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Construction and Properties of New Generalized Functions Space $\zeta(E(R^n))$ and Extended Fourier Transformation and Extended Convolution

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Abstract

In this paper, we will construct and study the space $\zeta(E(R^n))$**,** And we will define the the linear and bilinear operations on this space. As a special case we will define $\zeta(S(R^n))$, where $S(R^n)$ be the space of rapidly decreasing functions defined on R^n with topology defined by the family of semi norms $\rho(s,K,\varphi)=\sup\limits_{\substack{\bar{x}\in K\\ |\alpha|\le s}}\left| D^{\alpha}\varphi(\bar{x})\right|$. Also we will define the extended Fourier transform F^* and extended convolution $\bar{*}$ and we will give their properties.

Keywords: Generalized functions, Schwartz distributions, Rapidly decreasing distributions, locally convex algebra**.**

Introduction

The introduction of an associative multiplication in a space of distributions is impossible, numerous strengthenings in this direction have made it possible to isolate a class of pairs (μ,λ) of distributions for which the product is defined. Another approach is based on the use of algebras of new objects (instead of distributions) that possess the basic properties of distributions but allow a correct operation of multiplication, and the distributions or some of them are embedded in the classes of the new objects**.** In such algebras the operations of multiplication and differentiation are defined but the notions of Laplace and Fourier transforms and convolution are not always defined.

Following the general method of constructing algebras of new generalized functions proposed in **[1]**, were constructed algebras, as $L[E]$, $\xi(E)$, $\Pi(E), Z(E)$ **[2-5, 7]**, and the Fourier and Laplace Transforms and differentiation, convolution have been defined.

In this paper, we define the new spaces $(\zeta(E(R^n)), C^*)$ and $(\zeta(S(R^n)), C^*)$ such that**:**

$$E(R^n) \subset E^{/}(R^n) \subset \zeta(E(R^n)), \quad S(R^n) \subset S^{/}(R^n) \subset \zeta(S(R^n))$$

Let Ω be a separated complete locally convex algebra with topology defined by the family of semi norms $P_{i \in I}$ such that for each $i \in I$ there exist $j \in I$ and a constant $C_i > 0$ for which

$$p_i(xy) \le C_i p_j(x) p_j(y) \quad \forall x,\ y \in \Omega \tag{1, 1}$$

If $A : \Omega \to \Omega$ be continuous linear operator, then for any α there exist β and a constant $C_\alpha > 0$ such that

$$p_\alpha(Ax) \le C_\alpha p_\beta(x) \qquad \forall x \in \Omega \tag{1, 2}$$

And if $A : \Omega \times \Omega \to \Omega$ is continuous bilinear maping in Ω, then for any α there exist β, η and a constant $C_\alpha > 0$ such that

$$p_\alpha(B(x, y) \le C_\alpha p_\beta(x) p_\eta(x) \qquad \forall x, y \in \Omega \tag{1, 3}$$

Complex Generalized Numbers C^*

We define the generalized complex numbers as follows:

Let $G(C)$ be the set of all sequences of complex numbers. Define $G^*(C)$ as the set of all sequences $(z_k) \in G(C)$ such that there are a natural number $j \in N$ and a constant $\sigma_1 > 0$, such that $|z_k| < \sigma_1 k^j$ for each k. Define the set $I^*(C)$ as the set of all sequences $(z_k) \in G(C)$ such that for each natural number $i \in N$, there is a constant $\sigma_2 > 0$, such that $|z_k| < \sigma_2 k^{-i}$ for each $k \in N$.

Theorem 2. 1 (a) Each of sets $G(C)$ and $G^*(C)$ is an algebra; (b) The set $I^*(C)$ be an ideal in the algebra $G^*(C)$.

Proof We prove (b) Suppose that $\lambda = (\lambda_k)$ be an elements in $I^*(C)$ that is for each a natural number $i \in N$, there is a constant $\sigma_1 > 0$, such that

$|z_k| < \sigma_1 k^{-i}$ for each k, and let $\eta = (\eta_k) \in G^*(C)$ that is there are a natural number $j \in N$ and a constant $\sigma_2 > 0$, such that $|\eta_k| < \sigma_2 k^j$ for each k.

Now consider the inequality

$$|\ \eta_k \lambda_k\ | = |\ \eta_k\ |\ |\ \lambda_k\ | \leq \sigma_2\ k^j\ \sigma_1\ k^{-i} = k^{j-i}.$$

From which implies that $\eta\lambda = (\eta_k \lambda_k) \in I^*(C)$.
The proof of (a) is similar.

Define the algebra of generalized complex numbers as a factor algebras

$$C^* = C^*(C)/I^*(C).$$

We define the embeddings of the set of all real numbers R and the set of complex numbers C into the space of complex generalized numbers C^* by the following way:

$$f_1 : x \in R \to (x_k + 0\,i) \in C^*, \quad x_k = x \quad \forall\, k \ ;$$

$$f_2 : z \in C \to (z_k) \in C^* \qquad , \quad z_k = z \quad \forall\, k\ .$$

Construction of the Spaces $\zeta(E(R^n))$ and $\zeta(S(R^n))$

Let $E(R^n) = \{\varphi : \varphi : R^n \to C\ ,\ \varphi \in C^\infty(R^n)\}$ the set of all infinitely many differentiable functions on R^n.

Topology τ_1 in $E(R^n)$ we define by the following system of semi norms:

$$\rho(s,K,\varphi) = \sup_{\substack{\bar{x}\in K \\ |\alpha|\leq s}} \left|\ D^\alpha \varphi(\bar{x})\ \right| \tag{3, 1}$$

where

K is a compact set in R^n, and $s \in N \cup \{0\}$; $\bar{x} = (x_1, x_2, \ldots\ldots, x_n)$;

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots. + \alpha_n\ ;\ D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \ldots. \partial y^{\alpha_n}}.$$

Let $S(R^n)$ be the set of all infinitely many differentiable functions of rapid decay **[6]** $\theta : R^n \to C$. Topology τ_2 in $S(R^n)$ we define by the following system of semi norms:

$$\rho(s,h,\theta) = \sup_{\substack{|\eta| \le h \\ |\alpha| \le s}} \left| x_1^{\eta_1} x_2^{\eta_2} \ldots\ldots x_n^{\eta_n} \; D^\alpha \theta(\bar{x}) \right| \tag{3, 2}$$

Where $|\eta| = \eta_1 + \eta_2 + \ldots. + \eta_n$, and $\bar{x} = (x_1, x_2, \ldots\ldots, x_n)$

Theorem 3. 1 The spaces $(E(R^n), \tau_1)$ and $(S(R^n), \tau_2)$ are topological algebras and the corresponding semi norms satisfy the inequality (1, 1).

Proof. We give the proof for the space $(S(R^n), \tau_2)$.
Consider

$$\rho(s, h, (\varphi\theta)) = \sup_{\substack{|\eta| \le h \\ |\alpha| \le s}} \left| x_1^{\eta_1} x_2^{\eta_2} \ldots\ldots x_n^{\eta_n} D^\alpha (\varphi\theta) \right| \le$$

$$2^s . \sup_{\substack{|\eta| \le h \\ |\alpha| \le s}} \left| x_1^{\eta_1} x_2^{\eta_2} \ldots\ldots x_n^{\eta_n} \; D^\alpha (\varphi) \right| . \sup_{\substack{|\eta| \le h \\ |\alpha| \le s}} \left| x_1^{\eta_1} x_2^{\eta_2} \ldots\ldots x_n^{\eta_n} D^\alpha (\theta) \right| . \sup_{|\alpha| \le s} (\sum_{i=0}^{\infty} C_\alpha^i)$$

$$= \; C \, \rho\,(s,h,(\varphi)) \; . \, \rho\,(s,h,(\theta)).$$

The proof for $(E(R^n), \tau_1)$ is similar.

Now we define the following set:

$$\Gamma(E(R^n)) = \left\{ (\varphi_k)_{k=1}^{k-\infty} \; : \varphi_k \in E(R^n) \,, \forall s \,, \forall K \subset R^n \;, \exists m \in N \,, \; \exists d > 0 \;\; \rho(\alpha, K, \varphi_k) \le d \; k^m \; \forall k \right\}$$

$$\Lambda(E(R^n)) = \left\{ (\varphi_k)_{k=1}^{k-\infty} \; : \varphi_k \in E(R^n) \,, \forall s \,, \forall K \subset R^n \;, \forall m \in N \,, \; \exists d > 0 \;\; \rho(\alpha, K, \varphi_k) \le d \, k^{-m} \;\; \forall \, k \right\}$$

Theorem 3. 2 The sets $\Gamma(E(R^n))$ is an algebra**,** the set $\Lambda(E(R^n))$ is an Ideal of the algebra $\Gamma(E(R^n))$.

Proof Suppose $f, g \in \Gamma(E(R^n))$ and $h \in \Lambda(E(R^n))$. By using Theorem 1. 3 we find

$$\rho(s,K,(f_k g_k)) \le C\rho(s,K,(f_k)) \; \rho(s,K,(g_k)) \le (d_1 k^{m_1}) \, (d_2 k^{m_2}) \; = \; d_1 d_2 \; k^{m_1 + m_2}$$

that is $(fg) \in \Gamma(E(R^n))$ which means that the space $\Gamma(E(R^n))$ is an algebra; and from the inequalities $\rho(s,K,(f_k h_k)) \le C\rho(s,K,(f_k)) \; \rho(s,K,(h_k)) \le (c_1 d_1 k^{m_1} \;) \, (c_2 d_2 k^{-m}) \; = d \; k^{-m}$

we conclude that $\Lambda(E(R^n))$ is an ideal in $\Gamma(E(R^n))$.

The proof for $\Gamma^1(S(R^n))$ and $\Lambda^1(S(R^n))$ is similar.

Now we define the required algebra as a factor algebras such as:

$$\zeta(E(R^n)) = \Gamma(E(R^n)) / \Lambda(E(R^n)).$$

The embeddings $E(R^n) \subset E^/(R^n) \subset \zeta(E(R^n))$ show the importance of constructed space $\zeta(E(R^n))$.

As a special case if we take $E(R^n) = S(R^n)$, then we get The new space

$$\zeta(S(R^n)) = \Gamma\ (S(R^n)) / \Lambda\ (S(R^n))$$

And similarly

$$S(R^n) \subset S^/(R^n) \subset \zeta(S(R^n))$$

So in such spaces the multiplication of distributios is defined. Moreover if we connect these spaces with the generalized numbers($L(E(R^n)), C^*$) and ($\zeta(S(R^n)), C^*$) then in our new spaces we can study many mathematical models for example in ($L(S(R^n)), C^*$) the following model has a mathematical sense:

$$M = \begin{cases} Du = \delta^n v \ \ (\delta - Dirac\ \ function) \\ u(a) = b \qquad\qquad a, b \in C^* \\ v \in \zeta(S(R^n)), \end{cases}$$

Now if $T: S(R^n) \to S(R^n)$ is a linear transform, then we extend it to $T^*: L(S(R^n)) \to L\ (S(R^n))$ in the following way:

$T^*: f = (f_k\ ) + N(S(R^n) \in\ \zeta(S(R^n)) \to\ \ (\ T(f_k\ )) + N(S(R^n)\ \ \in\ \zeta(S(R^n))$

in particular if $F: S(R^n) \to S(R^n)$ is the Fourier transform, and $D: S(R^n) \to S(R^n)$ differential operator then we define

$F^*: f = (f_k\ ) + N(S(R^n) \in\ \zeta(S(R^n)) \to\ \ (F(f_k\ )) + N(S(R^n)\ \ \in\ \zeta(S(R^n))$

and

$D^*: f = (f_k\ ) + N(S(R^n) \in\ \zeta(S(R^n)) \to\ \ (D(f_k\ )) + N(S(R^n)\ \ \in\ \zeta(S(R^n))$

Now if $B: S(R^n) \times S(R^n) \to S(R^n)$ is a bilinear operation, then we define

$B: (\ (f_k\ ) + N(S(R^n)\ ,\ (g_k\ ) + N(S(R^n)\ ) \in \zeta(S(R^n)) \times \zeta(S(R^n)) \to\ \ (B(f_k, g_k\ )) + N(S(R^n)\ \ \in \zeta(S(R^n))$

so we may define the extended multiplication $\otimes$, and the extended convolution $\bar{*}$ by

$\otimes: (\ (f_k\ ) + N(S(R^n)\ ,\ (g_k\ ) + N(S(R^n)\ ) \in \zeta(S(R^n)) \times \zeta(S(R^n)) \to\ \ (\ (f_k . g_k\ )) + N(S(R^n)\ \ \in \zeta(S(R^n))$

$\bar{*}: (\ (f_k\ ) + N(S(R^n)\ ,\ (g_k\ ) + N(S(R^n)\ ) \in\ \zeta(S(R^n)) \times \zeta(S(R^n) \to\ \ (\ (f_k * g_k\ )) + N(S(R^n)\ \ \in\ \zeta(S(R^n)\ )$

It is worthy to note that the extended operations F^*, D^*, $\otimes$, $\bar{*}$ satisfy a good for applications the following properties:

1. $F^*(D^{*\,n}\ f\ ) = (-i\ \lambda)^n \otimes F^*\ (f)$,
2. $D^{*\,n}(F^*(\ f\ )) = F^*[(-i\ \lambda)^n \otimes\ f]$,
3. $F^*(\ f \otimes g\ ) = F^* f \,\bar{*}\, F^*\ g$,
4. $F^*(\ f\ \bar{*} g\ ) = F^* f \otimes F^*\ g$,
5. $D^{*\,n}\ (f \otimes g) = \sum_{i=0}^{n} C_n^i D^{*\,i} f \otimes D^{*\,n-i}\ g$,
6. $D^{*\,n}\ (f)\ \bar{*}\ g = (\ (D^{*\,n}\ f)\ \bar{*}\ g)\ =\ (f\ \bar{*}\ (D^{*\,n} g))$.

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