^*\right)^n T \\ &\Rightarrow \left(T^*\right)^n T \ge T\left(T^*\right)^n. \end{aligned}$$

Hence, T^* is co-n-power-hyponormal.

Corollary

If T and T^* are two n-power-hyponormal operators, then T is n-normal operator.

Theorem 1

If S, T are commuting n-power-hyponormal operators and $ST^* = T^*S$, then ST is an n-power-hyponormal operator.

Proof

Since $ST = TS$, so $S^n T^n = (ST)^n$ and $ST^* = T^* S$, so $S^n T^* = T^* S^n$.

Now,

$$ST^* = T^* S \Rightarrow TS^* = S^* T$$
$$\Rightarrow T^n S^* = S^* T^n.$$

We have

$$\begin{aligned}(ST)^n (ST)^* &= S^n T^n T^* S^* \\ &\le S^n T^* T^n S^* \\ &= T^* S^n S^* T^n \\ &\le T^* S^* S^n T^n.\end{aligned}$$

Hence

$$(ST)^n (ST)^* \le (ST)^* (ST)^n.$$

Then ST is an n-power-hyponormal operator.

Proposition 3

Let S, T are commuting n-power-hyponormal operators, such that $TS^* = S^* T$ and $(S+T)^*$ is commutes with $\sum_{k=1}^{n-1} C_n^k S^{n-k} T^k$. Then $(S+T)$ is an n-power-hyponormal operator

Proof

Since

$$\begin{aligned}(S+T)^n (S+T)^* &= \left(\sum_{k=0}^{n} C_n^k S^{n-k} T^k\right)(S^* + T^*) \\ &= S^n S^* + \sum_{k=1}^{n-1} C_n^k S^{n-k} T^k (S+T)^* \\ &\quad + T^n S^* + S^n T^* + T^n T^*.\end{aligned}$$

and since

$$TS^* = S^* T \Rightarrow T^n S^* = S^* T^n.$$

now

$$TS^* = S^* T \Rightarrow ST^* = T^* S$$
$$\Rightarrow S^n T^* = T^* S^n.$$

Since $(S+T)^*$ is commutes with $\sum_{k=1}^{n-1} C_n^k S^{n-k} T^k$. Hence

$$\begin{aligned}&(S+T)^n(S+T)^*\\&=S^nS^*+(S+T)^*\sum_{k=1}^{n-1}C_n^kS^{n-k}T^k+S^*T^n+T^*S^n+T^nT^*\\&\le S^*S^n+(S+T)^*\sum_{k=1}^{n-1}C_n^kS^{n-k}T^k+S^*T^n+T^*T^n\\&=(S+T)^*\left(\sum_{k=0}^{n}C_n^kS^{n-k}T^k\right)\\&=(S+T)^*(S+T)^n.\end{aligned}$$

This completes the proof.

Proposition 4

If S, $T\in B(H)$ are 2 -power-hyponormal operators, such that $TS^*=S^*T$ and $ST+TS=0$, then $T+S$ and ST are 2 -power-hyponormal.

Proof

Since $ST+TS=0$, hence $S^2T^2=T^2S^2$, so $(S+T)^2=S^2+T^2$.

$$\begin{aligned}(S+T)^2(S+T)^*&=(S^2+T^2)(S^*+T^*)\\&=S^2S^*+S^2T^*+T^2S^*+T^2T^*\\&=S^2S^*+T^*S^2+S^*T^2+T^2T^*\\&\quad(\text{for } TS^*=S^*T)\\&\le S^*S^2+T^*S^2+S^*T^2+T^*T^2\\&=(S+T)^*(S+T)^2.\end{aligned}$$

Now,

$$\begin{aligned}(ST)^2(ST)^*&=S^2T^2T^*S^*\\&\le S^2T^*T^2S^*\\&=T^*S^2S^*T^2\\&\le T^*S^*S^2T^2\\&=(ST)^*(ST)^2.\end{aligned}$$

Theorem 2

Let T_1, T_2,....., T_m be n-power-hyponormal operators in $B(H)$. Then $(T_1\oplus T_2\oplus......\oplus T_m)$ and $(T_1\otimes T_2\otimes......\otimes T_m)$ are n-power-hyponormal operators.

Proof

Since

$$\left(T_1 \oplus T_2 \oplus ......\oplus T_m\right)^n \left(T_1 \oplus T_2 \oplus ......\oplus T_m\right)^*$$
$$= \left(T_1^n \oplus T_2^n \oplus ......\oplus T_m^n\right)\left(T_1^* \oplus T_2^* \oplus ......\oplus T_m^*\right)$$
$$= T_1^n T_1^* \oplus ....\oplus T_m^n T_m^*$$
$$\leq T_1^* T_1^n \oplus ....\oplus T_m^* T_m^n$$
$$= \left(T_1^* \oplus T_2^* \oplus ......\oplus T_m^*\right)\left(T_1^n \oplus T_2^n \oplus ......\oplus T_m^n\right)$$
$$= \left(T_1 \oplus T_2 \oplus ......\oplus T_m\right)^* \left(T_1 \oplus T_2 \oplus ......\oplus T_m\right)^n.$$

Then $\left(T_1 \oplus T_2 \oplus ......\oplus T_m\right)$ is an n-power-hyponormal operator.

Now, for $x_1,.......,x_m \in H$

$$\left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^n \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^* \left(x_1 \otimes ......\otimes x_m\right)$$
$$= \left(T_1^n \otimes T_2^n \otimes ......\otimes T_m^n\right)\left(T_1^* \otimes T_2^* \otimes ....\otimes T_m^*\right)\left(x_1 \otimes ......\otimes x_m\right)$$
$$= T_1^n T_1^* x_1 \otimes ....\otimes T_m^n T_m^* x_m$$
$$\leq T_1^* T_1^n x_1 \otimes ....\otimes T_m^* T_m^n x_m$$
$$= \left(T_1^* \otimes T_2^* \otimes ......\otimes T_m^*\right)\left(T_1^n \otimes T_2^n ......\otimes T_m^n\right)\left(x_1 \otimes ......\otimes x_m\right)$$
$$= \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^* \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^n \left(x_1 \otimes ......\otimes x_m\right)$$

so

$$\left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^n \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^*$$
$$\leq \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^* \left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)^n.$$

Thus $\left(T_1 \otimes T_2 \otimes ......\otimes T_m\right)$ is an n-power-hyponormal operator.

Proposition 5

If T is 3 -power-hyponormal and $T^2 = -T^{*^2}$. Then T is 3 -normal operator.

Proof

Since

$$T^3 T^* = T T^2 T^* = -T T^{*^3}$$

and

$$T^* T^3 = T^* T^2 T = -T^{*^3} T.$$

T is 3 -power-hyponormal, then

$$T^3T^* \leq T^*T^3 \Rightarrow -TT^{*^3} \leq -T^{*^3}T$$
$$\Rightarrow TT^{*^3} \geq T^{*^3}T$$
$$\Rightarrow \left(TT^{*^3}\right)^* \geq \left(T^{*^3}T\right)^*$$
$$\Rightarrow T^3T^* \geq T^*T^3.$$

Hence $T^3T^* = T^*T^3$.

Proposition 6

If T is 4 -power-hyponormal and T is skew-normal operator, then T is 4 -normal operator.

Proof

T is skew-normal operator then $T^2 = T^{*^2}$. Since

$$T^4T^* = T^2T^2T^* = T^{*^5}$$

and

$$T^*T^4 = T^*T^2T^2 = T^{*^5}.$$

Hence $T^4T^* = T^*T^4$.

Proposition 7

If T is 2 -power-hyponormal operator and T is idempotent. Then T is hyponormal operator.

Proof

Since T is 2 -power-hyponormal operator, then

$$T^2T^* \leq T^*T^2$$

since T is idempotent $T^2 = T$, which implies

$$TT^* \leq T^*T.$$

Thus T is hyponormal operator.

Proposition 8

If T is 3 -power-hyponormal operator and T is idempotent. Then 2 -power-hyponormal operator.

Proof

Since T is 3 -power-hyponormal operator, then

$$T^3T^* \leq T^*T^3$$

since T is idempotent which implies

$$T^2T^* \leq T^*T^2.$$

Thus 2 -power-hyponormal operator.

Acknowledgments
The authors are grateful to the referee for his comments and suggestions.

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