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Analytical Methods of Research of Convection Process in A Drop

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Abstract

Calculations of the drop profile and the flow function have been carried out, based on the model of natural convection in a drop laying on a horizontal surface. The calculations have been carried out for the case of incomplete wetting. The paper presents analysis of the influence of separate system parameters on the drop spreading process. The results of the research and the conducted computing experiments have shown that change of the surface tension and of the liquid phase density have a maximum influence on the flow function. Thus, the both parameters depend on the temperature whose increase is accompanied by decrease of the contact wetting angle and increase of the three-phase contact point's abscissa.

Keywords-drop; horizontal substrate; drop profile; wetting angle; flow function.

INTRODUCTION

Processes of liquid phase wetting and spreading on a solid surface in various mediums are the initial and the most important stages of many physical and chemical phenomena accompanying modern technologies (for example, [15-17]) and the problem of controlling these processes through external fields (temperature, electric, magnetic, etc.) is one of the most urgent today [3, 4, 7, 14].

It is well known that the surface tension phenomenon can be described easily by the system of hydrodynamical equations [1, 10, 13]. The methods of experimental determination of the surface tension cause no difficulties as well [5, 6]. However these successes take place only in cases when the surface profile form is predetermined or stationary. Processes in free boundary liquids that depend substantially on the wetting phenomenon differ considerably from the ones mentioned above [11, 12, 18].

Let us consider research of one of the problems concerning wetting of solid surfaces by small drops, neglecting the forces that are tangential to the contact line between the liquid and solid mediums and assuming that the chemical composition of the contacting phases are constant. Such restrictions are reasonable for mathematical simplification of the capillarity theory problems and result in replacement of the drop by a cylinder that reduces three-dimensional problem to the two-dimensional one.

PROBLEM STATEMENT

Let us consider the process of natural convection in a liquid drop when it is spreading on a horizontal solid substrate with the temperature T_S. With incomplete wetting, by the liquid, of the limited volume V of the solid body surface and with the drop profile's plane symmetry $x=0$, in approximation of the lubrication theory, the researched process can be described by the following system of equations:

$$\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u_1}{\partial z^2}=0, \qquad (1)$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z}+g\cdot\left[1+\beta\cdot\left(T_S-T\right)\right]=0, \qquad (2)$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial z}=0, \qquad (3)$$

$$\frac{\partial^2 T}{\partial z^2}=0, \qquad (4)$$

where $T(x,z)$-temperature, $P(x,z)$-pressure, $u_1=\vartheta_x$, $u_2=\vartheta_z$-velocity components $\vec{\vartheta}(x,z)$, $z=h(x)$-free surface profile equation, $h_0=h(0)$-drop's apex height, $\pm x_0$-unknown abscissas of the three-phase contact points, g-gravitational acceleration, β-thermal expansion coefficient, ν,ρ-kinematic viscosity and density of the liquid drop.

Let us add the following conditions to the system (1)-(4):

$$\left.\left(P-P_0\right)\right|_{z=h(x)}=-\sigma h''(x), \qquad (5)$$

$$\left.\frac{\partial u_1}{\partial z}\right|_{z=h(x)}=0, \qquad (6)$$

$$\left.u_2(x,z)\right|_{z=h(x)}-\left.u_1(x,z)\right|_{z=h(x)}h'(x)=0, \qquad (7)$$

$$\left.\left(\frac{\partial T}{\partial z}+\alpha T\right)\right|_{z=h(x)}=0, \qquad (8)$$

$$u_1(x,z)\big|_{z=0} = u_2(x,z)\big|_{z=0} = 0\,, \tag{9}$$

$$\int_{-x_0}^{x_0} h(x)\,dx = S\,, \tag{10}$$

where P_0-external pressure, σ-interfacial surface tension at the liquid-environment boundary, α -heat-transfer coefficient, S -area of the drop's radial section.

TEMPERATURE SPREADING IN THE LIQUID PHASE

Let us obtain the law of temperature spreading inside of the liquid drop within the researched model.

By double integrating (4), taking into account (8) and that $T(x,z)\big|_{z=0} = T_S$, we have

$$T(x,z) = T_S - \alpha \cdot z \cdot T(x,h), \tag{11}$$

where the free surface temperature $T(x,h)$ is determined by the equality:

$$T(x,h) = \frac{T_S}{1+\alpha \cdot h}\,. \tag{12}$$

Thus, on the basis of (11) and (12), we can write:

$$T(x,z) = T_S \cdot \left(1 - \frac{\alpha \cdot z}{1+\alpha \cdot h}\right). \tag{13}$$

It obviously follows from the last relation that the temperature inside of the liquid drop is directly proportional to the substrate temperature and its spreading is a nonlinear process that depends essentially on the interfacial area profile.

DETERMINATION OF THE DROP PROFILE

Taking into account (13), let us present the relation (2) in the form:

$$\frac{\partial P}{\partial z} = -\rho g\left(1 + \frac{\alpha \beta T_S z}{1+\alpha h}\right). \tag{14}$$

By differentiating the equality (14) with respect to the variable x, we have:

$$\frac{\partial^2 P}{\partial z \partial x} = \frac{\alpha^2 \rho g \beta T_S}{(1+\alpha h)^2} z h'\,.$$

By integrating this between 0 and z, we obtain:

$$\frac{\partial P}{\partial x} = \frac{\alpha^2 \rho g \beta T_S}{2(1+\alpha h)^2} h' z^2 + f(h)\,, \tag{15}$$

where $f(h)$-the unknown function.

From (1), taking into account (15), we obtain:

$$\frac{\partial^2 u_1}{\partial z^2} = \frac{f(h)}{\nu\rho} + \frac{\alpha^2 g \beta T_S}{2\nu(1+\alpha h)^2} h' z^2\,.$$

As a result of integration between z and h, taking into account the boundary condition (6), the last equality takes the form:

$$\frac{\partial u_1}{\partial z}=\frac{(h-z)f(h)}{\nu\rho}+\frac{\alpha^2 g\beta T_S h'}{6\nu(1+\alpha h)^2}\left(h^3-z^3\right).$$

From here, by integrating between 0 and z, and taking into account (9), we have:

$$u_1(x,z)=\frac{\left(2hz-z^2\right)f(h)}{2\nu\rho}+ \\ +\frac{\alpha^2 g\beta T_S h'}{24\nu(1+\alpha h)^2}\left(4h^3z-z^4\right). \tag{16}$$

From (16), taking into account the continuity condition, we obtain:

$$\int_0^h u_1(x,z)\,dz=\frac{f(h)}{2\nu\rho}\left(2h\frac{h^2}{2}-\frac{h^3}{3}\right)+ \\ +\frac{\alpha^2 g\beta T_S h'}{24\nu(1+\alpha h)}\left(4h^3\frac{h^2}{2}-\frac{h^5}{5}\right)=0,$$

whence we find:

$$f(h)=-\frac{9\alpha^2\rho g\beta T_S h'h^2}{40(1+\alpha h)^2}. \tag{17}$$

Taking into account (17), the equation (15) takes the form:

$$\frac{\partial P}{\partial x}=\frac{\alpha^2\rho g\beta T_S h'}{40(1+\alpha h)^2}\left(20z^2-9h^2\right). \tag{18}$$

In turn, from (5), we have:

$$\left(P_0-\sigma h''\right)'=\left.\frac{\partial P}{\partial x}\right|_{z=h(x)}+\left.\frac{\partial P}{\partial z}\right|_{z=h(x)}h'.$$

From here, taking into account the relations (14) and (18), we find:

$$\sigma h'''=\rho g h'+\frac{\alpha\rho g\beta T_S h'h}{1+\alpha h}- \\ -\frac{11\alpha^2\rho g\beta T_S h'h^2}{40(1+\alpha h)^2}. \tag{19}$$

From (19), we obtain finally the free surface equation:

$$\sigma h'''-\rho g h'-\alpha\rho g\beta T_S h'h\times \\ \times\left(1+\frac{29}{40}\alpha h\right)[1+\alpha h]^{-2}=0. \tag{20}$$

To solve the equation (20), let us introduce the new function $w(h)=h'$, as a result of which the equation takes the form:

$$w\left(\frac{dw}{dh}\right)^2 + w^2\frac{d^2w}{dh^2} -$$
$$-\frac{w\rho g}{\sigma}\left[1+\alpha\beta T_S h\frac{1+\frac{29}{40}\alpha h}{\left(1+\alpha h\right)^2}\right]=0.$$

Because $h'(x)\big|_{x=\pm x_0} = \mp \operatorname{tg}\theta \neq 0$ and, therefore, $w \neq 0$, we can present the last equality in the form:

$$d\left(w\frac{dw}{dh}\right) = F_0(h)\,dh, \qquad (21)$$

where

$$F_0(h) = \frac{\rho g}{\sigma}\left[1+\alpha\beta T_S h\frac{1+\frac{29}{40}\alpha h}{\left(1+\alpha h\right)^2}\right].$$

By integrating (21), taking into account the agreed notation, we have:

$$d\left(w^2\right) = 2c\,dh + F_1(h)\,dh, \qquad (22)$$

where

$$F_1(h) = \frac{2\rho g h}{\sigma} + \frac{1.45\rho g\beta T_S}{\sigma}h +$$
$$+\frac{\rho g\beta T_S}{\sigma}\left[\frac{0.55}{\alpha(1+\alpha h)} - \frac{0.9}{\alpha}\ln\left|1+\alpha h\right|\right].$$

From (22), taking into account that $w\big|_{h=h_0} = 0$, we obtain:

$$w^2 = 2c\left(h-h_0\right) - \int_h^{h_0} F_1(h)\,dh. \qquad (23)$$

Carrying out the inverse substitution of the unknown function $w = \frac{dh}{dx}$, after the separation of variables, let us rewrite (23) in the form:

$$dx = \frac{dh}{\sqrt{2c\left(h-h_0\right)+F_2(h)}}, \qquad (24)$$

where $F_2(h) = -\int_h^{h_0} F_1(h)\,dh$.

From the relation (23), taking into account (10), we obtain:

$$V = SL = 2L\int_{-x_0}^{0} h(x)\,dx =$$
$$= \int_{0}^{h_0} \frac{2Lh(x)\,dh}{\sqrt{2c(h-h_0)+F_2(h)}},$$
where L -characteristic drop length.

The last equality demonstrates that, within the researched model, assignment of the drop volume V is equivalent to the assignment of the drop height h_0.
Then, taking into account that

$$h'(x)\big|_{x=-x_0} = \sqrt{2c\left(h\big|_{x=-x_0} - h_0\right) + F_2\left(h\big|_{x=-x_0}\right)} =$$
$$= \operatorname{tg}\theta,$$

we have:

$$\operatorname{tg}^2\theta = F_2(0) - 2ch_0$$

or

$$c = -\frac{\operatorname{tg}^2\theta}{2h_0} - \frac{\rho g h_0}{\sigma}\cdot(1+0.725\beta T_S) +$$
$$+\frac{\rho g\beta T_S}{2h_0\sigma}\left[\frac{0.9}{\alpha}\cdot h_0 - \right. \tag{25}$$
$$\left. -\left(\frac{0.35}{\alpha^2} + \frac{0.9}{\alpha}\cdot h_0\right)\cdot \ln|1+\alpha h_0|\right].$$

In turn:

$$F_2(h) = \frac{2\rho g + 1.45\rho g\beta T_S}{2\sigma}\cdot\left(h^2 - h_0^2\right) +$$
$$+\frac{\rho g\beta T_S}{\sigma}\cdot\left[\frac{0.9}{\alpha}\left(\frac{1}{\alpha}\cdot\ln\frac{(1+\alpha h_0)^{1+\alpha h_0}}{(1+\alpha h)^{1+\alpha h}} - \right.\right. \tag{26}$$
$$\left.\left. -h_0 + h\right) - \frac{0.55}{\alpha^2}\cdot\ln\left(\frac{|1+\alpha h_0|}{|1+\alpha h|}\right)\right].$$

By integrating (24) and taking into account that $h(x)\big|_{x=\pm x_0} = 0$, we obtain finally:

$$x = x_0 - \int_0^h \frac{dh}{\sqrt{2c(h-h_0)+F_2(h)}}, \tag{27}$$

where c and $F_2(h)$ are determined by the formulas (25), (26).

After transition to dimensionless variables $\xi = x/a$, $\eta = h/a$, $a^2 = \sigma/\rho g$ on the basis of (27), a series of computing experiments has been carried out. The results of one of the experiments are shown in Fig. 1.

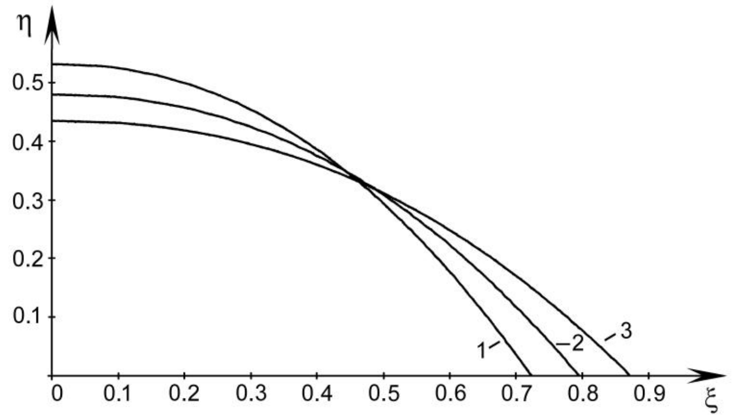

Figure 1: Deformation of the drop's free surface profile with change of the substrate temperature: 1-293°K; 2-298°K; 3-303°K.

The calculation profiles presented in Fig. 1 have been constructed in the MathCAD package. Experimental data for distilled water drops at various temperatures of the surface of the X230 barium borate glass, which is a constructional material of microchannel plates [9], have been used.

CALCULATION OF VELOCITY COMPONENTS AND THE FLOW FUNCTION

From (16), taking into account (17), we have:

$$u_1(x,z) = \frac{\alpha^2 g \beta T_S h'}{240\nu(1+\alpha h)^2} \times \\ \times\left(27h^2z^2 - 14h^3z - 10z^4\right). \tag{28}$$

By differentiating (28) with respect to x, we obtain:

$$\frac{\partial u_1}{\partial x} = \frac{\alpha^2 g \beta T_S}{240\nu(1+\alpha h)^3} \times \\ \times\left\{ \left[h''(1+\alpha h) - 2\alpha(h')^2\right] \times \\ \times\left(27h^2z - 14h^3 - 10z^3\right)z + \\ + 6zh(h')^2(1+\alpha h)(9z-7h)\right\}. \tag{29}$$

From the equation (3), taking into account (29) and the boundary condition (9), we have:

$$u_2(x,z)=\frac{\alpha^2 g\beta T_S}{240\nu(1+\alpha h)}\times$$
$$\times\left\{\left[h''(1+\alpha h)-2\alpha(h')^2\right]\times\right.$$
$$\times\left(9h^2z^3-7h^3z^2-2z^5\right)+$$
$$\left.+3h(h')^2\left(6z^3-7hz^2\right)\right\}.$$

Thus, the flow function $\Psi(x,z)$ connected with the velocity components by the relations $u_1=-\partial\Psi/\partial z$, $u_2=-\partial\Psi/\partial x$ is determined by the equality:

$$\Psi(x,z)=\frac{\alpha^2 g\beta T_S}{240a}\cdot\sqrt{c(h-h_0)+F_2(h)}\times$$
$$\times\frac{z^2\left(7h^3+2z^3-9zh^2\right)}{(1+\alpha h)^2}.$$

Figure 2 presents the graphs of the flow function for the distilled water-barium borate glass' system at the temperature $T=293°K$ in dimensionless variables.

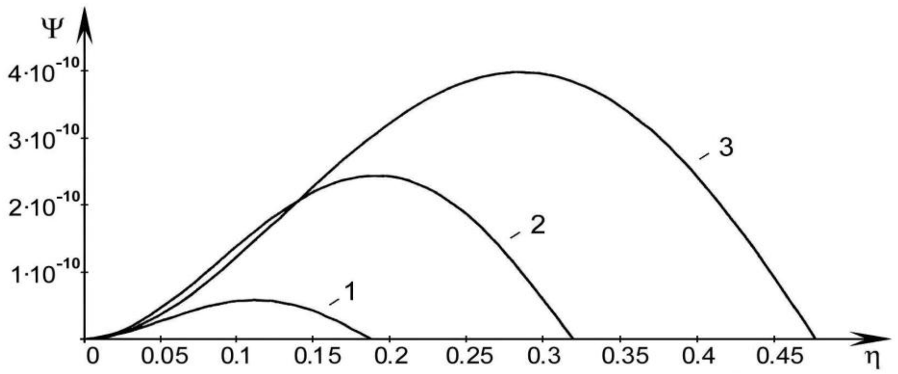

Figure 2: Change of the flow function with the drop height: 1-$\eta=0.186$; 2-$\eta=0.319$; 3-$\eta=0.479$.

CONCLUSION

Calculations show that increase of the solid surface temperature is accompanied by decrease of the contact wetting angle and the drop apex height as well as increase of the three-phase contact point's abscissa and the flow function is proportional to the dh/dx value. These results of the conducted computing experiment correspond to the experimental observations and demonstrate that intensity of movement of the liquid in the drop, with other conditions being equal, is determined by the wetting angle θ.

It should be also noted that the obtained results can be used for solving the problems [2] and [8] connected with calculation of temperature dependence of the three-phase system's physical and chemical parameters (a, σ, θ), based on deformation of the drop's free surface profile.

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