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The (mod, integral and exclusive) sum number of graph $K_{n,n}-((n+2)K_2)$

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Abstract

In this paper, we developed some new formulae for exclusive sum number of the complete bipartite graph $K_{n,m}$. Also we show that for $K_{n,n}-E((n+2)K_2), n\geq 6$ $\sigma=2n-4, \zeta=2n-6, \rho=n-3$ and $\in=2n-4$.

Keywords: (mod, integral, exclusive) sum graph; Complete Bipartite graph $K_{n,m}$; Graph $K_{n,n}-E(nK_2)$ and Graph $K_{n,n}-E((n+2)K_2)$.

Introduction

All graphs we considered in this paper are finite, simple, undirected graphs. We follow in general the graph-theoretic notation and terminology of Ref. [1] unless otherwise specified. The notion of a sum graph was introduced by Harary in 1990 [2]. In 1994 Harary introduced the notion of an integral sum graph [3]. Let $N(Z)$ be the set of all positive integers (integers). The sum graph $G^+(S)$ of a finite subset$S \subset N(Z)$ is the graph (S, E) with $uv \in E$ if and only if $u+v \in S$. A graph G is said to be an (integral) sum graph if it is isomorphic to the sum graph of some $\subset N(Z)$. The (integral) sum number $\sigma(G)\left(\zeta(G)\right)$ of G is the smallest number of isolated vertices which when added to G result in an (integral) sum graph. If G has a (integral) sum labeling then G is called a (integral) sum graph.

The notion of an exclusive sum labeling is introduced in [4]. A mapping L is called a sum labeling of a graph $H(V(H), E(H))$ if it is an injection from $V(H)$to a set of positive integers, such that $xy \in E(H)$ if and only if there exists a vertex $w \in V(H)$

such that $L(w) = L(x) + L(y)$. In this case, w is called a working vertex. We define L as an exclusive sum labeling of G if it is sum labeling of $G \cup rK_1$ for some non-negative integer r and G contain no working vertex. In general a graph G will require some isolated vertices to be labeled exclusively. The least numbers of isolated vertices is called exclusive sum number of G,denoted by $\in (G)$. An exclusive sum labeling of a graph G is said to be optimum if it labels G exclusively by using $\in (G)$ isolated vertices. In case $\in (G) = \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of vertices in G, the labeling is called $\Delta - optimum$ exclusive sum labeling.
Mod sum graph was introduced by Bolland, Laskar, Turner and Domke in 1990, as a generalization of sum labeling [5]. A graph $G = (V, E)$ is a mod sum graph if there exists a positive integer z and a labeling L of vertices of G with distinct elements from $\{1,2, \dots z-1\}$ so that $uv \in E(G)$ if and only if the sum, modulo z, of the labels assigned to u and v is the label of a vertex of G. The mod sum number $\rho(G)$of a connected graph G is the smallest nonnegative r such that $G \cup rK_1$ is a mod sum graph. Any sum graph can be considered as a mod sum graph by choosing a sufficiently large modulus z. The converse is not true. Unlike in the case of sum graphs, there exist mod sum graphs that are connected.

New formula on Exclusive sum labeling of $K_{n,m}$

Let $V(K_{n,m}) = (A, B)$be the bipartition of $K_{n,m}$ and $A = \{a_1, a_2, \dots a_n\}$ $B = \{b_1, b_2, \dots n_n\}$, $C = V(\in K_1)$ be the isolated vertices, $S = V((K_{n,m}) \cup \in K_1)$.

Let $\in = \in (K_{n,m})$for $n \geq 2$ and $m \geq 2$. Since $\in (K_n) = 2n-3$for $m \geq 3$, $\in (S_{n,m}) = \max\{n, m\}$, $\in (S_n) = n$ for $n \geq 2$, $\in (C_n) = 3$ for $n \geq 4$, $\in (P_n) = 2$.

Theorem 2. 1 $\in (K_{n,m}) = \mathrm{n} + \mathrm{m} - 1, for\ n \geq 2$ and $\mathrm{m} \geq 2$.

Proof. We consider the following labeling of the graph $(K_{n,m}) \cup (n+m-1)K_1$:
$a_i = (i-1)N + 1, i = 1,2, \dots, n,$
$b_j = (j-1)N + 3, j = 1,2, \dots, m,$
$c_k = (k-1)N + 4, k = 1,2, \dots, n+m-1,$
where $N \geq 8$ is an integer. It is obvious that the vertex is distinct for each vertex of $(K_{n,m}) \cup (n+m-1)K_1$.

Let $A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_m\}$, $C = \{c_1, c_2, \dots, c_{n+m+1}\}, S = A \cup B \cup C$. In fact it is easy to see that (a_i, b_j, c_k)strict increase, the remainder of dividing N is 1, (3, 4). Thus, we verify that the following assertions are true.

- $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.

- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \in S$ for any $a_i \in A$ and any $b_j \in B$.

Thus the above labeling is an exclusive sum labeling of

$$\left(K_{n,m}\right) \cup (n+m-1)K_1 \; for\; n \geq 2 \; and \; m \geq 2. \qquad \square$$

Exclusive sum labeling of $K_{n,n} - E(nK_2)$

Let $V\left(K_{n,n} - E(\mathrm{n}K_2)\right) = (A, B)$ be the bipartition of $K_{n,n} - E(\mathrm{n}K_2)$, and $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$, $\{a_1b_1, a_2b_2, \dots, a_nb_n\} = E(\mathrm{n}K_2), C = V(\in K_1)$ be the isolated vertices and $S = V\left(\left(K_{n,n} - E(nK_2)\right) \cup \in K_1\right)$. Let $\in = \in \left(K_{n,n} - E(nK_2)\right), for\; n \geq 6$, we will give some properties of the exclusive sum graph $\left(K_{n,n} - E(nK_2)\right), for\; n \geq 6$. It is clear that $K_{n,n} - E(\mathrm{n}K_2)$ is two independent edges for $n = 2$ and $K_{n,n} - E(\mathrm{n}K_2)$ is a 6-cycle for $n = 3$. In this paper, we only consider the case of $n \geq 6$.

Theorem 3. 1 $\in \left(K_{n,n} - \mathrm{E}(\mathrm{nK_2})\right) = 2\mathrm{n} - 3, for\; n \geq 6$.

Proof.

We consider the following labeling of the graph $\left(K_{n,n} - (nK_2)\right) \cup (2n-3)K_1$:

$a_i = (i-1)N + 1, i = 1,2, \dots, n,$
$b_j = (j-1)N + 3, j = 1,2, \dots, n,$
$c_k = (k-1)N + 4, k = 1,2, \dots, n-2,$
$c_k = kN + 4, k = n-1, \dots, 2n-3,$

where $N \geq 8$ is an integer.

Let $V\left(K_{n,n} - E(nK_2)\right) = (A, B)$ be the bipartition of $K_{n,n} - E(nK_2)$, and $A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\} C = V\left((2n-3)K_1\right) = \{c_1, c_2, \dots, c_{2n-3}\}, S = V\left(\left(K_{n,n} - E(nK_2)\right) \cup (2n-3)K_1\right) = A \cup B \cup C$. It is easy to verify that the following assertions are true.

- $S \subset Z_m \backslash \{0\}$.
- $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \notin S$ if and only if $i + j = n \; or \; i = j = n$.

So $a_1b_{n-1}, a_2b_{n-2}, \dots, a_{n-1}b_1, a_nb_n$ is $E(nK_2)$.

Thus the above labeling is an exclusive sum labeling of
$\left(K_{n,n} - E(nK_2)\right) \cup (2n-3)K_1 \; for \; n \geq 6.$ □

The sum number and integral sum number of $K_{n,n} - E\left((n+2)K_2\right)$

Let $V\left(K_{n,n} - E\left((n+2)K_2\right)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\left((n+2)K_2\right)$, and $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$, $\{a_1b_1, a_2b_2, \dots, a_nb_n, a_{n+1}b_{n+1}, a_{n+2}b_{n+2}\} = E\left((n+2)K_2\right)$, $C = V(\sigma K_1)$ be the isolated vertices, $S = V\left(\left(K_{n,n} - E\left((n+2)K_2\right)\right) \cup \sigma K_1\right)$.

Let $\sigma = \sigma\left(K_{n,n} - E\left((n+2)K_2\right)\right), for \; n \geq 6$,we will give some properties of the (integral) sum graph $\left(K_{n,n} - E\left((n+2)K_2\right)\right), for \; n \geq 6$. It is clear that $K_{n,n} - E\left((n+2)K_2\right)$ is four independent edges for $n = 2$ and $K_{n,n} - E\left((n+2)K_2\right)$ is a 10-cycle for $n = 3$. Since $\sigma\left(K_{2,2} - E(2K_2)\right) = 1$, $\zeta\left(K_{2,2} - E(2K_2)\right) = \rho\left(K_{2,2} - E(2K_2)\right) = 0$ and $\sigma(C_6) = 2$ and $\rho(C_6) = 0$. In this paper, we only consider the case of $n \geq 6$.

Lemma 4. 1 In a (integral) sum labeling of $(K_{n,n} - E\left((n+2)K_2\right)) \cup sK_1 \; (s \geq r)$ if there exists $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_i + b_q \in B$ for any $a_i \in A (i \neq q)$.

Proof. Since $a_p + b_q \in B$,we may assume without loss of generality that $a_p + b_q = b_r$. It is obvious that $a_i + b_q \in S$ for any $a_i \in A$ and $i \neq q$. Since $a_p + b_q = b_r \in B$and$a_i \in A$, we have that $a_i + b_r = a_p + \left(a_i + b_q\right) \in S$ for $i \neq r, q$. Thus we can obtain that $a_i + b_q \in B \cup \{a_p\}$ for $i \neq r, q$. We will prove that for $a_i + b_q \notin A$ for any $a_i \in A$ and $i \neq q$. If $a_i + b_q \in A$ we may assume without loss of generality that$a_i + b_q = a_s$. Then $a_s + a_j = a_i + \left(a_j + b_q\right) \notin S$ for $j \neq s, q$. So $a_j + b_q \in A \cup \{b_i\} \cup Cforj \neq s, q$. Hence we have that $a_j + b_q \in (A \cup \{b_i\} \cup C) \cap \left(B \cup \{a_p\}\right) = \{a_p, b_i\}$ for $j \neq r, s, q$. So $n \leq 5$, contradicting the fact that $n \geq 6$. Therefore, $a_i + b_q \notin A$ for any $a_i \in A$ and $i \neq q$. Thus we have that $a_r + b_q \in B \cup C, a_i + b_q \in B$ for $i \neq r, q$. So we only need to prove that$a_r + b_q \notin C$. If $a_r + b_q \in C$ then $a_i + (a_r + b_q) = a_r + (a_i + b_q) \in S$ for $i \neq q$. So $a_i + b_q \in (A \cup \{b_r\} \cup C)$ for$i \neq q$. Hence we have that $a_i + b_q = b_r$for $j \neq r, q$. So $n \leq 3$, contracting the fact that $n \geq 6$. From the above we have that the lemma holds. □

In a similar way we have the following lemma.

Lemma 4. 2. In a (integral) sum labeling of $(K_{n,n} - E\left((n+2)K_2\right)) \cup sK_1 \; (s \geq r)$ if there exists $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in A$ then $a_p + b_i \in A$ for any $b_i \in B (i \neq p)$.

Lemma 4. 3. In a (integral) sum labeling of $(K_{n,n} - E((n+2)K_2)) \cup sK_1\ (s \geq r)$ if there exists $a_p \in A$ and $b_q \in B(q \neq p)$ such that $a_p + b_q \in B$ then $a_p + b_i \in B \cup C$ for any $b_i \in B(i \neq p)$.

Proof. By contradiction, If there exists $r \neq p$ such that $a_p + b_r \in A$ then we have that $a_p + b_i \in A$ for any $b_i \in B$ and $i \neq p$ by Lemma 4. 2. So $a_p + b_q \in A$, contradicting the fact that $a_p + b_q \in B$. □

Lemma 4. 4 $\sigma\left(K_{n,n} - E\left((n+2)K_2\right)\right) \geq 2n-4$ and $\zeta\left(K_{n,n} - E\left((n+2)K_2\right)\right) \geq 2n-6$ for $n \geq 6$.

Proof. If $a_i + b_j \in C$ for any $a_i \in A$ and any $b_j \in B\ (j \neq i)$ we assume that $b_1 < b_2 < \ldots < b_n$ and a_i is the least integer of $A - \{a_n\}$. Then we have that $\sigma\left(K_{n,n} - E\left((n+2)K_2\right)\right) \geq \zeta\left(K_{n,n} - E\left((n+2)K_2\right)\right) \geq 2n-4$ by the distinctness of $a_i + b_1, a_i + b_2, \ldots a_i + b_{i-1}, a_i + b_{i+1}, \ldots, a_i + b_n, a_1 + b_n, \ldots, a_{i-1} + b_n, \ldots, a_{i+1} + b_n, \ldots, a_{n-2} + b_n$. If there exists $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in \mathrm{A} \cup B$ .We assume without loss of generality that $a_p + b_q \in B$. Then we have that $a_i + b_q \in B$ for any $a_i \in A(i \neq q)$, $a_p + b_i \in B \cup C$ for any $b_i \in B(i \neq p)$ and $B = \{b_q, a_1 + b_q, \ldots, a_{q-1} + b_q, a_{q+1} + b_q, a_{n-1} + b_q\}$. Suppose that $a_1 < a_2 < \cdots < a_n$ and b_t is the largest integer of $A - \{b_q\}$.Then $b_t = a_n + b_q$. Firstly we will prove that $a_i + b_j \in C$ for any $a_i \in A(i \neq p)$ and for any $b_j \in B(j \neq \mathrm{i}, q)$.

If there exists $a_r \in A$ and $b_s \in B\ (s \neq r, q)$ such that $a_r + b_s \in B$ then $a_i + b_s \in B$ for any $a_i \in A(i \neq s)$and $B = \{b_s, a_1 + b_s, \ldots, a_{s-1} + b_s, a_{s+1} + b_s, \ldots, a_n + b_s\}$. we may assume without loss of generality that $b_q < b_s$. Then since $a_1 + b_q < a_1 + b_s \leq a_k + b_s$for $s \neq k$ then $a_1 + b_q = b_s$. So $0 < a_1 < a_2 < \cdots < a_n$ and $b_q < a_1 + b_q = b_s < a_1 + b_s \leq a_k + b_s$for $s \neq k$. So $b_q \notin \mathrm{B} = \{b_s, a_1 + b_s, \ldots., a_{s-1} + b_s, a_{s+1} + b_s, \ldots, a_n + b_s\}$, which is a contradiction. If there exists $a_r \in A\ (r \neq q)$ and $b_s \in B(s \neq \mathrm{r}, \mathrm{q})$ such that $a_r + b_s \in A$ then $a_r + b_i \in A$ for any $b_i \in B(i \neq r)$. So $a_r + b_q \in A$, contracting the fact $a_r + b_q \in B$. So $a_i + b_j \in C$ for any $a_i \in A(i \neq q)$ and any $b_j \in B(j \neq \mathrm{i}, q)$ and $a_q + b_j \notin \mathrm{B}$ for any $b_j \in B(j \neq \mathrm{i}, q)$. Hence if $q \neq 1$ then we have that $\zeta\,(K_{n,n} - E((n+2)K_2)) \geq 2n - 6$ by the distinctness of $\{a_1 + b_2,\ a_1 + b_3, \ldots,\ a_1 + b_{q-1},\ a_1 + b_{q+1}, \ldots, a_1 + b_n,\ a_2 + b_t, \ldots,\ a_{n-1} + b_t\} - \{\ a_q + b_t\} \cup \{\ a_t + b_t\}$, if $q = 1$ then we have that $\zeta\,(K_{n,n} - E((n+2)K_2)) \geq 2n - 6$ by the distinctness of $\{a_2 + b_3, \ldots,\ a_2 + b_n,\ a_3 + b_t, \ldots, a_{n-1} + b_t\} - \{\ a_t + b_t\}$. Thus, we only need to prove that $\sigma\left(K_{n,n} - E((n+2)K_2)\right) \geq 2n - 4$. Hence we may assume that $a_i, b_i > 0$ for any $1 \leq i \leq n$ in the following proof. If there exists $b_s \in B\ (s \neq q)$ such that $a_q + b_s \in A$ then $a_q + b_i \in A$ for any $b_j \in B(i \neq q)$ and $A = \{a_q,\ a_q +$

$b_1, \ldots, a_q + b_{q-1}, a_q + b_{q+1}, \ldots, a_q + b_n\}$. Hence $B = \{b_q, a_q + b_q + b_1, \ldots, a_q + b_q + b_{q-1}, a_q + b_q + b_{q+1}, \ldots, a_q + b_q + b_n\}$. We have that $(n-1)(a_q + b_q) = 0$. However, $a_q + b_q > 0$ for $a_q, b_q > 0$, which is a contradiction. So $a_i + b_j \in C$ for any $b_j \in B (j \neq q)$ and any $a_i \in A(i \neq j)$. Since $a_1 + b_1, a_1 + b_2, \ldots, a_1 + b_{q-1}, a_1 + b_{q+1}, \ldots, a_1 + b_n, a_2 + b_t, \ldots, a_{n-1} + b_t$ are distinct. Let $R = \{a_1 + b_1, a_1 + b_2, \ldots, a_1 + b_{q-1}, a_1 + b_{q+1}, \ldots, a_1 + b_n, a_2 + b_t, \ldots, a_{n-1} + b_t\}$. Then $R = \{2a_1 + b_q, a_1 + a_2 + b_q, \ldots, a_1 + a_{q-1} + b_q, a_1 + b_{q+1} + b_q, \ldots, a_1 + a_n + b_q, a_2 + a_n + b_q, \ldots, a_{n-1} + a_n + b_q,\}$ where $2a_1 + b_q < a_1 + a_2 + b_q < \cdots < a_1 + a_{q-1} + b_q < a_1 + b_{q+1} + b_q, \ldots < a_1 + a_n + b_q < a_2 + a_n + b_q < \cdots < a_{n-1} + a_n + b_q$ and $R - \{a_1 + b_1\} \cup \{a_t + b_t\} \subset C$. Since $a_1 + b_q \neq b_q$ we have that $a_q + (a_1 + b_q) = a_1 + a_q + b_q \in C$. By $a_1 + a_{q-1} + b_q < a_1 + a_q + b_q < a_1 + a_{q+1} + b_q$ we can obtain that $a_1 + a_q + b_q \notin$ R. So $|C| \geq |R - \{a_1 + b_1\} \cup \{a_t + b_t\}| + 1 \geq 2n - 4$. □

Lemma 4. 5. $\sigma\left(K_{n,n} - E((n+2)K_2)\right) \leq 2n - 4 \; for \; n \geq 6$.

Proof.
We consider the following labeling of the graph

$\left(K_{n,n} - E((n+2)K_2)\right) \cup (2n-4)K_1$:

$a_i = (i-1)N + 1, i = 1,2, \ldots, n,$
$b_j = (j-1)N + 3, j = 1,2, \ldots, n,$
$c_k = (k-1)N + 4, k = 1,2, \ldots, n-2,$
$c_k = kN + 4, k = n-1, n, \ldots, 2n-4,$
where $N \geq 8$ is an integer.

Let $V\left(K_{n,n} - E((n+2)K_2)\right) = (A, B)$ be the bipartition of $K_{n,n} - E((n+2)K_2)$, and
$A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\} C = V((2n-4)K_1) = \{c_1, c_2, \ldots, c_{2n-4}\}, S = V\left(\left(K_{n,n} - E((n+2)K_2)\right) \cup (2n-4)K_1\right) = A \cup B \cup C$. It is easy to verify that the following assertions are true.

- $S \subset Z_m \backslash \{0\}$.
- $a_i + a_j \notin S$ for any $a_i, a_j \in A(i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \notin S$ if and only if $i + j = n \; or \; 2\,n - 1 \; or \; i = j = n$.

So $a_1 b_{n-1}, a_2 b_{n-2}, \ldots, a_{n-1} b_1, a_n b_{n-1}, a_{n-1} b_n, a_n b_n$ is $E((n+2)K_2)$.

Thus the above labeling is a sum labeling of

$$\left(K_{n,n} - E(n + 2K_2)\right) \cup (2n - 4)K_1 \; for \; n \geq 6. \qquad \square$$

Lemma 4. 6. $\zeta\;(K_{n,n} - E\left((n + 2)K_2\right)) \leq 2n - 6 \; for \; n \geq 6.$

Proof. We consider the following labeling of the graph$\left(K_{n,n} - E\left((n + 2)K_2\right)\right) \cup (2n - 6)K_1$:

$a_i = (i - 1)N + 7, i = 1,2, \dots, n - 1, a_n = 3,$
$b_j = (j - 1)N + 4, j = 1,2, \dots, n - 1, b_n = -3,$
$c_k = (k - 1)N + 11, k = 1,2, \dots, n - 3,$
$c_k = kN + 11, k = n - 2, n - 1, \dots, 2n - 6,$

where $N \geq 30$ is an integer. Let $V\left(K_{n,n} - E\left((n + 2)K_2\right)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\left((n + 2)K_2\right)$, and $A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\} C = V\left((2n - 6)K_1\right) = \{c_1, c_2, \dots, c_{2n-6}\}, S = V\left(\left(K_{n,n} - E\left((n + 2)K_2\right)\right) \cup (2n - 6)K_1\right) = A \cup B \cup C$. It is easy to verify that the following assertions are true.

- $S \subset Z_m \backslash \{0\}$.
- $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \notin S$ if and only if $i + j = n - 1 \; or \; 2n - 3 \; or \; n - 1 \leq i = j \leq n$.

$a_1 b_{n-2}, a_2 b_{n-3}, \dots, a_{n-2} b_1, a_{n-2} b_{n-1} a_{n-1} b_{n-2},, a_{n-1} b_{n-1}, a_n b_n$ is $E\left((n+2)K_2\right)$.

Thus the above labeling is a sum labeling of

$$\left(K_{n,n} - E\left((n + 2)K_2\right)\right) \cup (2n - 6)K_1 \; for \; n \geq 6. \qquad \square$$

We have the following theorem by Lemmas 4. 4-4. 6.

Theorem 4.1. $\zeta\left(K_{n,n} - E\left((n+2)K_2\right)\right) = 2n - 6$ and $\sigma\left(K_{n,n} - E\left((n+2)K_2\right)\right) = 2n - 4$ for $n \geq 6$.

The mod sum number of $K_{n,n} - E\left((n + 2)K_2\right)$

Let $\rho = \rho\left(K_{n,n} - E\left((n + 2)K_2\right)\right), for \; n \geq 6$,we will give some properties of the mod sum graph $\left(K_{n,n} - E\left((n + 2)K_2\right)\right) \cup \rho K_1, for \; n \geq 6$. Let $V\left(K_{n,n} - E\left((n + 2)K_2\right)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\left((n + 2)K_2\right)$ and $A = \{a_1, a_2, \dots, a_n\}$, $B = \{b_1, b_2, \dots, b_n\}$, $\{a_1 b_1, a_2 b_2, \dots, a_{n+2} b_{n+2}\} = E\left((n + 2)K_2\right)$, $C =$

$V(\rho K_1)$ be the isolated vertices, $S = V\left(\left(K_{n,n} - E\big((n+2)K_2\big)\right) \cup \rho K_1\right)$ and the modulus be m. Lemmas 5. 1-5. 3 have been established for (integral) sum graph labeling of $K_{n,n} - E\big((n+2)K_2\big)$ in section 4.

Lemma 5. 1. If there exists $a_p \in A$ and $b_q \in B(q \neq p)$ such that $a_p + b_q \in B$ then $a_i + b_q \in B$ for any $a_i \in A(i \neq q)$.

Lemma 5. 2. If there exists $a_p \in A$ and $b_q \in B(q \neq p)$ such that $a_p + b_q \in A$ then $a_p + b_i \in A$ for any $b_i \in B(i \neq p)$.

Lemma 5. 3. If there exists $a_p \in A$ and $b_q \in B(q \neq p)$ such that $a_p + b_q \in B$ then $a_p + b_i \in B \cup C$ for any $b_i \in B(i \neq p)$.

Lemma 5. 4. $K_{n,n} - E\big((n+2)K_2\big)$is not a mod sum graph for $n \geq 6$.

Proof. By contradiction. If $K_{n,n} - E\big((n+2)K_2\big)$ is a mod sum graph then $C = \emptyset$ and $a_i + b_j \in A \cup B$for any $a_i \in A$ and any $b_j \in B(i \neq j)$. We may assume without loss of generality that there exist $a_p \in A$ and $b_q \in B(q \neq p)$ such that$a_p + b_q \in B$. Then $a_i + b_q \in B$ for any $a_i \in A(i \neq q)$ by Lemma 5. 1. Hence $a_i + b_j \in B$ for any $a_i \in A(i \neq q)$ and any $b_j \in B(i \neq j)$ by Lemma 5. 3. if there exists $b_r \in B(r \neq q)$such that $a_q + b_r \in A$ then $a_q + b_k \in A$ for any $b_k \in B$and $k \neq q$. Thus $A = \{a_q,\ a_q + b_1, \dots,\ a_q + b_{q-1},\ a_q + b_{q+1}, \dots,\ a_q + b_n\}$. So there exists an integer $l_i \neq q$ such that $a_i = a_q + b_{l_i}$ for any $i \neq q$. Since $a_i + a_k = \big(a_k + b_{l_i}\big) + a_q \notin$ S for any $k \neq i, l_i$ we have that $a_k + b_{l_i} \in A \cup \{b_q\} \cup C, k \neq i, l_i$. From the above, we know that $a_k + b_{l_i} \in \{b_q\}, k \neq$ q$, i, l_i$. So $n \leq 4$, contradicting the fact that $n \geq 6$. Hence $a_i + b_j \in B$ for any $a_i \in A$ and any $b_j \in B(j \neq i)$. Let $R = \{a_i + b_1, \dots, a_i + b_{i-1}, a_i + b_{i+1}, \dots, a_i + b_n\} \subset B$. Since $|B \backslash R| = 1$ we may assume without loss of generality that $B \backslash R = \{b_i\}$. Thus $B = \{b_t, a_i + b_1, \dots, a_i + b_{i-1}, a_i + b_{i+1}, \dots, a_i + b_n\}$. Hence $b_t + (n-1)a_i = b_i$.

If $ka_i \neq 0$ for any $k(2 \leq k \leq n-1)$ then $B = \{b_t, b_t + a_i, \dots, b_t + 2a_i, b_t + (n-1)a_i\} = \{a_1 + b_t, \dots, a_{t-1} + b_t, a_{t+1} + b_t, \dots, a_n + b_t, b_t\}$. Hence $\{a_1, \dots, a_{t-1}, a_{t+1} \dots . a_n\} = \{a_i, 2a_i \dots, (n-1)a_i\}$. Since $n \geq 6$ we have that $a_i, 2a_i, 3a_i$ are distinct vertices of A and $a_i + 2a_i = 3a_i$. So a_i is adjacent to $2a_i$, which is a contradiction.

If there exists a least integer l_i $(2 \leq l_i \leq n-1)$ such that $l_i a_i = 0$. Suppose that there exist integers s_i and $r_i (0 \leq r_i \leq l_i)$ such that $-1 = s_i l_i + r_i$. We have that elements of B can be partitioned into $\{b_t, b_t + a_i, \dots, b_t + r_i a_i\}$ and $s_i (\geq 1)$ sets of equal size l_i such that the elements of each set from an l_i −cycle under addition of a_i. Assume that one of the l_i −cycles is $\{b_k, b_k + a_i, \dots, b_k + (l_i - 1)a_i\}$. Since $B = \{a_1 + b_k, \dots, a_{k-1} + b_k, a_{k+1} + b_k, \dots, a_n + b_k, b_k\}$ we have that $\{a_i, 2a_i \dots, (l_i - 1)a_i\} \subset A$. If $l_i \geq 4$ then $a_i, 2a_i, 3a_i$ are distinct vertices of A and $a_i + 2a_i = 3a_i$. So a_i is adjacent to $2a_i$, which is a contradiction. Hence $2 \leq l_i \leq 3$. If $l_i = 2$ then $a_i =$

$m/2$. If $l_i = 3$ then $a_i = m/3$ or $2m/3$. So $n = |A| \leq 3$, contradicting the fact that $n \geq 6$. □

Lemma 5. 5. $\rho\left(K_{n,n} - E\left((n+2)K_2\right)\right) \geq n-3$ for $n \geq 6$.

Proof. By Lemma 5. 4 we have that there exists $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in C$. If $a_i + b_q \in C$ for any $a_i \in A\ (i \neq q)$ then the lemma holds by the distinctness of $a_1 + b_q, \ldots, a_{q-1} + b_q, a_{q+1} + b_q, \ldots, a_n + b_q$. If there exists $a_r \in A\ (r \neq p, q)$ such that $a_r + b_q \notin C$ then $a_r + b_q \in A \cup B$. If $a_r + b_q \in B$ then by Lemma 5. 1 we have that $a_i + b_q \in B$ for any $a_i \in A\ (i \neq q)$, contradicting the fact that $a_p + b_q \in C$. If $a_r + b_q \in A$ then $a_r + b_j \in A$ for any $b_j \in B\ (j \neq r)$ by Lemma 5. 2. Hence $a_i + b_j \notin B$ for any $b_j \in B\ (j \neq r)$ and any $a_i \in A\ (i \neq j)$(If there exists $a_s \in A(s \neq j)$ such that $a_s + b_j \in B$ then we have that $a_k + b_j \in B$ for any $a_k \in A(k \neq j)$, contradicting the fact that $a_r + b_j \in A$.). So $a_i + b_j \in A \cup C$ for any $a_i \in A$ and any $b_j \in B\ (j \neq r, i)$. If there exists $a_p + b_t \in A$ then $a_p + b_j \in A$ for any $b_j \in B\ (j \neq p)$ by Lemma 5. 2, contradicting the fact that $a_p + b_q \in C$. Hence $a_p + b_j \in C$ for any $b_j \in B\ (j \neq r, p)$. So $\left\{ a_p + b_1, \ldots, a_p + b_{p-1}, a_p + b_{p+1}, \ldots, a_p + b_{n-1} \right\} - \left\{ a_p + b_r \right\} \subset C$. The lemma holds. □

Theorem 5. 1. $\rho\left(K_{n,n} - E\left((n+2)K_2\right)\right) = n-3$ for $n \geq 6$.

Proof. By Lemma 5. 5 we only need to prove that $\rho\left(K_{n,n} - E\left((n+2)K_2\right)\right) \leq n-3$. Mod sum labeling of the graph $\left(K_{n,n} - E\left((n+2)K_2\right)\right) \cup (n-3)K_1$ are as follows.

$a_i = (i-1)N + 7, i = 1,2, \ldots, n-1, a_n = 3,$
$b_j = (j-1)N + 4, j = 1,2, \ldots, n-1, b_n = m-3,$
$c_k = (k-1)N + 11, k = 1,3,4, \ldots, n-3,$
And take the modulus $m = (n-1)N$, Where $N \geq 30$ is an integer.
Let $V\left(K_{n,n} - E\left((n+2)K_2\right)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\left((n+2)K_2\right)$, and
$A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}\ C = V\left((n-3)K_1\right) = \{c_1, c_2, \ldots, c_{n-3}\}, S = V\left(\left(K_{n,n} - E\left((n+2)K_2\right)\right) \cup (n-3)K_1\right) = A \cup B \cup C.$
It is easy to verify that the following assertions are true.

- $S \subset Z_m \backslash \{0\}$.
- $a_i + a_j \notin S$ for any $a_i, a_j \in A(i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \notin S$ if and only if $i + j = 3\ or\ n\ or\ i = j = n$.

So $a_1b_2, a_2b_1, a_1b_{n-1}, a_2b_{n-2}, \dots, a_{n-1}b_1, a_nb_n$is $E\big((n+2)K_2\big)$.
Thus the above labeling is a sum labeling of
$\left(K_{n,n} - E\big((n+2)K_2\big)\right) \cup (n-3)K_1 \; for \; n \geq 6.$ □

Exclusive sum labeling of $K_{n,n} - E\big((n+2)K_2\big)$

Let $V\left(K_{n,n} - E\big((\mathrm{n}+2)K_2\big)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\big((\mathrm{n}+2)K_2\big)$, and $A = \{a_1, a_2, \dots, a_n\}$,
$B = \{b_1, b_2, \dots, b_n\}, \{a_1b_1, a_2b_2, \dots, a_nb_n, a_{n+1}b_{n+1}, a_{n+2}b_{n+2}\} = E\big((\mathrm{n}+2)K_2\big), C = V(\in K_1)$ be the isolated vertices, $S = V\left(\left(K_{n,n} - E\big((n+2)K_2\big)\right) \cup \in K_1\right)$. Let $\in = \in \left(K_{n,n} - E\big((n+2)K_2\big)\right), for \; n \geq 6$,we will give some properties of an exclusive sum graph $\left(K_{n,n} - E\big((n+2)K_2\big)\right), for \; n \geq 6$.

Theorem 6. 1 $\in\left(K_{n,n} - E\big((n+2)K_2\big)\right) = 2n-4$ for $n \geq 6$.

Proof. We consider the following labeling of the graph $\left(K_{n,n} - \big((n+2)K_2\big)\right) \cup (2n-4)K_1$:
$a_i = (i-1)N + 1, i = 1,2, \dots, n,$
$b_j = (j-1)N + 3, j = 1,2, \dots, n,$
$c_k = (k-1)N + 4, k = 1,2, \dots, n-2,$
$c_k = kN + 4, k = n-1, \dots, 2n-4,$
where $N \geq 8$ is an integer.
Let $V\left(K_{n,n} - E\big((n+2)K_2\big)\right) = (A, B)$ be the bipartition of $K_{n,n} - E\big((n+2)K_2\big)$, and
$A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\} C = V\big((2n-4)K_1\big) = \{c_1, c_2, \dots, c_{2n-4}\}, S = V\left(\left(K_{n,n} - E\big((n+2)K_2\big)\right) \cup (2n-4)K_1\right) = A \cup B \cup C$. It is easy to verify that the following assertions are true.

- $S \subset Z_m \backslash \{0\}$.
- $a_i + a_j \notin S$ for any $a_i, a_j \in A (i \neq j)$.
- $b_i + b_j \notin S$ for any $b_i, b_j \in B (i \neq j)$.
- $c_i + c_j \notin S$ for any $c_i, c_j \in C (i \neq j)$.
- $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
- $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
- $a_i + b_j \notin S$ if and only if $i + j = n \; or \; 2n-1 \; or \; i = j = n$.

So $a_1b_{n-1}, a_2b_{n-2}, \dots, a_{n-1}b_1, a_nb_{n-1}, a_{n-1}b_n, a_nb_n$ is $E\big((n+2)K_2\big)$.

Thus the above labeling is an exclusive sum labeling of
$\left(K_{n,n} - E\big((n+2)K_2\big)\right) \cup (2n-4)K_1 \; for \; n \geq 6.$ □

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