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Derivation of Block Predictor–Block Corrector Method for Direct Solution of Third Order Ordinary Differential Equations

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Abstract

This paper proposes a new numerical method with stepnumber k=5 for solving third order initial value problems of ordinary differential equations directly. Interpolation and collocation technique is applied in developing the method where power series approximate solution is used as a basis function and its third derivative as a collocation equation. The new method developed possesses the following properties namely zero–stability, consistency, convergence and having order six for the main method. The application of the method to some third order initial value problems confirmed its superiority over the existing method when compared in terms of error.

Keywords: Interpolation, Collocation, Block Predictor-Corrector method, Direct solution, Third order initial value problem.

Introduction

In this paper, the third initial value problems of the form

$$y''' = f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \qquad (1)$$

is considered. The usual **c**onventional method of solving higher order initial value problems by reducing to its equivalent system of first order ordinary differential equations (ODEs) was found to have some setbacks which include the difficulty in writing the computer code and burden of computing which affects the accuracy of the method in terms of error (Awoyemi, 1992).Direct methods of solving (1) were developed to overcome the challenges in reduction method, and one of these is predictor–corrector method which approximates the numerical solution of ODEs at one point at a time.

Subsequently, block predictor-corrector methods were proposed to enhance the former method by approximating the numerical solution of ODEs at several points at the same time. Scholars have applied methods developed via this approach to initial value problems and the numerical results generated proved the efficiency of the method in terms of accuracy. Some researchers like Adesanya, et al. (2012) and Odekunle, et al. (2014) to mention a few developed block predictor-corrector methods for direct solution of second order initial value problems of ordinary differential equations with a few concentration on third order initial value problems.

This article proposes the block predictor-block corrector method with stepnumber k=5 for direct solution of third order initial value problem of the form (1).

Derivation of the Block Predictor

Power series of the form

$$y(x)=\sum_{j=0}^{k+3} a_j x^j \tag{2}$$

is considered as an approximate solution to (1). Where k =5 is the stepnumber. Equation (3) is derived bydifferentiating (2) thrice and this gives

$$y'''=\sum_{j=3}^{k+3} j(j-1)(j-2)a_j x^{j-3} \tag{3}$$

Equation (2) is interpolated at $x=x_{n+i}, i=0,1,4$ and (3) is collocated at $x=x_{n+i}, i=0(1)4$. As a result, we get

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+4} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \tag{4}$$

The values of *a's* in (4) is derived by Gaussian elimination method which are substituted into (2) to produce a continuous explicit scheme of the form

$$y(t)=\alpha_4(t)y_{n+4}+\sum_{j=0}^{k-4}\alpha_j(t)y_{n+j}+h^3\sum_{j=0}^{k-1}\beta_j(t)f_{n+j} \tag{5}$$

where $t=\dfrac{x-x_n-4h}{h}$,

$$\alpha_0(t)=\frac{3}{4}t+\frac{t^2}{4}$$
$$\alpha_1(t)=-\frac{4}{3}t-\frac{t^2}{3}$$
$$\alpha_4(t)=\frac{7}{12}t+\frac{t^2}{12}$$
$$\beta_0(t)=h^3(\frac{t}{840}-\frac{11t^2}{1440}+\frac{t^4}{96}+\frac{11t^5}{1440+}\frac{t^6}{480}+\frac{t^7}{5040})$$
$$\beta_1(t)=h^3(\frac{53t}{140}+\frac{41t^2}{240}-\frac{t^4}{18}-\frac{7t^5}{180}-\frac{7t^6}{720}-\frac{t^7}{1260})$$
$$\beta_2(t)=h^3(\frac{113t}{140}+\frac{53t^2}{240}+\frac{t^4}{8}+\frac{19t^5}{240}+\frac{t^6}{60}+\frac{t^7}{840})$$
$$\beta_3(t)=h^3(\frac{313t}{420}+\frac{89t^2}{144}-\frac{t^4}{6}-\frac{13t^5}{180}-\frac{t^6}{80}-\frac{t^7}{1260})$$
$$\beta_4(t)=h^3(\frac{19t}{280}+\frac{79t^2}{480}+\frac{t^3}{6}+\frac{7t^5}{288}+\frac{t^6}{288}+\frac{t^7}{5040})$$

(6)

Evaluating (6) at the non- interpolating points i.e.at t = -2, -1 and 1. This gives the following schemes

$$y_{n+2}=-\frac{1}{6}y_{n+4}-\frac{4}{3}y_{n+1}+\frac{1}{2}y_n-h^3(\frac{1}{360}f_n-\frac{43}{180}f_{n+1}-\frac{7}{20}f_{n+2}-\frac{13}{180}f_{n+3}-\frac{1}{360}f_{n+4})$$
$$y_{n+3}=-\frac{1}{2}y_{n+4}-y_{n+1}+\frac{1}{2}y_n-h^3(\frac{1}{240}f_n-\frac{7}{30}f_{n+1}-\frac{21}{40}f_{n+2}-\frac{7}{30}f_{n+3}-\frac{1}{240}f_{n+4})$$

(7)

$$y_{n+5}=-\frac{5}{3}y_{n+4}+\frac{5}{3}y_{n+1}-y_n+h^3(\frac{1}{72}f_n+\frac{4}{9}f_{n+1}+\frac{5}{4}f_{n+2}+\frac{10}{9}f_{n+3}+\frac{37}{72}f_{n+4})$$

The first and second derivatives of (6) are evaluated at all the points within the specified interval, i.e. t = -4, -3, -2, -1, 0 and 1. This yields the derivatives of (7). Equation (7) and its derivatives at points x_n are then combined in a matrix to produce a block displayed below

$$y_{n+1}=y_n+hy'_n+\frac{1}{2}h^2y''_n+h^3(\frac{113}{1120}f_n+\frac{107}{1008}f_{n+1}-\frac{103}{1680}f_{n+2}+\frac{43}{1680}f_{n+3}-\frac{47}{10080}f_{n+4})$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + h^3(\frac{331}{630} f_n + \frac{332}{315} f_{n+1} - \frac{8}{21} f_{n+2} + \frac{52}{315} f_{n+3} - \frac{19}{630} f_{n+4})$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2} h^2 y''_n + h^3(\frac{1431}{1120} f_n + \frac{1863}{560} f_{n+1} - \frac{243}{560} f_{n+2} + \frac{45}{112} f_{n+3} - \frac{81}{1120} f_{n+4}) \quad (8)$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + h^3(\frac{248}{105} f_n + \frac{2176}{315} f_{n+1} + \frac{32}{105} f_{n+2} + \frac{128}{105} f_{n+3} - \frac{8}{63} f_{n+4})$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2} h^2 y''_n + h^3(\frac{1789}{473} f_n + \frac{2203}{187} f_{n+1} + \frac{625}{336} f_{n+2} + \frac{1547}{499} f_{n+3} + \frac{625}{2016} f_{n+4})$$

The corresponding derivatives are

$$y'_{n+1} = hy'_n + h^2 y''_n + h^3(\frac{-1399}{10080} f_n + \frac{8}{5} f_{n+1} - \frac{1}{3} f_{n+2} + \frac{8}{45} f_{n+3} - \frac{1}{30} f_{n+4})$$

$$y'_{n+2} = hy'_n + 2h^2 y''_n + h^3(\frac{53}{90} f_n + \frac{117}{40} f_{n+1} + \frac{27}{80} f_{n+2} + \frac{3}{8} f_{n+3} - \frac{1}{30} f_{n+4})$$

$$y'_{n+3} = hy'_n + 3h^2 y''_n + h^3(\frac{147}{160} f_n + \frac{117}{40} f_{n+1} + \frac{27}{80} f_{n+2} + \frac{3}{8} f_{n+3} - \frac{9}{160} f_{n+4}) \quad (9)$$

$$y'_{n+4} = hy'_n + 4h^2 y''_n + h^3(\frac{56}{45} f_n + \frac{64}{15} f_{n+1} + \frac{16}{15} f_{n+2} + \frac{64}{45} f_{n+3} + 0 f_{n+4})$$

$$y'_{n+5} = hy'_n + 5h^2 y''_n + h^3(\frac{475}{288} f_n + \frac{125}{24} f_{n+1} + \frac{125}{48} f_{n+2} + \frac{125}{72} f_{n+3} + \frac{125}{96} f_{n+4})$$

$$y''_{n+1} = h^2 y''_n + h(\frac{251}{270} f_n + \frac{323}{360} f_{n+1} - \frac{11}{30} f_{n+2} + \frac{53}{360} f_{n+3} - \frac{19}{720} f_{n+4})$$

$$y''_{n+2} = h^2 y''_n + h(\frac{29}{90} f_n + \frac{62}{45} f_{n+1} + \frac{4}{15} f_{n+2} + \frac{2}{45} f_{n+3} - \frac{1}{90} f_{n+4})$$

$$y''_{n+3} = h^2 y''_n + h(\frac{27}{80} f_n + \frac{51}{40} f_{n+1} + \frac{9}{10} f_{n+2} + \frac{21}{40} f_{n+3} - \frac{3}{80} f_{n+4}) \quad (10)$$

$$y''_{n+4} = h^2 y''_n + h(\frac{14}{45} f_n + \frac{64}{45} f_{n+1} + \frac{8}{15} f_{n+2} + \frac{64}{45} f_{n+3} + \frac{14}{45} f_{n+4})$$

$$y''_{n+5} = h^2 y''_n + h(\frac{95}{144} f_n - \frac{25}{72} f_{n+1} + \frac{25}{6} f_{n+2} - \frac{175}{72} f_{n+3} + \frac{425}{144} f_{n+4})$$

Derivation of the Block Corrector

In developing the block corrector, equation (2) is interpolated at $x = x_{n+i}, i = 0,1,4$ and (3) is collocated $x = x_{n+i}, i = 0(1)5$. The same procedure in Section 2 is adopted and this gives the block and its derivatives as follows

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + h^3\left(\frac{576}{5911} f_n + \frac{354}{2869} f_{n+1} - \frac{209}{2182} f_{n+2} + \frac{173}{2880} f_{n+3} - \frac{80}{3653} f_{n+4} + \frac{139}{40320} f_{n+5}\right)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + h^3\left(\frac{317}{630} f_n + \frac{367}{315} f_{n+1} - \frac{38}{63} f_{n+2} + \frac{122}{315} f_{n+3} - \frac{89}{630} f_{n+4} + \frac{1}{45} f_{n+5}\right)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2 y''_n + h^3\left(\frac{783}{640} f_n + \frac{716}{199} f_{n+1} - \frac{784}{803} f_{n+2} + \frac{423}{448} f_{n+3} - \frac{337}{981} f_{n+4} + \frac{243}{4480} f_{n+5}\right)$$

(11)

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + h^3\left(\frac{712}{315} f_n + \frac{2336}{315} f_{n+1} - \frac{32}{45} f_{n+2} + \frac{704}{315} f_{n+3} - \frac{40}{63} f_{n+4} + \frac{32}{315} f_{n+5}\right)$$

$$y_{n+5} = y_n + 5hy'_n + \frac{25}{2}h^2 y''_n + h^3\left(\frac{1293}{358} f_n + \frac{16916}{1339} f_{n+1} + \frac{625}{4032} f_{n+2} + \frac{543}{113} f_{n+3} - \frac{625}{1152} f_{n+4} + \frac{584}{3425} f_{n+5}\right)$$

$$y'_{n+1} = y'_n + hy''_n + h^2\left(-\frac{113}{853} f_n - \frac{736}{911} f_{n+1} - \frac{162}{883} f_{n+2} - \frac{682}{3671} f_{n+3} + \frac{577}{15120} f_{n+4} - \frac{114}{18049} f_{n+5}\right)$$

$$y'_{n+2} = y'_n + 2hy''_n + h^2\left(\frac{71}{126} f_n + \frac{544}{315} f_{n+1} - \frac{37}{63} f_{n+2} + \frac{136}{315} f_{n+3} - \frac{101}{630} f_{n+4} + \frac{8}{315} f_{n+5}\right)$$

$$y'_{n+3} = y'_n + 3hy''_n + h^2\left(\frac{123}{140} f_n + \frac{1316}{421} f_{n+1} - \frac{9}{140} f_{n+2} + \frac{87}{112} f_{n+3} - \frac{9}{35} f_{n+4} + \frac{9}{224} f_{n+5}\right) \qquad (12)$$

$$y'_{n+4} = y'_n + 4hy''_n + h^2\left(\frac{376}{315} f_n + \frac{1424}{315} f_{n+1} + \frac{176}{315} f_{n+2} + \frac{608}{315} f_{n+3} - \frac{16}{63} f_{n+4} + \frac{16}{315} f_{n+5}\right)$$

$$y'_{n+5} = y'_n + 5hy''_n + h^2\left(\frac{1525}{1008} f_n + \frac{2203}{374} f_{n+1} + \frac{625}{504} f_{n+2} + \frac{1547}{499} f_{n+3} + \frac{625}{1008} f_{n+4} + \frac{275}{2016} f_{n+5}\right)$$

$$y''_{n+1} = y''_n + h(\frac{95}{288} f_n + \frac{1427}{1440} f_{n+1} - \frac{133}{240} f_{n+2} + \frac{241}{720} f_{n+3} - \frac{173}{1440} f_{n+4} + \frac{3}{160} f_{n+5})$$

$$y''_{n+2} = y''_n + h(\frac{14}{45} f_n + \frac{43}{30} f_{n+1} + \frac{7}{45} f_{n+2} + \frac{7}{45} f_{n+3} - \frac{1}{15} f_{n+4} + \frac{1}{90} f_{n+5})$$

$$y''_{n+3} = y''_n + h(\frac{51}{160} f_n + \frac{219}{160} f_{n+1} + \frac{57}{80} f_{n+2} + \frac{57}{80} f_{n+3} - \frac{21}{160} f_{n+4} + \frac{3}{160} f_{n+5}) \quad (13)$$

$$y''_{n+4} = y''_n + h(\frac{14}{45} f_n + \frac{64}{45} f_{n+1} + \frac{8}{15} f_{n+2} + \frac{64}{45} f_{n+3} + \frac{14}{45} f_{n+4})$$

$$y''_{n+5} = y''_n + h(\frac{95}{288} f_n + \frac{125}{96} f_{n+1} + \frac{125}{144} f_{n+2} + \frac{125}{144} f_{n+3} + \frac{125}{96} f_{n+4} + \frac{95}{288} f_{n+5})$$

Properties of the Block Corrector

Order of the Block Corrector

The method proposed by Lambert (1973) is applied in finding the order of the block (11) whereby y and f functions are expanded in Taylor series about x .This gives the method to have order $[6,6,6,6,6]^T$ with error constants

$$[\frac{-1}{3603}, \frac{-1}{9007}, \frac{125}{168}, \frac{1}{1058}, \frac{-1}{1307}]^T.$$

Zero Stability of the Method

A block method (11) is said to be *zero-stable* if the first characteristic polynomial $\rho(r) = \det[rA^0 - A^1] = 0$ satisfying $|r| \le 1$ and for $|r| = 1$ the multiplicity must not exceed the order of differential equation considered.
Therefore, applying the above rule, we have

$$p(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \right| = 0$$

Thus, $r = 0,0,0,0,1$. Hence our method is zero-stable. The new method (11) is *consistent* because the order is greater than one. Furthermore, the developed method is *convergent* since it is consistent and zero stable as defined by Henrici (1962).

Numerical Experiment

In order to examine the accuracy of the new method, the following third order initial value problems are considered.

Problem 1: $y''' = e^x$
$y(0) = 3, y'(0) = 1, y''(0) = 5, h = 0.1$

Theoretical solution: $y(x) = 2 + 2x^2 + e^x$

Table 1: *Comparing the new method with Olabode (2009)*

x	Exact-solution	Computed-solution	Error in new method *k*=5	Error in Olabode (2009), *k*=5
0.1	3.125170918075647700	3.125170918079017000	3.369305E-12	9.24352E-10
0.2	3.301402758160169700	3.301402758181770200	2.160050E-11	8.3983E-10
0.3	3.529858807576003300	3.529858807629335800	5.333245E-11	4.23997E-10
0.4	3.811824697641270600	3.811824697741157000	9.988632E-11	3.58729E-10
0.5	4.148721270700128200	4.148721270860027000	1.598988E-10	2.99872E-10
0.6	4.542118800390508900	4.542118800641649300	2.511404E-10	3.90509E-10
0.7	4.993752707470476600	4.993752707866625500	3.961489E-10	1.47048E-09
0.8	5.505540928492468600	5.505540929085150900	5.926823E-10	2.49247E-09
0.9	6.079603111156950000	6.079603111999866800	8.429168E-10	0.15695E-09
1.0	6.718281828459045500	6.718281829603648200	1.144603E-09	3.54096E-09

Problem 2: $y''' = 3\sin x$
$y(0) = 1, y'(0) = 0, y''(0) = -2$

Theoretical solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 2: *Comparing the new method with Olabode (2009)*

x	Exact-solution	Computed-solution	Error in new method *k*=5	Error in Olabode (2009), *k*=5
0.1	0.990012495834077020	0.990012495834077360	3.330669E-16	1.65922E-10
0.2	0.960199733523725120	0.960199733523724900	2.220446E-16	4.76275E-10
0.3	0.911009467376818090	0.911009467376818090	0.000000E+00	6.23182E-10
0.4	0.843182982008655380	0.843182982008655270	1.110223E-16	19.9134E-10
0.5	0.757747685671118280	0.757747685671118280	0.000000E+00	3.28882E-10
0.6	0.656006844729035250	0.656006844729035030	2.220446E-16	1.27096E-09
0.7	0.539526561853465480	0.539526561853465370	1.110223E-16	4.84653E-09
0.8	0.410120128041496110	0.410120128041496280	1.665335E-16	1.09585E-08
0.9	0.269829904811992980	0.269829904811993150	1.665335E-16	2.0188E-08
1.0	0.120906917604418850	0.120906917604418710	1.387779E-16	3.53956E-08

Conclusion

A block predictor-block corrector of order six for solving third order initial value problems has been developed via interpolation and collocation approach in this paper. The numerical results generated when the new method was applied to third order ODEs are better than existing method of the same stepnumber k = 5 and these are evidently shown in Tables 1 and 2.

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