On Some Labelings of Barbell Graph

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Abstract

The graphs considered here are finite, undirected and simple. A Barbell graph $B(p, n)$ is the graph obtained by connecting $n$-copies of a complete graph $K_p$ by a bridge. In this paper the ways to construct Square Sum, Square Difference, Strongly Multiplicative, Even Mean and Odd Mean labelings for Barbell graphs are reported.

Keywords: Barbell graph, Square Sum labeling, Square Difference labeling, Strongly Multiplicative labeling, Even Mean labeling and Odd Mean labeling.

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1. Introduction

The concept of graph labeling was introduced by A. Rosa in 1967. Let $G(V, E)$ be a $(p, q)$ graph. The Square Sum and Square Difference labelings were introduced by Ajitha, Arumugam and Germina [2009]. They proved that $K_p$ $(p \leq 5)$ have Square-Sum and Square Difference labeling. The concept of Strongly Multiplicative graphs was introduced by Beineke and Hegde [2001]. They proved that $K_p$ $(p \leq 5)$ is strongly Multiplicative graph. The concept of Mean labeling was introduced by Somasundaram and Ponraj [2003]. They studied the Mean labeling of Cycle, Path and
some of their related graph like $C_m \cup P_n$, $P_m \times P_n$, $P_m \times C_n$ etc. In this paper we prove that the existence of Square sum, Square Difference, Strongly Multiplicative, Odd and Even Mean labelings of Barbell graph $B(p, n)$ for $p=5$ and $n \geq 2$.

**Definition 1.1:**
Let $G$ be a $(p, q)$ graph. A one-one map $f: V(G) \rightarrow \{0, 1, \ldots, p-1\}$ is said to be a Square Sum labeling if the induced map $f^*(uv) = (f(u))^2 + (f(v))^2$ is injective. It is said to be a Square Difference labeling if the induced map $f^*(uv) = (f(u))^2 - (f(v))^2$ is injective.

**Definition 1.2:**
A $(p, q)$ graph is said to be Strongly Multiplicative if there exist a one-one map $f: V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the induced map $f^*(uv) = f(u)f(v)$ are distinct.

**Definition 1.3:**
A $(p, q)$ graph is called Even Mean graph, if there exist a one-one map $f: V(G) \rightarrow \{2, 4, 6, 8, \ldots, 2q\}$ such that the induced map $f^*(uv) = (f(u)+f(v))/2$ are distinct.

**Definition 1.4:**
A $(p, q)$ graph is called Odd Mean graph, if there exist a one-one map $f: V(G) \rightarrow \{1, 3, 5, \ldots, (2q-1)\}$ such that the induced map $f^*(uv) = (f(u)+f(v))/2$ are distinct.

**2. Main Results**

**Theorem 2.1:**
The Barbell graph $B(p, n)$ is a Square Sum graph for all $3 \leq p \leq 5$ and for all $n \geq 2$.

**Proof:**
Let $V$ be the vertex set and $E$ be the edge set of $B(p, n)$. Denote the vertex set $V= \{v_1, v_2, \ldots, v_{np}\}$ and edge set as $E= \{e_1, e_2, \ldots, e_{(p^2-p+22n-22)/2}\}$. Define a map $f: V \rightarrow \{0, 1, 2, \ldots, (np-1)\}$ such that $f(v_i) = i-1$, $1 \leq i \leq np$. Define the induced function on edges as $f^*: E \rightarrow \mathbb{N}$ such that

$f^*(v_iv_j) = f(v_i)^2 + f(v_j)^2$
$= (i-1)^2 + (j-1)^2$

To prove $f^*$ is injective we have to prove $f^*(v_iv_j) \neq f^*(v_{i+1}v_{j+1})$, when $i \neq j$. Now $f^*(v_{i+1}v_{j+1}) = i^2 + j^2$

Therefore $(i-1)^2 + (j-1)^2 \neq i^2 + j^2$, when $i \neq j$. Therefore, all edge labels are distinct. Hence $B(p, n)$ admits Square Sum labeling.

**Example 2.1.1:**
Square Sum labeling of $B(5, 2)$, $B(5, 3)$, $B(4, 2)$ are given in the following figure1.
Theorem 2.2:
The Barbell graph $B(p, n)$ is a Square Difference graph for all $3 \leq p \leq 5$ and for all $n \geq 2$.

Proof:
Let $V$ be the vertex set and $E$ be the edge set of $B(p, n)$.
Denote the vertex set $V = \{v_1, v_2, ..., v_{np}\}$ and edge set as $E = \{e_1, e_2, ..., e_{(p^2 - p + 2n - 2)/2}\}$.
Define a map $f: V \rightarrow \{0, 1, 2, ..., (np - 1)\}$ such that $f(v_i) = i - 1$, $1 \leq i \leq np$.
Define the induced function on edges as $f^*: E \rightarrow \mathbb{Z}$ such that $f^*(v_1 v_j) = f(v_i)^2 - f(v_j)^2 = (i - 1)^2 - (j - 1)^2$.

To prove $f^*$ is injective we have to prove $f^*(v_1 v_j) \neq f^*(v_{i+1} v_{j+1})$ where $i \neq j$.
Now $f^* (v_{i+1} v_{j+1}) = i^2 - j^2$.
Therefore $(i-1)^2 - (j-1)^2 \neq i^2 - j^2$, when $i \neq j$. 
Therefore, all edge labels are distinct. Hence $B(p, n)$ admits Square Difference labeling.

**Example 2.2.1:**
Square Difference labeling of $B(4, 3)$, $B(5, 2)$, $B(5, 4)$ are shown in following figure 2.

![Graphs](image)

**Figure 2:** The Square Difference labeling of $B(4, 3)$, $B(5, 2)$ and $B(5, 4)$

**Theorem 2.3:**
The Barbell graph $B(p, n)$ is a Strongly Multiplicative graph for all $3 \leq p \leq 5$ and for all $n \geq 2$.

**Proof:**
Let $G(V, E)$ be a graph.
Denote the vertex set $V = \{v_1, v_2, \ldots, v_{np}\}$ and edge set as $E = \{e_1, e_2, \ldots, e_{(p^2 - p + 22n - 22y)/2}\}$. Define a map $f: V \rightarrow \{1, 2, \ldots, np\}$ such that $f(v_i) = i$, $1 \leq i \leq np$.
Define the induced function on edges as $f^* : E \rightarrow \mathbb{N}$ such that $f^*(v_iv_j) = ij$. 
To prove $f^*$ is injective we have to prove $f^* (v_i v_j) \neq f^* (v_{i+1} v_{j+1})$ when $i \neq j$.

Now $f^* (v_{i+1} v_{j+1}) = (i+1) (j+1)$

Therefore $ij \neq (i+1) (j+1)$ when $i \neq j$.

Therefore all edge labels are distinct.

Hence $B (p, n)$ admits Strongly Multiplicative labeling.

**Example 2.3.1:**

Strongly Multiplicative labeling of $B (5, 2)$ and $B (5, 3)$ are given in following figure 3.

![Figure 3: The Strongly Multiplicative labeling of $B (5, 2)$ and $B (5, 3)$](image)

**Theorem 2.4:**

The Barbell graph $B (p, n)$ admits Even-Mean labeling for $p=5$ and for all $n \geq 2$.

**Proof:**

Let $V$ be the vertex set and $E$ be the edge set of $B (p, n)$

Denote the vertex set $V= \{v_1^1, v_2^1, ..., v_p^1\}$ and edge set as $E= \{e_1, e_2, ..., e_{(p^2-p+2n-2)/2}\}$.

Define $f: V \rightarrow \{2, 4, 6, ..., 2q\}$ as follows.

Label the vertices $v_1^1, v_2^1, v_3^1, v_4^1, v_5^1$ as follows.

$f (v_1^1)= 2$

$f (v_2^1)= 4$

$f (v_3^1)= 16$

$f (v_4^1)= 6$

$f (v_5^1)= 10$

To label the remaining vertices of Barbell graph namely $v_1^2, v_2^2, v_3^2, v_4^2, v_5^2, ..., v_1^n, v_2^n, ..., v_5^n$ consider the following cases.
Case 1:
When \( n \equiv 0 \pmod{2} \)
The vertices are labeled by the formula,
\[
f(v^n_j) = f(v^{n-1}_j) + n + (n-1)PC_2 \quad \text{for } n \geq 2, 1 \leq j \leq 5
\]

Case 2:
When \( n \equiv 1 \pmod{2} \)
The vertices are labeled by the formula,
\[
f(v^n_j) = f(v^{n-1}_j) + (n-1) + PC_2(n-1) \quad \text{for } n \geq 3, 1 \leq j \leq 5
\]
Define the induced function on edges as \( f^*: E \to \mathbb{N} \) such that
\[
f^*(v^n_i, v^m_j) = \frac{f(v^n_i) + f(v^m_j)}{2}
\]
To prove \( f^* \) is injective we have to prove \( f^* (v^n_i, v^m_j) \neq f^* (v^n_{i+1}, v^m_{j+1}) \), when \( m=n \), \( m=n+1 \) and \( i \neq j \)
When \( m=n \);
Now \( f^*(v^n_i, v^m_j) = \frac{f(v^n_i) + f(v^m_j)}{2} \)
Therefore \( \frac{f(v^n_i) + f(v^m_j)}{2} \neq \frac{f(v^n_i) + f(v^m_j)}{2} \) when \( i \neq j \)

When \( m=n+1 \);
Now \( f^*(v^n_{i+1}, v^{m+1}_j) = \frac{f(v^n_{i+1}) + f(v^{m+1}_j)}{2} \)
Therefore \( \frac{f(v^n_{i+1}) + f(v^{m+1}_j)}{2} \neq \frac{f(v^n_{i+1}) + f(v^{m+1}_j)}{2} \) when \( i \neq j \)

Therefore all edge labels are distinct.
Hence \( f^* \) is Even Mean labeling.

Example 2.4.1:
Even Mean labeling of \( B(5, 3) \) is shown in the following figure 4.
Theorem 2.5:
The Barbell graph \( B(p, n) \) admits Odd Mean labeling for \( p=5 \) and for all \( n \geq 2 \).

Proof:
Let \( V \) be the vertex set and \( E \) be the edge set of \( B(p, n) \)
Denote the vertex set \( V= \{v_1^1, v_2^1, ..., v_p^1\} \) and edge set as \( E= \{e_1, e_2, ..., e_{(p^2-p+2n-22)/2}\} \).
Define \( f: V \rightarrow \{1, 3, 5, ..., (2q-1)\} \) as follows.
Label the vertices \( v_1^1, v_2^1, v_3^1, v_4^1, v_5^1 \) as follows.
\[ f(v_1^1) = 1 \]
\[ f(v_2^1) = 3 \]
\[ f(v_3^1) = 15 \]
\[ f(v_4^1) = 5 \]
\[ f(v_5^1) = 9 \]
To label the remaining vertices of Barbell graph namely \( v_1^n, v_2^n, v_3^n, v_4^n, v_5^n, ..., v_1^n, v_2^n, ..., v_5^n \) consider the following cases.

Case 1:
When \( n \equiv 0 \pmod{2} \)
The vertices are labeled by the formula,
\[ f(v_j^n) = f(v_{j-1}^{n-1}) + n + (n-1)PC \]
for \( n \geq 2, 1 \leq j \leq 5 \)

Case 2:
When \( n \equiv 1 \pmod{2} \)
The vertices are labeled by the formula,
\[ f(v_j^n) = f(v_{j-1}^{n-1}) + (n-1) + PC \]
for \( n \geq 3, 1 \leq j \leq 5 \)
Define the induced function on edges as \( f^*: E \rightarrow \mathbb{N} \) such that
\[ f^*(v_i^n, v_j^m) = \frac{f(v_i^n) + f(v_j^m)}{2} \]
To prove \( f^* \) is injective we have to prove \( f^*(v_i^n, v_j^m) \neq f^*(v_{i+1}^n, v_{j+1}^m) \), when \( m=n \), \( m=n+1 \) and \( i \neq j \)
When \( m=n; \)
Now \( f^*(v_{i+1}^n, v_{j+1}^m) = \frac{f(v_{i+1}^n) + f(v_{j+1}^m)}{2} \)
Therefore \( \frac{f(v_i^n) + f(v_j^m)}{2} \neq \frac{f(v_{i+1}^n) + f(v_{j+1}^m)}{2} \) when \( i \neq j \)
When \( m=n+1; \)
Now \( f^*(v_{i+1}^n, v_{j+1}^m) = \frac{f(v_{i+1}^n) + f(v_{j+1}^{n+1})}{2} \)
Therefore \( \frac{f(v_i^n) + f(v_{j+1}^{n+1})}{2} \neq \frac{f(v_{i+1}^n) + f(v_{j+1}^{n+1})}{2} \) when \( i \neq j \)
Therefore all edge labels are distinct.
Hence \( f \) is Odd Mean labeling.
Example 2.5.1:
Odd Mean labeling of $B(5, 3)$ is shown in the following figure 5.

![Figure 5: The Odd Mean labeling of B (5, 3)](image)

Conclusion
In this paper we have examined the existence of Square sum, Square difference, Strongly Multiplicative and Even Mean, Odd Mean labelings for Barbell graph for all $n \geq 2$. Further investigation can be done to obtain the above labeling for some class of graphs.

References
