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Generalizations of δ-primary gamma-ideal of gamma-rings

Mohamed Youssfi Elkettani[1] **and Abdulbakee Kasem**[2]

[1-2]Department of Mathematics Faculty of Sciences,
University Ibn Tofail, B. P 242, Kenitra, Morocco.

Abstract

let M be a Γ-ring, $\varphi: \mathrm{J}(M) \to \mathrm{J}(M)) \cup \{\varnothing\}$ be a function where $\mathrm{J}(M)$ denotes the set of all Γ-ideal of M and δ be a mapping from $\mathrm{J}(M)$ into $\mathrm{J}(M)$ such that for all $I \in \mathrm{J}(M), I \subseteq \delta(I)$ and for all $I, J \in \mathrm{J}(M), I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$. In this paper we introduced the notion of a φ-δ-primary Γ-ideal which is a generalizations of δ-primary Γ-ideals of Γ-ring and study some of it properties.

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Introduction

The concept of Γ-ring has a special place among generalization of rings. For example in Barnes [2], Kyuno [5] and Luch [6] studied the stricture of Γ-ring and obtained various generalization analogous to corresponding parts in rings theory. Zhao [8] investigated the possibilities of a unified approach to studding such tow ideals, and introduced the notion of δ-primary ideals for a mapping δ that assigns to each ideal I an ideal $\delta(I)$ of the same ring, such that the following conditions are satisfied: $I \subseteq \delta(I)$ and $I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$. In [3] Jun and all introduced the notion of Γ-ideal expansions in Γ-rings, let M be a Γ-ring with $\mathrm{J}(M)$ its set of Γ-ideal. A Γ-ideal expansion is a function $\delta: \mathrm{J}(M) \to \mathrm{J}(M)$, which satisfies the following condition: $I \subseteq \delta(I)$ for each Γ-ideal I of M and $I \subseteq J$ implies $\delta(I) \subseteq \delta(J)$, for all Γ-ideals I, J of M. In [1] Anderson and Batanieh give a generalization of prime ideals, let $\phi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\varnothing\}$ be a function. A proper ideal I of M is siad to be ϕ-prime if for $a, b \in M$ with $ab \in I \setminus \phi(I)$, either $a \in I$ or $b \in I$. In [9] Darani give a generalization of primary ideals, let $\phi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\varnothing\}$ be a function where $\mathfrak{J}(M)$ denotes the set of all ideals of M. A proper ideal I of M is called ϕ-primary if whenever $a, b \in M, ab \in I \setminus \phi(I)$ implies that either $a \in I$ or $b \in \sqrt{I}$. So if we take $\phi_{\varnothing}(I) = \varnothing$ (resp., $\phi_0(I) = 0$), a ϕ-primary ideal is primary (resp., weakly primary). Motivated by these generalization in [1], [9], Zhao's idea in [8] and [4]. In this paper we give some more generalization of δ-primary Γ-ideals of Γ-rings and study the properties of these classes of Γ-ideals.

Preliminaries

In this section, we will give some basic concepts about Γ-ring which you need later.

Definition 2. 1 *([3]) Let M and Γ be two abelian groups and for all $x, y \in M$ and all $\alpha, \beta \in \Gamma$ the conditions:*

1. $x\alpha y \in M$;
2. $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$;
3. $(x\alpha y)\beta z = x\alpha(y\beta z)$;

are satisfied, then we call M a Γ-ring.

Definition 2. 2 *([3]) A right (resp. left) Γ-ideal of a Γ-ring M we mean an additive subgroup U of M such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U isboth a right and a left Γ-ideal, then we say that U is a Γ-ideal of M.*

Definition 2. 3 *([2]) A proper Γ-ideal I of M is prime if a $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$ for all $a, b \in M$.*

Definition 2. 4 *([3]) Let M' be a Γ-ring. A mapping $\sigma : M \to M'$ of Γ-rings is called a Γ-ring homomorphism if it satisfies:*

1. $\sigma(a+b) = \sigma(a) + \sigma(b)$ for all $a, b \in M$;
2. $\sigma(a\gamma b) = \sigma(a)\gamma\sigma(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 2. 5 *([3]) A proper Γ-ideal I of M is primary if a $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in \sqrt{I}$ for all $a, b \in M$* .where $\sqrt{I} := \{x \in M \mid (x\gamma)^{n-1}x \in I$ for some $n \in \mathbb{N}$ and $\gamma \in \Gamma\}$, and $(x\gamma)^{n-1}x = x$ where $n = 1$.

Definition 2. 6 *([3]) Let M be a Γ-ring with $\mathcal{J}(M)$ its set of Γ-ideal. A Γ-ideal expansion is a function $\delta:\mathcal{J}(M)\to\mathcal{J}(M)$, which satisfies the following condition:*

(1) $I\subseteq\delta(I)$ for each Γ-ideal I of M

(2) $I\subseteq J$ implies $\delta(I)\subseteq\delta(J)$ for all Γ-ideals I,J of M

Example 1. *([3, Example 3])*

1. The identity function $1_d:\mathcal{J}(M)\to\mathcal{J}(M)$ is a Γ-ideal expansion of M.
2. Denote $\mathcal{M}:=\bigcap\{J\mid I\subseteq J$ and J is a maximal Γ-ideal of $M\}$. A function $g:\mathcal{J}(M)\to\mathcal{J}(M)$ given by $g(I)=\mathcal{M}(I)$ for all $I\in\mathcal{J}(M)$ is a Γ-ideal expansion of M.
3. The constant function $c:\mathcal{J}(M)\to\mathcal{J}(M)$. such that $c(I)=M$, is Γ-ideal expansion of M.

Definition 2. 7 *Given a Γ-ideal expansion δ of M. A proper Γ-ideal I of M is δ-primary if $a\Gamma b\subseteq I$ implies $a\in I$ or $b\in\delta(I)$ for all $a,b\in M$.*

Example 2. *([3, Example 5]) Every Γ-ideal $I\in\mathcal{J}(M)$ is c-primary, where c is a Γ-ideal expansion of M in example 1(3).*

Main Results

In this section we extend the concept of δ-primary Γ-ideal of Γ-ring and we shall show the extend δ-primary Γ-ideal enjoy analogy many of the properties δ-primary Γ-ideal of Γ-ring.

Definition 3. 1 *Let M be a Γ-ring and let $\varphi:\mathcal{J}(M)\to\mathcal{J}(M)\cup\{\varnothing\}$ be a function such that $\varphi(I)\subseteq I$, and δ be an expansion Γ-ideal of M. A proper Γ-ideal I of M is called φ-δ-primary provided that for $a,b\in M$, $a\Gamma b\subseteq I\setminus\varphi(I)$ implies $a\in I$ or $b\in\delta(I)$.*

Example 3 *Let M be a Γ-ring. Define the map $\varphi_\alpha:\mathcal{J}(M)\to\mathcal{J}(M)\cup\{\varnothing\}$ as follows:*

1. $\varphi_\varnothing:\varphi(I)=\varnothing$ defines δ-primary Γ-ideals.
2. $\varphi_0:\varphi(I)=0$ defines weakly δ-primary Γ-ideals.
3. $\varphi_2:\varphi(I)=I\Gamma I$ defines almost δ-primary Γ-ideals.
4. $\varphi_n(n\geq 2):\varphi(I)=(I\Gamma I)^{n-1}I$ defines n-almost δ-primary Γ-ideals.
5. $\varphi_\omega:\varphi(I)=\bigcap_{n=1}^{\infty}(I\Gamma I)^{n-1}I$ defines ω-δ-primary Γ-ideals.
6. $\varphi_1:\varphi(I)=I$ defines any Γ-ideals.

We will begin by giving preliminary theorem from which will show some of the relations between the definitions given in Example 3. Note that the following Theorem is an extension of [9, Lemma 2. 5] and [9, Theorem 2. 6]. The proof of Theorem 3. 3, we need the following Lemma. Recall that An expansion Γ-ideal δ is said to be global if for any Γ-ring homomorphism $f: R \to S$, we have $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in \mathrm{J}(M)$.

Lemma 3. 2 *Let δ be an expansion Γ-ideal, and I, J are Γ-ideals of M. If δ is global then $\delta(I/J) = \delta(I)/J$*

Proof.
Let $i: M \to M/J$ be the natural quotient homomorphism, since δ is global $\delta(I/J) = \delta(i(I)) = i(\delta(I)) = \delta(I)/J$ as desired.

Theorem 3. 3 *let M be a Γ-ring, and let $\varphi: \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be a function and δ be a global expansion Γ-ideal. The following assertions hold.*

(i) If δ is global. Then an Γ-ideal I of M is φ-δ-primary if and only if $I/\varphi(I)$ is a weakly δ-primary of $M/\varphi(I)$.
(ii) If $\psi_1, \psi_2 : \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be function with $\psi_1 \leq \psi_2$. Then if I is ψ_1-δ-primary, it is ψ_2-δ-primary too.
(iii) I δ-primary $\Rightarrow I$ weakly δ-primary $\Rightarrow I$ ω-δ-primary $\Rightarrow I$ (n+1) almost δ-primary $\Rightarrow I$ n-almost δ-primary $\Rightarrow I$ almost δ-primary.
(iv) I is ω-δ-primary if and only if I is n-almost δ-primary for all $n \geq 2$.

Proof.

(i) Assume that I is $\varphi-\delta$-primary Γ-ideal of M. let $a, b \in M$ such that $0 \neq (a+\varphi(I))\Gamma(b+\varphi(I)) \subseteq I/\varphi(I)$ then $a\Gamma b \in I \setminus \varphi(I)$ implies $a \in I$ or $b \in \delta(I)$, Hance $a+\varphi(I) \subseteq I/\varphi(I)$ or $b+\varphi(I) \subseteq \delta(I)/\varphi(I)$, So $a+\varphi(I) \subseteq I/\varphi(I)$ or $b+\varphi(I) \subseteq \delta(I/\varphi(I))$ by Lemma 3. 2. consequently $I/\varphi(I)$ is a weakly δ-primary of $M/\varphi(I)$.conversely, assume that $I/\varphi(I)$ is a weakly δ-primary of $M/\varphi(I)$. let $a, b \in M, a\Gamma b \subseteq I \setminus \varphi(I)$ then $0 \neq (a+\varphi(I))\Gamma(b+\varphi(I)) \in I/\varphi(I)$. since $I/\varphi(I)$ is a weakly δ-primary of $M/\varphi(I)$. So $a+\varphi(I) \in I/\varphi(I)$ or $b+\varphi(I) \in \delta(I/\varphi(I)) = \delta(I)/\varphi(I)$ by Lemma 3. 2. Hence $a \in I$ or $b \in \delta(I)$ as desired.
(ii) Let $a, b \in M$ such that $a\Gamma b \subseteq I \setminus \psi_2(I)$ implies $a, b \subseteq I \setminus \psi_1(I)$, since I is ψ_1-δ-primary Γ-ideal of M then $a \in I$ or $b \in \delta(I)$, as required.
(iii) It follows from (ii) and the fact $\varphi_\varnothing \leq \varphi_0 \leq \varphi_\omega \leq \varphi_{n+1} \leq \varphi_n \leq \varphi_2 \leq \varphi_1$.
(iv) It is similar of (iii).

Let J be an Γ-ideal of M and $\varphi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\varnothing\}$ a function. Define $\varphi_J : \mathfrak{J}(M/J) \to \mathfrak{J}(M/J) \cup \{\phi\}$ by $\varphi_J(I/J) = (\varphi(I)+J)/J$ for every Γ-ideal $I \subseteq \mathfrak{J}(M)$ with $J \subseteq I$ (and $\varphi_J(I/J) = \varnothing$ if $\varphi(I) = \varnothing$). In the following we show that if I is a φ-δ-primary Γ-ideal of M, then I/J is a φ_J-δ-primary Γ-ideal of M/J.

Theorem 3. 4 *Let I be a proper Γ-ideal of the Γ-ring M, and let $\varphi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\varnothing\}$ be a function, δ be a global expansion Γ-ideal. Assume that I is a φ-δ-primary Γ-ideal of M. Then*

1. If J is an Γ-ideal of M with $J \subseteq I$ then I/J is a φ_J-δ-primary Γ-ideal of M/J.
2. If in addition $J \subseteq \varphi(I)$, and I/J is φ_J-δ-primary, then I is φ-δ-primary.
3. If $\varphi(I) \subseteq J$ and I is a φ-δ-primary Γ-ideal of M, then I/J is a weakly δ-primary Γ-ideal of M/J.
4. If $\varphi(I) \subseteq J$, J is a φ-δ-primary Γ-ideal of M and I/J is a weakly δ-primary Γ-ideal of M/J, then I is a φ-δ-primary Γ-ideal of M.

Proof.

1. Suppose that $a,b \in M$ such that $(a+J)\Gamma(b+J) \subseteq I/J \setminus \varphi_J(I/J) = I/J \setminus (\varphi(I)+J)/J$. then $a\Gamma b \subseteq I \setminus \varphi(I)$ and I is φ-δ-primary, gives $a \in I$ or $b \in \delta(I)$. Therfore, $a/J \in I/J$ or $b/J \in \delta(I)/J = \delta(I/J)$. By lemma 3. 2. This shows that I/J is φ_J-δ-primary.
2. Assume that $a\Gamma b \subseteq I \setminus \varphi(I)$ for some $a,b \in M$. Then $(a+J)\Gamma(b+J) \subseteq I/J \setminus \varphi_J(I/J) = I/J \setminus \varphi_J(I/J)$. Since I/J is assumed to be φ_J-δ-primary, we get $a+J \in I/J$ or $b+J \in \delta(I/J) = \delta(I)/J$ by lemma 3. 2. consequently, either $a \in I$ or $b \in \delta(I)$, that is I is φ-δ-primary.
3. Is a direct consequence of part(1).
4. Let $a\gamma b \in I \setminus \varphi(I)$ where $a,b \in M$. Note that $a\Gamma b \subseteq \varphi(I)$ because $\varphi(J) \subseteq \varphi(I)$. If $a\Gamma b \subseteq J$, then either $a \in J \subseteq I$ or $a \in \delta(J) \subseteq \delta(I)$, since J is a φ-δ-primary Γ-ideal. If $a\Gamma b \subseteq J$, then $(a+J)\Gamma(b+I) \subseteq (I/J) \setminus \{0\}$ and so either $a+J \subseteq I/J$ or $b+I \subseteq \delta(I/J) = \delta(I)/J$. Therefore, either $a \in I$ or $b \in \delta(I)$ consequently I φ-δ-primary Γ-ideal of M.

Corollary 3. 5 *Let M be a Γ-ring and let $\varphi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\varnothing\}$ be a function. An Γ-ideal of M is φ-δ-primary if and only if $I/\varphi(I)$ is a weakly δ-primary Γ-ideal of $M/\varphi(I)$.*

Proof.
In partes (2) and (3) of Theorem 3. 4 set $J = \varphi(I)$.

Proposition 3. 6 *Let I be a Γ-ideal of a Γ-ring M such that $\varphi(I)$ be a δ-primary Γ-ideal of M. If I is a φ-δ-primary Γ-ideal of M, then I is an δ-primary Γ-ideal of M.*

Proof.
Assume that $a\Gamma b \subseteq I$ for some elements $a,b \in M$ such that $a \notin I$. If $a\Gamma b \subseteq \varphi(I)$, then $\varphi(I)$ δ-primary and $a \notin I$ implies that $b \in \delta(\varphi(I)) \subseteq \delta(I)$ and so we are done. When $a\Gamma b \not\subset \varphi(I)$ clearly the result follows.
Recall that, for two ideals I and J of a Γ-ring M, the residual division of I and J is defined to be the ideal $I : J = \{x \in M \mid x\Gamma y \subseteq I \ \ for \ all \ either \ \ y \in J\}$.

Theorem 3. 7 *Let I be a proper Γ-ideal of M, let $\varphi : \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be a function and δ is a global expension. Then the following statement are equivalent:*

(i) I is φ-δ-primary
(ii) For every $a \notin \delta(I)$, $(I :_M a) = I \cup (\varphi(I) :_M a)$
(iii) For every $a \notin \delta(I)$, $(I :_M a) = I$ or $(I :_M a) = (\varphi(I) :_M a)$
(iv) For the ideals A and B of M, $A\Gamma B \subseteq I \setminus \varphi(I)$ imply $A \subseteq I$ or $B \subseteq \delta(I)$.

Proof.
(i)$\Rightarrow$(ii) Assume that I is φ-δ-primary. We show that $I \cup (\varphi(I) :_M a) \subseteq (I :_M a)$, let $x \in I$ imply $x\Gamma a \subseteq I$ so $x \in (I :_M a)$. Let $x \in (\varphi(I) :_M a)$ imply $x\Gamma a \subseteq \varphi(I) \subseteq I$ so $x \in (I :_M a)$. Hence $I \cup (\varphi(I) :_M a) \subseteq (I :_M a)$. one the other hand, for every $r \in (I :_M a)$, if $r\Gamma a \subseteq \varphi(I)$, then $r \in (\varphi(I) :_M a)$. otherwise. from $r\Gamma a \subseteq I \setminus \varphi(I)$ and $a \notin \delta(I)$ we get $r \in I$. Hence $(I :_M a) \subseteq I \cup (\varphi(I) :_M a)$. Then $(I :_M a) = I \cup (\varphi(I) :_M a)$.
(ii)$\Rightarrow$(iii) Is clear because $(I :_M a)$ is an Γ-ideal of M
(iii)$\Rightarrow$(iv) Let A and B are Γ-ideals of M with $A\Gamma B \subseteq I$. suppose that $A \not\subset I$ and $B \not\subset \delta(I)$. We will show that $A\Gamma B \subseteq \varphi(I)$. Let $b \in B$ we have tow cases $b \notin \delta(I)$ or $b \in \delta(I)$.
case one: If $b \notin \delta(I)$ we have $(I :_M b) = I$ or $(I :_M b) = (\varphi(I) :_M b)$ by (iii). Now from $A\Gamma b \subseteq A\Gamma B \subseteq I$ we have $A \subseteq (I :_M b)$. choose $a \in A \setminus I$, then from $a \in (I :_M b) \setminus I$ and (iii)we get $(I :_M b) = (\varphi(I) :_M b)$. Therefore, $A \subseteq (I :_M b) = (\varphi(I) :_M b)$, that is $A\Gamma b \subseteq \varphi(I)$. So $A\Gamma B \subseteq \varphi(I)$.
case tow : $b \in \delta(I)$, $b \in B \cap \delta(I)$. choose $b' \in B \setminus \delta(I)$. Then $b + b' \in B \setminus \delta(I)$, and hence we have $A\Gamma b' \subseteq \varphi(I)$, and $A\Gamma(b + b') \subseteq \varphi(I)$. Let $a \in A$. There $a\Gamma b = a\Gamma(b + b') - a\Gamma b' \in \varphi(I)$. Hence, $A\Gamma b \subseteq \varphi(I)$. So $A\Gamma B \subseteq \varphi(I)$. contradiction with assumption $A\Gamma B \not\subset \varphi(I)$. Hence $A \subseteq I$ or $B \subseteq \delta(I)$.

(iv) $\Rightarrow$ (i) Let $a\Gamma b \subseteq I \setminus \varphi(I)$, where $a,b \in M$. Then $(a)\Gamma(b) \subseteq I \setminus \varphi(I)$. By (iv) $(a) \subseteq I$ or $(b) \subseteq \delta(I)$ so $a \in I$ or $b \in \delta(I)$.

Let M be a Γ-ring, and let $\varphi : \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be a function. Recall that, every δ-primary Γ-ideal of M is φ-δ-primary. Theorem 3. 3 and 3. 7 provide some condition under which a φ-δ-primary Γ-ideal is δ-primary.

Theorem 3. 8 *Let M be a Γ-ring, and let $\varphi : \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be a function, and let I be a φ-δ-primary of M.*

(i) If $I\Gamma I \subseteq \varphi(I)$, then I is δ-primary.

(ii) If I is not δ-primary and $\delta(I\Gamma I) = \delta(I)$, then $\delta(I) = \delta(\varphi(I))$

Proof.

(i) Assume that $a,b \in M$ such that $a\Gamma b \subseteq I$. If $a\Gamma b \subseteq \varphi(I)$, since I is φ-δ-primary, either $a \in I$ or $b \in \delta(I)$. Hence we may assume that $a\Gamma b \subseteq \varphi(I)$. If $a\Gamma I \not\subset \varphi(I)$, then there exist an element $a_0 \in I$ such that $a\gamma a_0 \notin \varphi(I)$. Now $a\Gamma(a_0 + b) = a\Gamma a_0 + a\Gamma b \subseteq I \setminus \varphi(I)$ and I is φ-δ-primary that either $a \in I$ or $a_0 + b \in \delta(I)$. But $a_0 \in I \subseteq \delta(I)$. So, either $a \in I$ or $b \in \delta(I)$. Similarly, if $b\Gamma I \subseteq \varphi(I)$, we can show that either $a \in I$ or $b \in \delta(I)$. So we may assume that $a\Gamma I \subseteq \varphi(I)$ and $b\Gamma I \subseteq \varphi(I)$. Since $I\Gamma I \not\subset \varphi(I)$, then exist $c,d \in I$ with $c\Gamma d \not\subset \varphi(I)$. Now $(a+c)\Gamma(b+d) = a\Gamma b + a\Gamma d + c\Gamma b + c\Gamma d \in I \setminus \varphi(I)$, imply that either $a + c \in I$ or $b + d \in \delta(I)$. Therefore, either $a \in I$ or $b \in \delta(I)$. Consequently, I is δ-primary.

(ii) Since $\varphi(I) \subseteq I$, we have $\delta(\varphi(I)) \subseteq \delta(I)$. On the other hand, it follows form part(1)that $I\Gamma I \subseteq \varphi(I)$. Hence $\delta(I) = \delta(I\Gamma I) \subseteq \delta(\varphi(I))$. So $\delta(\varphi(I)) = \delta(I)$

Corollary 3. 9 *Let I be a φ-δ-primary Γ-ideal where $\varphi \le \varphi_3$. Then I is ω-δ-primary.*

Proof.

If I is δ-primary, then it is ω-δ-primary. By theorem 3. 3(iii). Assume that I is not δ-primary. Then $I\Gamma I \subseteq \varphi(I) \subseteq I\Gamma I\Gamma I$ by theorem 3. 8(1). Hence $\varphi(I) = (I\Gamma I)^{n-1}\Gamma I$ for all $n \ge 2$. consequently I is n-almost-δ-primary for every $n \ge 2$ and hence it is ω-δ-primary by theorem 3. 3(iv).

Theorem 3. 10 *Let M be a Γ-ring, let $\varphi : \mathrm{J}(M) \to \mathrm{J}(M) \cup \{\varnothing\}$ be a function and δ is a global expansion. Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of Γ-ideal of M such that* *for every* $\lambda, \lambda' \in \Lambda$, $\sqrt{\varphi(I_\lambda)} = \sqrt{\varphi(I_{\lambda'})}$, $\varphi(I_\lambda) \subseteq \varphi(I)$ *and* $\delta(I_\lambda \Gamma I_\lambda) = \delta(I_\lambda)$. *If for every* $\lambda \in \Lambda$, I_λ *is a* φ-δ*-primary* Γ*-ideal of* M *that is not* δ*-primary, then* $I = \bigcap_{\lambda\in\Lambda} I_\lambda$ *is a* φ-δ*-primary* Γ*-ideal of* M.

Proof.
Since I_λ is a φ - δ -primary but are not δ -primary, then for every $\lambda \in \Lambda, \delta(I_\lambda) = \delta(\varphi(I_\lambda))$ by Theorem 3. 8. on the other hand $\varphi(I_\lambda) \subseteq \varphi(I)$ for every $\lambda \in \Lambda$, and so $\delta(\varphi(I_\lambda)) \subseteq \delta(I)$. we have $I \subseteq I_\lambda \Rightarrow \delta(I) \subseteq \delta(I_\lambda) = \delta(\varphi(I_\lambda))$. Hence $\delta(I) = \delta(I_\lambda) = \delta(\varphi(I_\lambda))$ for every $\lambda \in \Lambda$. Let $a\Gamma b \subseteq I \setminus \varphi(I)$, $a, b \in M$ and $a \notin I$. Therefore there is a $\lambda \in \Lambda$ such that $a \notin I_\lambda$. Since I_λ is φ - δ -primary and $a\Gamma b \subseteq I_\lambda \setminus \varphi(I)$, then $b \in \delta(I_\lambda) = \delta(I)$. consequently I is a φ-δ-primary Γ-ideal of M.

Corollary 3. 11 *Let* M *be* Γ*-ring, and let* $\varphi : \mathfrak{J}(M) \to \mathfrak{J}(M) \cup \{\emptyset\}$ *be a function. If* I *is* φ-δ*-primary* Γ*-ideal of* M *with* $\sqrt{\varphi(I)} = \varphi(\sqrt{I})$ *and* $\sqrt{\delta(I)} = \delta(\sqrt{I})$, *then* $\sqrt{I}$ *is* φ-δ*-primary.*

Proof.
Let $a, b \in M$ such that $a\Gamma b \subseteq \sqrt{I} \setminus \varphi(\sqrt{I})$ but $a \notin \sqrt{I}$. If $(a\gamma b\gamma)^{n-1} a\gamma b \in \varphi(I)$ for all $\gamma \in \Gamma$, then $a\Gamma b \subseteq \sqrt{\varphi(I)} = \varphi(\sqrt{(I)})$ a contradiction. So $(a\gamma b\gamma)^{n-1} a\gamma b \in I \setminus \varphi(I)$. Since I is φ-δ-primary Γ-ideal of M and $a \notin \sqrt{I}$, so $(b\Gamma b)^{n-1} b \subseteq \delta(I)$. Hence $b \in \sqrt{\delta(I)} = \delta(\sqrt{I})$. So $\sqrt{I}$ is φ-δ-primary.
Next we give a definition of additive expansion ideal function. An expansion ideal δ is called additive if $\delta(I + J) = \delta(I) + \delta(J)$ for every Γ-ideals I and J of Γ-ring M. Note that in Theorem 3. 12 δ is an expansion ideal additive.

Theorem 3. 12 *Let* I, J *are* φ-δ*-primary* Γ*-ideals of* Γ*-ring* M *that is not* δ*-primary* Γ*-ideals such that* $\delta(I\Gamma I) = \delta(I)$, *suppose that the two* Γ*-ideals* $\varphi(I)$ *and* $\varphi(J)$ *are not coprime. Then*

1. $\delta(I + J) = \delta(\varphi(I) + \varphi(J))$
2. If $\varphi(I) \subseteq J$ and $\varphi(J) \subseteq \varphi(I + J)$ then $I + J$ is a φ-δ-primary Γ-ideal of M.

Proof.

1. By Theorem 3. 8 we have $\delta(I) = \delta(\varphi(I))$ and $\delta(J) = \delta(\varphi(J))$. Also we have $\varphi(I) \subseteq I$ and $\varphi(J) \subseteq J$ implies
$\delta(\varphi(I) + \varphi(J)) \subseteq \delta(I + J) = \delta(I) + \delta(J) = \delta(\varphi(I)) + \delta(\varphi(J)) = \delta(\varphi(I) + \varphi(J))$
Hence $\delta(I + J) = \delta(\varphi(I) + \varphi(J))$.

2. Assume that $\varphi(I)\subseteq J$ and $\varphi(J)\subseteq\varphi(I+J)$. Since $\varphi(I)+\varphi(J)\neq M$, then $I+J$ is proper Γ-ideal of M, by part(1). Since $(I+J)/J\cong I/I\cap J$ and I is φ-δ-primary, we get that $(I+J)/J$ is a weakly δ-primary Γ-ideal of M/J. By Theorem 3. 4(3). On the other hand J is also φ-δ-primary, by Theorem 3. 3(i). Now, the assertion follows from theorem 3. 4(4).

In the end of this short paper we give the following result related of strongly φ-δ-primary. We called a proper strongly Γ-ideal I of M to be a φ-δ-primary Γ-ideal of M if $I_1\Gamma I_2\subseteq I\setminus\varphi(I)$ for Γ-ideal I_1,I_2 of M implies that either $I_1\subseteq I$ or $I_2\subseteq\delta(I)$.

Theorem 3. 13 *Let I be a proper Γ-ideal of Γ-ring M. Then the following conditions are equivalent:*

1. I is strongly φ-δ-primary.
2. For every Γ-ideals I_1,I_2 of M such that $I\subseteq I_1, I_1\Gamma I_2\subseteq I\setminus\varphi(I)$ implies that either $I_1=I$ or $I_2\subseteq\delta(I)$.

Proof.

1. (1) $\Rightarrow$ (2) Is obviously.
2. (2) $\Rightarrow$ (1) Let J,I_2 be Γ-ideals of M such that $J\Gamma I_2\subseteq I\setminus\varphi(I)$, then we have that $(J+I)\Gamma I_2=J\Gamma I_2+I\Gamma I_2\subseteq I\setminus\varphi(I)$, set $I_1=J+I$. Then, by the hypothesis either $I_1\subseteq I$ or $I_2\subseteq\delta(I)$. Therefore, either $J\subseteq I$ or $I_2\subseteq\delta(I)$. So I is strongly φ-δ-primary Γ-ideal of M.

Corollary 3. 14 *Let I,J are two proper Γ-ideals of Γ-ring M. Such that I is strongly φ-δ-primary and $I\subseteq J$. Then J is strongly φ-δ-primary.*

Proof.
By take $I_1=I_2=J$ in Theorem 3. 13 we have $J\subseteq I$ as required.

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