Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2015), pp. 125–145

http://www.ripublication.com/gjpam.htm

Equitable, restrained and k-domination in intuitionistic fuzzy graphs

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Abstract

Graph theory is one of the most flourishing branches of modern mathematics and computer applications. The theory of domination has been the nucleus of research activity in graph theory in recent times. This is largely due to a variety of new parameters that can be developed from the basic definition of domination.

In this paper, the concept of equitable domination, k-domination, restrained domination, connected equitable domination, total k-domination and global restrained domination in intuitionistic fuzzy graphs(IFGs) have been introduced. The lower and upper bound for the equitable, restrained and k-domination numbers of an IFG are obtained. Further, some properties of the domination numbers d_e, d_k, d_r and d_{gr} with known parameters of G are investigated. Also, domination parameters such as equitable domination, k-domination on direct, cartesian and strong products of two IFGs are analyzed.

AMS subject classification:
Keywords: Equitable domination, k-domination, restrained domination.

1. Introduction

Graph theory has numerous applications in modern science and technology. Domination in graph theory has many interesting applications in real world applications such as locating radar stations, nuclear power plants, communication networks and voting situations. Finding the minimal dominating set can be used to optimize time and distance while traveling, to optimize the performance of computer communication networks.

The study of dominating sets in graphs was started by Ore and Berge [4, 10] and the domination number was introduced by Cockayne and Hedetniemi [6]. Presently, science and technology are featured with complex processes and phenomena for which complete and precise information is not always available. For such cases, mathematical models are developed to handle the types of systems containing elements of uncertainty.

The notion of fuzzy sets was introduced by Zadeh [18] as a method of representing uncertainty and vagueness. The first definition of fuzzy graphs was proposed by Kaufmann [8] from the fuzzy relations introduced by Zadeh. The concept of domination in fuzzy graphs introduced by A.Somasundaram and S.Somasundaram [15]. Intuitionistic fuzzy models give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. In 1984, Atanassov [1] introduced intuitionistic fuzzy sets as a generalization of fuzzy sets added a new component which determines the degree of non-membership. Intuitionistic fuzzy graph was introduced in [1]. In [11], the concept of domination, total domination, connected domination have been introduced. In this way, the authors got motivated to work further on the theory of domination and to introduce new concepts such as equitable dominating set, k-dominating set, restrained and global restrained dominating set which have since been introduced. This paper is organized as follows: section 2 contains basic notations and definitions required for this work. In section 3, the definition of equitable dominating set, k-dominating set, restrained and global restrained dominating set of an IFG is given. Also, analyzed some properties of the domination numbers d_e, d_k, d_r and d_{gr} with known parameters of G. In section 4, equitable domination, k-domination is studied on direct, strong and cartesian products of two IFGs. Section 5 concludes the paper. For other notations and terminologies not mentioned in the paper, the readers may read [7, 11, 16, 17].

2. Preliminaries

In this section, some basic definitions relating to IFGs are given. Throughout this paper, simple and undirected IFGs are taken into consideration.

Definition 2.1. [1] Let a set E be fixed. An *intuitionistic fuzzy set* (IFS) A in E is an object of the form $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in E\}$, where the function $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$ respectively, and for every $x \in E$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

Notations

μ_i, γ_i	Degrees of membership and non-membership of the vertex v_i of G
μ_{ij}, γ_{ij}	Degrees of membership and non-membership of the edge e_{ij} of G
$\mu_{ij}^{\infty}, \gamma_{ij}^{\infty}$	μ, γ-strength of connectedness between the vertex v_i and v_j in G
$deg_{\mu_i}, deg_{\gamma_i}$	μ, γ-degree of a vertex v_i in G
$d_e(G)$	Equitable domination number of G
$d_{ie}(G)$	Independent equitable domination number of G

EN_v^i	Equitable neighborhood of a vertex v_i in G.
EN_μ^i, EN_γ^i	μ, γ-equitable neighborhood of a vertex v_i in G.
$deg_{EN^i\mu}$, $deg_{EN_\gamma^i}$	μ, γ-equitable neighborhood degree of a vertex v_i in G
$d_k(G)$	k-domination number of G
$d_{ik}(G)$	k-independent domination number of G
$d_r(G)$	Restrained domination number of G
$d_{gr}(G)$	Global restrained domination number of G
$d_{se}(G)$,$d_{we}(G)$	Strong (weak) equitable domination number of G
$d_{ce}(G)$	Connected equitable domination number of G
$d_{tk}(G)$	Total k-domination number of G
$d_{rc}(G)$	Restrained connected domination number of G.

Definition 2.2. [12] Let X be a universal set and let V be an IFS over X in the form $V = \{\langle v_i, \mu_i, \gamma_i\rangle \,|v_i \in V\}$ such that $0 \leq \mu_i + \gamma_i \leq 1$. *Six types of cartesian products* of n elements of V over X are defined as

$$\begin{aligned}
v_1 \times_1 v_2 \times_1 v_3 ... \times_1 v_n = \{\langle (v_1, v_2, \ldots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \rangle \\
|v_1, v_2, \cdots, v_n \in V\},
\end{aligned}$$

$$\begin{aligned}
v_1 \times_2 v_2 \times_2 v_3 ... \times_2 v_n = \{\langle (v_1, v_2, \ldots, v_n), \sum_{i=1}^{n} \mu_i - \sum_{i \neq j} \mu_i \mu_j \\
+ \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \mu_i \mu_j \mu_k \cdots \mu_n \\
+ (-1)^{n-1} \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \rangle |v_1, v_2, \cdots, v_n \in V\}
\end{aligned}$$

$$\begin{aligned}
v_1 \times_3 v_2 \times_3 v_3 ... \times_3 v_n = \{\langle (v_1, v_2, \ldots, v_n), \prod_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \gamma_i - \sum_{i \neq j} \gamma_i \gamma_j \\
+ \sum_{i \neq j \neq k} \gamma_i \gamma_j \gamma_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \gamma_i \gamma_j \gamma_k \cdots \gamma_n \\
+ (-1)^{n-1} \prod_{i=1}^{n} \gamma_i \rangle |v_1, v_2, \cdots, v_n \in V\}
\end{aligned}$$

$$\begin{aligned}
v_1 \times_4 v_2 \times_4 v_3 ... \times_4 v_n = \{\langle (v_1, v_2, \cdots, v_n), \min(\mu_1, \mu_2, \ldots, \mu_n), \\
\max(\gamma_1, \gamma_2, \cdots, \gamma_n) \rangle |v_1, v_2, \cdots, v_n \in V\}
\end{aligned}$$

$$\begin{aligned}
v_1 \times_5 v_2 \times_5 v_3 ... \times_5 v_n = \{\langle (v_1, v_2, \ldots, v_n), \max(\mu_1, \mu_2, \ldots, \mu_n), \\
\min(\gamma_1, \gamma_2, \cdots, \gamma_n) \rangle |v_1, v_2, \cdots, v_n \in V\}.
\end{aligned}$$

$$\begin{aligned}
v_1 \times_6 v_2 \times_6 v_3 ... \times_6 v_n = \{\langle (v_1, v_2, \cdots, v_n), \frac{\sum_{i=1}^{n} \mu_i}{n}, \frac{\sum_{i=1}^{n} \gamma_i}{n} \rangle \\
|v_1, v_2, \cdots, v_n \in V\}.
\end{aligned}$$

It must be noted that $v_i \times_t v_j$ is an IFS, where $t = 1, 2, 3, 4, 5, 6$ such that the sum of their degrees of membership and non-membership lies in [0, 1].

Definition 2.3. [7] An *intuitionistic fuzzy graph (IFG)* is of the form $G = (V, E)$ where (i) $V = \{v_1, v_2, ...v_n\}$ such that $\mu_i : V \to [0, 1]$ and $\gamma_i : V \to [0, 1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and

$$0 \leq \mu_i(v_i) + \gamma_i(v_i) \leq 1$$

for every $v_i \in V, i = 1, 2, ..., n$
(ii) $E \subseteq V \times V$ where $\mu_{ij} : V \times V \to [0, 1]$ and $\gamma_{ij} : V \times V \to [0, 1]$ are such that

$$\mu_{ij} \leq \mu_i \oslash \mu_j$$
$$\gamma_{ij} \leq \gamma_i \oslash \gamma_j$$

and

$$0 \leq \mu_{ij} + \gamma_{ij} \leq 1$$

where μ_{ij} and γ_{ij} are the degrees of membership and non-membership of the edge (v_i, v_j); the values $\mu_i \oslash \mu_j$ and $\gamma_i \oslash \gamma_j$ can be determined by one of the six cartesian products $\times_t,\ t = 1, 2, 3, 4, 5, 6$ for all i and j given in Definition 2.2.

Definition 2.4. [11] Let $G = (V, E)$ be an IFG, then the *cardinality* of a subset S of V is defined as $|S| = \sum_{v_i \in S} \left(\frac{1 + \mu_i - \gamma_i}{2} \right)$ for all $v_i \in S$.

Definition 2.5. [11] The number of vertices in G is called as *order* of an IFG, $G = (V, E)$, denoted by $o(G)$, and is defined as $o(G) = \sum_{v_i \in V} \left(\frac{1 + \mu_i - \gamma_i}{2} \right)$ for all $v_i \in V$.

Definition 2.6. [11] If $v_i, v_j \in V \subseteq G$, the μ-strength of connectedness between v_i and v_j is $\mu_{ij}^{\infty} = \sup\{\mu_{ij}^k \mid k = 1, 2, \ldots, n\}$ and γ-strength of connectedness between v_i and v_j is $\gamma_{ij}^{\infty} = \inf\{\gamma_{ij}^k \mid k = 1, 2, \ldots, n\}$.
If v_i, v_j are connected by means of paths of length k then μ_{ij}^k is defined as $\sup\{\mu_{i1} \wedge \mu_{12} \wedge \mu_{23} \ldots \wedge \mu_{k-1j} \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$ and γ_{ij}^k is defined as $\inf\{\gamma_{i1} \vee \gamma_{12} \vee \gamma_{23} \ldots \vee \gamma_{k-1j} \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$.

Definition 2.7. [11] An edge (v_i, v_j) is said to be a *strong edge* of an IFG $G = (V, E)$, if $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$.

Definition 2.8. [11] An IFG, $G = (V, E)$ is said to be *connected* IFG if there exists a path between every pair of vertices v_i, v_j in V. Connected IFG is also defined using strength of connectedness as follows:

(i) $\mu_{ij}^{\infty} > 0$, and $\gamma_{ij}^{\infty} > 0$

(ii) $\mu_{ij}^{\infty} = 0$, and $\gamma_{ij}^{\infty} > 0$

(iii) $\mu_{ij}^{\infty} > 0$, and $\gamma_{ij}^{\infty} = 0$ for all $v_i, v_j \in V$.

Definition 2.9. [7] The *complement* of an IFG, $G = (V, E)$ is an IFG, $\overline{G} = (\overline{V}, \overline{E})$, where

(i) $\overline{V} = V$,

(ii) $\overline{\mu_i} = \mu_i$ and $\overline{\gamma_i} = \gamma_i$, for all $i = 1, 2, \ldots, n$,

(iii) $\overline{\mu_{ij}} = \min(\mu_i, \mu_j) - \mu_{ij}$ and
$\overline{\gamma_{ij}} = \max(\gamma_i, \gamma_j) - \gamma_{ij}$ for all $i, j = 1, 2, \ldots, n$.

Definition 2.10. [16] The *cartesian product* of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \Box G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle), V_1 \cap V_2 = \phi$ where

(i) $V = \left\{ v_i u_p \mid v_i \in V_1, u_p \in V_2 \right\}, i = 1, 2, \ldots m, p = 1, 2, \ldots n$

(ii) $E = (v_i u_p, v_j u_q)$, such that either one of the following is true :

- $(u_p, u_q) \in E_2$, when $i = j$
- $(v_i, v_j) \in E_1$, when $p = q$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of G, and is given by $\langle \mu_r, \gamma_r \rangle = \left\langle \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \right\rangle$ for all $v_r \in V$, $r = 1, 2, 3, \ldots, mn$

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of G, and is given by

$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} \left\langle \min(\mu_i, \mu_{pq}), \max(\gamma_i, \gamma_{pq}) \right\rangle & \text{if } i = j, (u_p, u_q) \in E_2 \\ \left\langle \min(\mu_p, \mu_{ij}), \max(\gamma_p, \gamma_{ij}) \right\rangle & \text{if } p = q, (v_i, v_j) \in E_1 \\ \langle 0, 0 \rangle & \text{otherwise} \end{cases}$$

Definition 2.11. [16] The *direct product* of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \otimes G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle, V_1 \cap V_2 = \phi)$ where

(i) $V = \left\{ v_i u_p \mid v_i \in V_1, u_p \in V_2 \right\}, i = 1, 2, \ldots m, p = 1, 2, \ldots n$

(ii) $E = (v_i u_p, v_j u_q)$ if $i \neq j,\ p \neq q, (v_i, v_j) \in E_1$ and $(u_p, u_q) \in E_2$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of G, and is given by
$\langle \mu_r, \gamma_r \rangle = \big\langle \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \big\rangle$ for all $v_r \in V$, $r = 1, 2, 3, \ldots, mn$

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of G, and is given by

$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} \big\langle \min(\mu_{ij}, \mu_{pq}), \max(\gamma_{ij}, \gamma_{pq}) \big\rangle & \text{if } i \neq j,\ p \neq q, (v_i, v_j) \in E_1 \\ & \quad \text{and } (u_p, u_q) \in E_2 \\ \langle 0, 0 \rangle & \text{otherwise} \end{cases}$$

Definition 2.12. **[16]** The *strong product* of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \boxtimes G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle)$, $V_1 \cap V_2 = \phi$ where

(i) $V = \big\{ v_i u_p \mid v_i \in V_1, u_p \in V_2 \big\}$, $i = 1, 2, \ldots m$, $p = 1, 2, \ldots n$

(ii) $E = (v_i u_p, v_j u_q)$, such that either one of the following is true :

- $(u_p, u_q) \in E_2$, when $i = j,\ p \neq q$
- $(v_i, v_j) \in E_1$, when $p = q, i \neq j$
- $(v_i, v_j) \in E_1$ and $(u_p, u_q) \in E_2$, when $i \neq j, p \neq q$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of G, and is given by
$\langle \mu_r, \gamma_r \rangle = \big\langle \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \big\rangle$ for all $v_r \in V$, $r = 1, 2, 3, \ldots, mn$

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of G, and is given by

$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} \big\langle \min(\mu_i, \mu_{pq}), \max(\gamma_i, \gamma_{pq}) \big\rangle & \text{if } i = j, (u_p, u_q) \in E_2 \\ \big\langle \min(\mu_p, \mu_{ij}), \max(\gamma_p, \gamma_{ij}) \big\rangle & \text{if } p = q, (v_i, v_j) \in E_1 \\ \big\langle \min(\mu_{ij}, \mu_{pq}), \max(\gamma_{ij}, \gamma_{pq}) \big\rangle & \text{if } i \neq j,\ p \neq q, (v_i, v_j) \in E_1, \\ & \quad (u_p, u_q) \in E_2 \end{cases}$$

3. Equitable, k-domination and restrained domination in IFG

In this section, the concepts such as equitable dominating set, equitable independent set, k-dominating set restrained dominating set and global restrained dominating set of an IFGs are given. Also, we analyzed some properties of the domination numbers d_e, d_k, d_r and d_{gr} with known parameters of G.

Definition 3.1. Let $G = (V, E)$ be an IFG. The *degree of a vertex* v_i in G, is defined to be sum of the weights of the strong edges incident at v_i. The *μ-degree* of a vertex v_i in G, denoted by deg_{μ_i}, is defined as $deg_{\mu_i} = \sum_{i \neq j} \mu_{ij}$, $j = 1, 2 \cdots, n$. The *γ-degree* of a vertex v_i in G, denoted by deg_{γ_i}, is defined as $deg_{\gamma_i} = \sum_{i \neq j} \gamma_{ij}$, $j = 1, 2 \cdots, n$. The degree of a vertex v_i in G, denoted by $deg(v_i)$ takes the form $deg(v_i) = (deg_{\mu_i}, deg_{\gamma_i})$
The *minimum degree* of G is $\delta(G) = (\delta_{\mu_i}, \delta_{\gamma_i})$, where $\delta_{\mu_i} = \min(deg_{\mu_i} | v_i \in V)$ and $\delta_{\gamma_i} = \min(deg_{\gamma_i} | v_i \in V)$.
The *maximum degree* of G is $\Delta(G) = (\Delta_{\mu_i}, \Delta_{\gamma_i})$, where $\Delta_{\mu_i} = \max(deg_{\mu_i} | v_i \in V)$ and $\Delta_{\gamma_i} = \max(deg_{\gamma_i} | v_i \in V)$.

Note 3.2. In an IFG, it is not always possible for a vertex to have both minimum μ-degree and minimum γ-degree and also maximum μ-degree and minimum γ-degree.

Definition 3.3. Let $G = (V, E)$ be an IFG. A subset S of V is called an *equitable dominating set* in G if for every $v_i \in (V - S)$, there exists $v_j \in D$ such that $(v_i, v_j) \in E$, $\left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1$, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1$, $\gamma_{ij} \geq \gamma_{ij}^{\infty}$. $'-'$ refers to set difference.

Definition 3.4. The minimum cardinality of a equitable dominating set is called the *equitable domination number* of G, and is denoted by $d_e(G)$.

Definition 3.5. An IFG $G = (V, E)$ is said to be *degree equitable IFG*, for every $v_i \in V$ there exists $(v_i, v_j) \in$, such that $\left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1$, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1$, $\gamma_{ij} \geq \gamma_{ij}^{\infty}$.

Definition 3.6. An equitable dominating set S of an IFG is said to be a *minimal equitable dominating set* if no proper subset of S is a equitable dominating set.

Definition 3.7. A subset S of V is said to be *independent equitable set* of G, $\left|deg_{\mu_i} - deg_{\mu_j}\right| > 1$, $\mu_{ij} < \mu_{ij}^{\infty}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| > 1$, $\gamma_{ij} < \gamma_{ij}^{\infty}$ for all $v_i, v_j \in S$.

Definition 3.8. The minimum cardinality of a independent equitable dominating set is called the *independent equitable domination number* of G, and is denoted by $d_{ie}(G)$.

Definition 3.9. The *μ-equitable neighborhood* of a vertex v_i in V, denoted by EN_{μ}^{i}, is defined as

$EN^i_\mu = \left\{v_j \in V | v_j \in N(v_i), \left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1, \mu_{ij} \geq \mu^\infty_{ij}\right\}$.

The γ*-equitable neighborhood* of a vertex v_i in V, denoted by EN^i_γ is defined as $EN^i_\gamma = \left\{v_j \in V | v_j \in N(v_i), \left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1, \gamma_{ij} \geq \gamma^\infty_{ij}\right\}$. The *equitable neighborhood* of a vertex v_i in V, denoted by EN^i_v, is defined as $EN^i_v = (EN^i_\mu, EN^i_\gamma)$.

Definition 3.10. A vertex v_i of an equitable IFG is said to be an *equitable isolated vertex* if a vertex $v_j \in V$ be such that $\left|deg_{\mu_i} - deg_{\mu_j}\right| > 1, \mu_{ij} < \mu^\infty_{ij}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| > 1, \gamma_{ij} < \gamma^\infty_{ij}$ for all $v_j \in V - \{v_i\}$. i.e. $EN^i_v = \phi$.

Definition 3.11. The *equitable neighborhood degree* of a vertex is defined as $deg_{EN^i_v} = (deg_{EN^i_\mu}, deg_{EN^i_\gamma})$ where

$deg_{EN^i_\mu} = \sum_{v_j \in N(v_i)} \mu_{ij}$ and $deg_{EN^i_\gamma} = \sum_{v_j \in N(v_i)} \gamma_{ij}$

The minimum equitable neighborhood degree is defined as

$\delta_{EN} = (\delta_{EN^i_\mu}, \delta_{EN^i_\gamma})$ where

$\delta_{EN^i_\mu} = \min\left\{deg_{EN^i_\mu}, v_i \in V\right\}$ and $\delta_{EN^i_\gamma} = \min\left\{deg_{EN^i_\gamma}, v_i \in V\right\}$.

The maximum equitable neighborhood degree is defined as $\Delta_{EN} = (\Delta_{EN^i_\mu}, \Delta_{EN^i_\gamma})$ where $\Delta_{EN^i_\mu} = \max\left\{deg_{EN^i_\mu}, v_i \in V\right\}$ and $\Delta_{EN^i_\gamma} = \max\left\{deg_{EN^i_\gamma}, v_i \in V\right\}$.

Definition 3.12. Let $G = (V, E)$ be an IFG. Then $D \subseteq V$ is said to be a *strong (weak) equitable dominating set* of G if every vertex $v_j \in (V - D)$ is strongly (weakly) dominated by any vertex v_i in D.

Definition 3.13. The minimum cardinality of a strong(weak)equitable dominating set is called the *strong (weak)equitable domination number* of G, and is denoted by $d_{se}(G)$ and $d_{we}(G)$.

Definition 3.14. Let G be a connected IFG. A subset V' of V is called a *connected equitable dominating set* of G, if

(i) For every $v_j \in (V - V')$, there exists $v_i \in V'$ such that $(v_i, v_j) \in E$, $\left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1, \mu_{ij} \geq \mu^\infty_{ij}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1, \gamma_{ij} \geq \gamma^\infty_{ij}$.

(ii) The sub graph $H = (V', E')$ of $G = (V, E)$ induced by V' is connected.

The minimum cardinality of a connected equitable dominating set is called the *connected equitable domination number* of G, and is denoted by $d_{ce}(G)$.

Definition 3.15. Let $k \geq 1$ be an integer. A set $D \subseteq V$ of an IFG is a *k-dominating set* if for every $v_i \in (V - D)$ there exists a path (v_i, v_j), which contains at least k-strong edges for $v_j \in D$.

The minimum cardinality of a k-dominating set is called the *k-domination number* of G, and is denoted by $d_k(G)$.

Definition 3.16. Let $G = (V, E)$ be an IFG without isolated vertices. A subset D of V is a *total k-dominating set* if for every vertex $v_i \in V$, there exists atleast k-strong edges $(v_j, v_i) \in E(G)$ for $v_j \in D$, $v_i \neq v_j$. The minimum cardinality of a total k-dominating set is called the *total k-domination number* of G, and is denoted by $d_{tk}(G)$.

Definition 3.17. A set $D \subseteq V$ of an IFG is a *k-independent dominating set* if for every $v_i \in (V - D)$ there exists a path (v_i, v_j), which contains at least $k + 1$ strong edges for $v_j \in D$.

The minimum cardinality of a k-independent dominating set is called the *k-independent domination number* of G, and is denoted by $d_{ik}(G)$.

Definition 3.18. Let $G = (V, E)$ be an IFG. A subset $S \subseteq V$ is said to be a restrained dominating set of G, if every vertex in $V - S$ dominates a vertex in S and also a vertex in $V - S$. The minimum cardinality of a restrained dominating set is called the *restrained domination number* of G, and is denoted by $d_r(G)$.

Definition 3.19. Let $G = (V, E)$ be an IFG. A subset $S \subseteq V$ is said to be a restrained global dominating set of G, if it is a restrained dominating set of both G and $\overline{G}$. The minimum cardinality of a restrained global dominating set is called the *global restrained domination number* of G, and is denoted by $d_{gr}(G)$.

Definition 3.20. Let G be a connected IFG. A subset V' of V is called a *connected restrained dominating set* of G, if

(i) For every $v_j \in (V - V')$, there exists $v_i \in V'$ such that $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$. Also there exists $v_k \in (V - V')$ such that $\mu_{kj} \geq \mu_{kj}^{\infty}$ and $\gamma_{kj} \geq \gamma_{kj}^{\infty}$

(ii) The sub graph $H = (V', E')$ of $G = (V, E)$ induced by V' is connected.

Definition 3.21. The minimum cardinality of a restrained connected dominating set is called the *restrained connected domination number* of G, and is denoted by $d_{rc}(G)$.

Example 3.22. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3), (v_2, v_3), (v_1, v_4)\}$, as given in Figure 1.

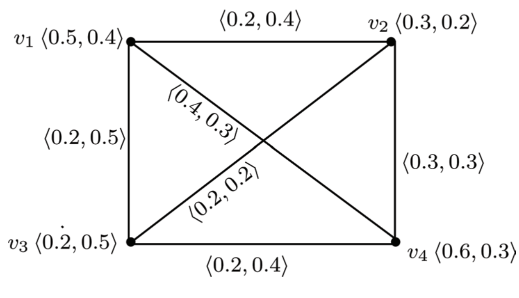

Figure 1: Equitable dominating set of G.

Here, $\{v_3\}$ is a minimal equitable dominating set of G. The equitable domination number is 0.35.

Example 3.23. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_5, v_4)\}$, as given in Figure 2.

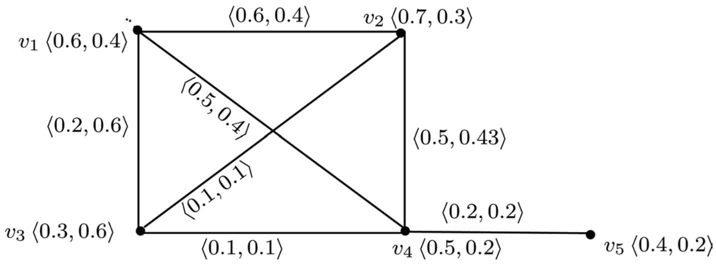

Figure 2: Independent equitable dominating set of G.

Here, $\{v_1, v_5\}$ is a minimal independent equitable dominating set of G. The independent equitable domination number is 1.25.

Example 3.24. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3)\}$, as given in Figure 3 is an degree equitable IFG.

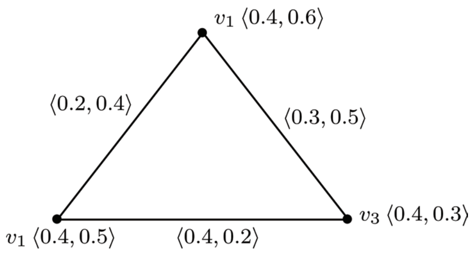

Figure 3: Degree Equitable IFG.

Example 3.25. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_1, v_3), (v_4, v_3), (v_1, v_4)\}$, as given in Figure 4.

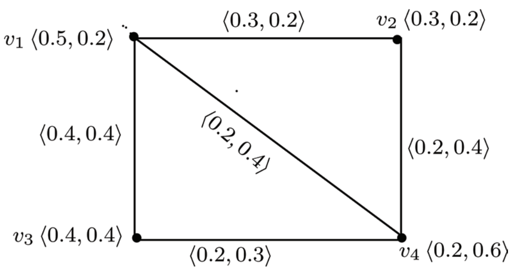

Figure 4: 3-dominating set.

Here, $\{v_2\}$ is a minimal 3-dominating set of G. The 3-domination number is 0.55.

Example 3.26. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_4, v_3), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8), (v_8, v_5)\}$, as given in Figure 5.

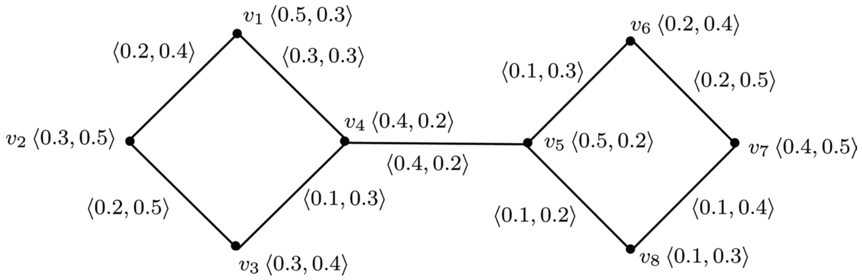

Figure 5: k-total dominating set.

Here, $\{v_3, v_4, v_7\}$ is an minimal 2-independent dominating set of G. The 2-independent domination number is d_{ik} is 1.5. $\{v_2, v_4, v_6\}$ is an minimal 2-total dominating set of G. The 2-total domination number is 1.35.

Example 3.27. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_2, v_4), (v_4, v_5), (v_5, v_3)\}$, as given in Figure 6.

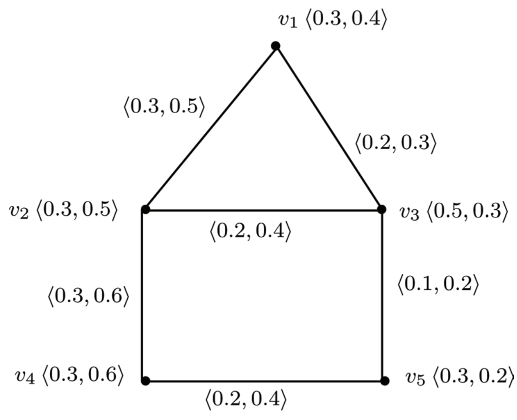

Figure 6: Restrained dominating set.

Here, $\{v_2, v_5\}$ is a minimal restrained dominating set of G. The restrained domination number is 0.95. $\{v_3, v_5\}$ is a minimal global restrained dominating set of G. The global restrained domination number is 1.15.

Theorem 3.28. Any equitable dominating set D is a minimal equitable dominating set of an IFG G if and only if for each $v_i \in D$ one of the following conditions is satisfied:

(i) v_i is an isolated vertex in D for some $v_j \in D$, $\mu_{ij} = 0$ and $\gamma_{ij} = 0$

(ii) $EN_v^i \cap (V - D) \neq \phi$

(iii) There exists a vertex $v_k \in (V - D)$ such that $EN(v_k) \cap D = \{v_i\}$.

Proof. Let D be a minimal equitable dominating set of an IFG G. Then for every vertex $v_i \in D$, $D - \{v_i\}$ is not an
equitable dominating set. This means that some vertex v_k in $(V - D) \cup \{v_i\}$ is not dominated by any vertex in $D - \{v_k\}$. Then there arises two cases:

Case (i) : If $v_k = v_i$, then v_i is not adjacent to any vertex v_j in D such that $\mu_{ij} = 0$ and $\gamma_{ij} = 0$. v_i is an isolated vertex in D. The neighborhood of each vertex in D is a strong neighbor in $(V - D)$. Then $EN_v^i \cap (V - D) \neq \phi$.

Case (ii)*:* If $v_k \in (V - D)$, then v_k is not dominated by any vertex in $D - \{v_i\}$, but it is dominated by some vertex v_i in D then v_j is adjacent to only vertex v_k in $V - D$. Thus $EN_v^k \cap D = \{v_i\}$.

Conversely, Let D be an equitable dominating set and for each vertex $v_i \in D$, one of the above conditions holds. Assume that D is not a minimal equitable dominating set. Then, there exists a vertex $v_i \in d$ such that $D - \{v_i\}$ is an equitable dominating set. Hence, v_i is adjacent to atleast one vertex in $D - \{v_i\}$, and therefore the condition (i) does not hold. Also if $D - \{v_i\}$ is a dominating set, then every vertex of $(V - D)$ is adjacent to at least one vertex in $D - \{v_i\}$, which implies that conditions (ii) and (iii) do not hold, which is a contradiction. ■

Theorem 3.29. Let $G = (V, E)$ be an IFG of order $o(G)$, then

(i) $d_e(G) \leq d_{se}(G) \leq o(G) - \Delta_{EN_\mu}(G)$

(ii) $d_e(G) \leq d_{we}(G) \leq o(G) - \delta_{EN_\mu}(G)$.

Proof. Every strong equitable dominating set is an equitable dominating set of G, $d_e(G) \leq d_{se}(G)$ and every weak equitable dominating set is a equitable dominating set of G, $d_e(G) \leq d_{we}(G)$. Let $v_i, v_j \in V$. If $deg_{EN_\mu^i} = \Delta_{EN_\mu}(G)$ and $deg_{EN_\mu^j} = \delta_{EN_\mu}(G)$. Clearly $V - EN_v^i$ is a strong equitable dominating set and $V - EN_v^j$ is a weak equitable dominating set. Therefore, $d_{se}(G) \leq \left|V - EN_\mu^i\right|$ and $d_{we}(G) \leq \left|V - EN_\mu^j\right|$. i.e $d_{se}(G) \leq o(G) - \Delta_{EN_\mu}(G)$ and $d_{we}(G) \leq o(G) - \delta_{EN_\mu}(G)$. ■

Theorem 3.30. Any connected equitable dominating set of an IFG G is an equitable dominating set.

Proof. Let S be a connected equitable dominating set of an IFG G. For every $v_i \in (V-S)$, there exists a vertex $v_j \in S$, such that $(v_i v_j) \in E$, $\left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1$, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1$, $\gamma_{ij} \geq \gamma_{ij}^{\infty}$ and the subgraph S is connected. Thus, S is an equitable dominating set of an IFG G. ■

Theorem 3.31. Let $G = (V, E)$ be an IFG without isolated vertices. Let D be a minimal equitable dominating set of G. Then $(V - D)$ is an equitable dominating set of G.

Proof. Let v_i be any vertex in D. For any two vertices $v_i, v_j \in V$, there exists an edge $(v_i, v_j) \in E$ such that $\mu_{ij} > 0$ and $\gamma_{ij} > 0$. There is an vertex $v_k \in EN_v^i$. By Theorem 3.1, $v_k \in (V - D)$. Thus every element of D is dominated by some element of $(V - D)$. Then $(V - D)$ is an equitable dominating set of G. ■

Theorem 3.32. Let $G = (V, E)$ be a connected IFG and H is a spanning subgraph of H. Then $d_e(G) \geq d_e(H)$.

Proof. Let $G = (V, E)$ be a connected IFG. and let $H = (V', E')$ is the spanning subgraph of H. S is the minimum equitable dominating set of H. Then S is also equitable dominating set in $V(H) - S$. That is, S is an equitable dominating set in G. Thus $d_e(G) \geq d_e(H)$.

Theorem 3.33. For any complete IFG, $d_r(G) = \min\left\{\sum_{v_i \in V}\left(\frac{1+\mu_i-\gamma_i}{2}\right)\right\}$ for all $v_i \in G$.

Proof. Let $G = (V, E)$ be a complete IFG. For all (v_i, v_j) in G, $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\gamma_{ij} = \max(\gamma_i, \gamma_j)$. Every edge in G is an strong edge. That is, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$. So every vertex dominates all other vertices. Let $D \subset V$ be a minimal restrained dominating set of G. Then every vertex in $(V - D)$ dominates a vertex in D and another vertex in $(V - D)$. Thus , $d_r(G) = \min\left\{\sum_{v_i \in V}\left(\frac{1+\mu_i-\gamma_i}{2}\right)\right\}$ for all $v_i \in G$. ■

Note 3.34. For a complete bipartite IFG K_{V_1,V_2} with $|V_1| \geq 2$ and $|V_2| \geq 2$,

$$d_r(G) = \min_{v_i \in V_1}\left(\frac{1+\mu_i-\gamma_i}{2}\right) + \min_{v_i \in V_2}\left(\frac{1+\mu_i-\gamma_i}{2}\right)$$

Theorem 3.35. For an IFG $G = (V, E)$, $|V| = n$ then, $d_{gr}(G) = o(G)$ if and only if $G = K_V$ or K_V^C, where $o(G)$ is the cardinality of all the vertices in G.

Proof. To prove $d_{gr}(G) = o(G)$. Assume that $G = K_V$ or K_V^C. To prove $d_{gr}(G) = o(G)$. For every edge (v_i, v_j) in K_V $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\gamma_{ij} = \max(\gamma_i, \gamma_j)$. Then K_V^C contains no edges. All the vertices in K_V^C is an isolated vertices. For all $v_i \in K_V^C$

$\mu_{ij} = 0, \gamma_{ij} = 0$. Each vertex dominates itself. Let $\{v_1, v_2, v_3, \dots v_i, v_n\}$ be the vertices of G. Suppose $\{v_i, v_{i+1}, \dots, v_n\}$ is the restrained dominating set of G, but it is not the dominating set of K_V^C. so the only dominating set of both K_V and K_V^C is the set containing all the vertices in G. The global restrained dominating set of K_V or K_V^C contains n vertices. Hence, $d_{gr}(G) = o(G)$.

Conversely, suppose $d_{gr}(G) = o(G)$. To prove $G = K_V$ or K_V^C, suppose $G \neq K_V$ or $G \neq K_V^C$, then G has atleast one vertex v_i which is not dominated by v_j. $V - \{v_i\}$ is a global restrained dominating set containing $n-1$ vertices. This implies that $d_{gr}(G) \neq o(G)$ which is a contradiction. Hence $d_{gr}(G) = o(G)$. ■

Note 3.36. For a complete bipartite IFG $G = K_{V_1,V_2}$,

(i) $d_{gr}(G) = o(G)$, if $|V_1| \leq 2$ and $|V_2| \leq 2$

(ii) $d_{gr}(G) = \min\limits_{v_i \in V_1}\left(\frac{1+\mu_i-\gamma_i}{2}\right) + \min\limits_{v_i \in V_2}\left(\frac{1+\mu_i-\gamma_i}{2}\right)$, if $|V_1| > 2$ and $|V_2| > 2$

(iii $d_{gr}(G) = \min\limits_{v \in V_1}\left(\frac{1+\mu_i-\gamma_i}{2}\right) + \min\limits_{v_i \in V_2}\left(\frac{1+\mu_i-\gamma_i}{2}\right)$, if $|V_1| = 2$ and $|V_2| > 2$.

Theorem 3.37. Let $G = (V, E)$ be an IFG, then the following hold:

(i) If G be an intuitionistic fuzzy cycle and $|V| \geq 6$, then $d_r(G) = d_{gr}(G)$.

(ii) If G be a path and $|V| \geq 9$, then $d_r(G) = d_{gr}(G)$.

Note 3.38. For $|V| < 6$ and $|V| < 8$, the result is verified with suitable examples.

Example 3.39. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $E = \{(v_1, v_2), (v_2, v_4), (v_4, v_5), (v_5, v_6), (v_3, v_1)\}$, as given in Figure 7.

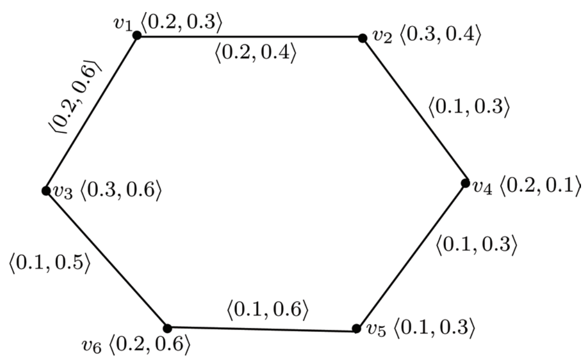

Figure 7: Intuitionistic fuzzy cycle G.

Here, $D = \{v_1, v_4, v_5\}$ is restrained dominating set of G it also dominating set of $\overline{G}$. Then, $d_r(G) = d_{gr}(G) = 1.36$.

Example 3.40. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7)(v_7, v_8)$ $(v_8, v_9), (v_9, v_{10})\}$, as given in Figure 8.

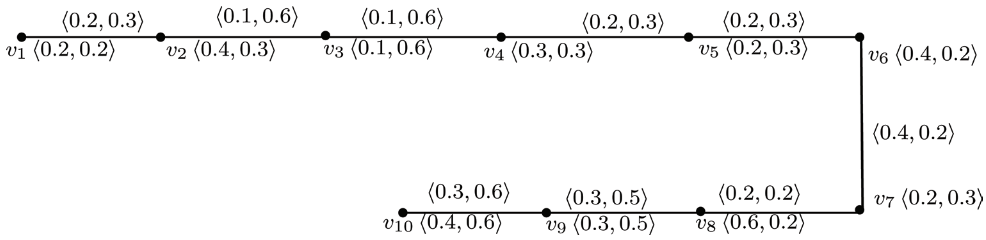

Figure 8: Intuitionistic fuzzy path G.

Here, $D = \{v_1, v_4, v_7, v_{10}\}$ is restrained dominating set of G it also dominating set of $\overline{G}$. Then $d_r(G) = d_{gr}(G) = 1.85$.

Theorem 3.41. Let $G = (V, E)$ be an IFG with $o(G)$. If every vertex $v_i \in V$ dominates at least two vertices in V and for vertex v_j of maximum μ-degree Δ_μ, $N(v_j) \neq \phi$,then $d_r(G) \leq o(G) - \Delta_\mu$.

Proof. Let v_j be a vertex of maximum μ-degree Δ_μ. If $\Delta_\mu = o(G) - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$ for a vertex $v_i \in V$. Then $\{v_i\}$ is a restrained dominating set of G and so, $d_r(G) \leq \left(\frac{1 + \mu_i - \gamma_i}{2}\right) = o(G) - \Delta_\mu$. Hence, $\Delta_\mu < o(G) - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$. Let $D = V - N(v_i)$. Let $|D| = o(G) - \Delta_\mu - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$. Since $N(v_i)$ is without isolated vertex, then $D \cup \{v_i\}$ is a restrained dominating set of G. ■

Therefore, $d_r(G) \leq o(G) - \Delta_\mu$.

Theorem 3.42. Let $G = (V, E)$ is an intuitionistic fuzzy cycle and $|V| \geq 6$, then $d_r(\overline{G}) \leq d_r(G)$.

Proof. Let S be a restrained dominating set of G. Then $(V - S)$ has at least two components. Let $\{v_i, v_{i+1}\}$ and $\left\{v_j, v_{j+1}\right\}$ be two arbitrary components of $(V - S)$. Then, v_i dominates $v_{i+2} \in S$, in $\overline{G}$. Therefore, $d_r(\overline{G}) \leq d_r(G)$. ■

4. Equitable domination and k-domination in product of intutionistic fuzzy graphs

In this section, equitable domination and k-domination are defined for cartesian, direct and strong products of two intuitionistic fuzzy graphs.

Theorem 4.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs with $V_1 \cap V_2 = \phi$. Then the strong product $G = G_1 \boxtimes G_2$ remains connected even after removal of all weak edges in it.

Proof. Let $G = G_1 \boxtimes G_2$ be the strong product of two IFGs G_1 and G_2. Let $e = (v_i u_p, v_j u_q)$ be a weak edge in G such that $\mu_{ip,jq} < \mu^{\infty}_{ip,jq}$ and $\gamma_{ip,jq} < \gamma^{\infty}_{ip,jq}$. To prove $G^{'} = G - e$ is connected. Assume that, $G^{'}$ is a disconnected IFG. The edge $e = (v_i u_p, v_j u_q)$ disconnect the graph into more than one components. This implies that, there is no path between $v_i u_p$ and $v_j u_q$ except the edge $e = (v_i u_p, v_j u_q)$ in $G^{'}$. Then $\mu_{ip,jq} \geq \mu^{\infty}_{ip,jq}$ and $\gamma_{ip,jq} \geq \gamma^{\infty}_{ip,jq}$, which is a contraction. Then $G^{'}$ is connected. ∎

Theorem 4.2. If a vertex v_i dominates v_j in G_1 and the vertex u_p dominates u_q in G_2, then the vertex $v_i u_p$ does not dominate the vertex $v_j u_q$ in $G_1 \Box G_2$.

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs. If v_i dominates v_j in G_1, there exists an strong edge (v_i, v_j) in G_1 such that $\mu_{ij} \geq \mu^{\infty}_{ij}$ and $\gamma_{ij} \geq \gamma^{\infty}_{ij}$. Similarly, if u_p dominates u_q in G_2, there exists an strong edge (up, u_q) in G_2 such that $\mu_{pq} \geq \mu^{\infty}_{pq}$ and $\gamma_{pq} \geq \gamma^{\infty}_{pq}$. By Definition 2.10, there is no edge between $v_i u_p$ and $v_j u_q$ in $G_1 \Box G_2$. That is, $\mu_{ip,jq} = 0$ and $\gamma_{ip,jq} = 0$. Then $v_i u_p$ does not dominate $v_j u_q$ in $G_1 \Box G_2$. ∎

Note 4.3. If v_i dominates v_j in G_1 and u_p dominates u_q in G_2, then the vertex $v_i u_p$ dominates the vertex $v_j u_q$ in $G_1 \otimes G_2$.

Theorem 4.4. For any two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $d_{te}(G_1 \otimes G_2) \leq \min(|D_1 \times V_2|, |D_2 \times V_1|)$.

Proof. Let $D_1 \subset V_1$ and $D_2 \subset V_2$ be the minimum equitable total dominating sets of G_1 and G_2 respectively. Suppose $D = D_1 \times D_2$ be the minimum equitable total dominating set of $G_1 \otimes G_2$. Let $v_i u_p$ be an arbitrary vertex of $G_1 \otimes G_2$. Then, there exists $v_j \in D_1$ such that $\left|deg_{\mu_i} - deg_{\mu_j}\right| \leq 1, \mu_{ij} \geq \mu^{\infty}_{ij}$ and $\left|deg_{\gamma_i} - deg_{\gamma_j}\right| \leq 1, \gamma_{ij} \geq \gamma^{\infty}_{ij}$. Also, $u_q \in D_2$, $\left|deg_{\mu_p} - deg_{\mu_q}\right| \leq 1, \mu_{pq} \geq \mu^{\infty}_{pq}$ and $\left|deg_{\gamma_p} - deg_{\gamma_q}\right| \leq 1, \gamma_{pq} \geq \gamma^{\infty}_{pq}$. That is, v_j dominates v_i and u_q dominates u_p. Thus, $\left|deg_{\mu_{ip}} - deg_{\mu_{jq}}\right| \leq 1, \mu_{ip,jq} \geq \mu^{\infty}_{ip,jq}$ and $\left|deg_{\gamma_{ip}} - deg_{\gamma_{jq}}\right| \leq 1, \gamma_{ip,jq} \geq \gamma^{\infty}_{ip,jq}$. $v_j u_q$ dominates $v_i u_p$ in $G_1 \otimes d_2$. Then, D is an equitable dominating set of $G_1 \otimes G_2$. Therefore $d_{te}(G_1 \otimes G_2) \leq min(|D_1 \times V_2|, |D_2 \times V_1|)$. ∎

Theorem 4.5. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two complete IFGs. Then $G = G_1 \otimes G_2$ is also a complete IFG.

Example 4.6. Consider the IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{v_1, v_2\}, E_1 = \{(v_1, v_2)\}$ and $V_2 = \{u_1, u_2\}, E_1 = \{(u_1, u_2)\}$. The graph of $G = G_1 \otimes G_2$ is displayed in Figure 9.

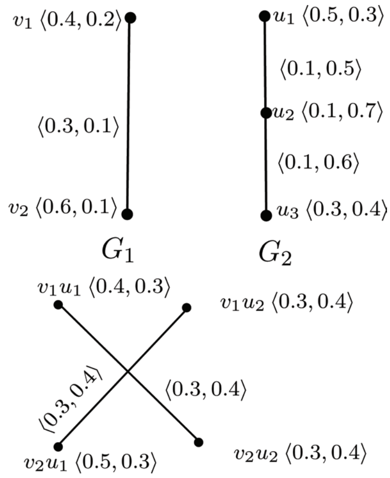

Figure 9: $G_1 \otimes G_2$.

The graph of $G_1 \otimes G_2$ is a complete IFG. The converse of the theorem is not true in general.

Theorem 4.7. Let D_1 and D_2 be k-dominating sets of connected IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Then

(i) $G_1 \Box G_2$ is connected

(ii) If D_1 is connected, then $D_1 \times V_2$ is a connected k-dominating set of $G_1 \Box G_2$.

(iii) If D_2 is connected, then $V_1 \times D_2$ is a connected k-dominating set of $G_1 \Box G_2$.

Proof. To prove $G_1 \Box G_2$ is connected, it is enough to prove that for any two arbitrary distinct vertices $v_i u_p, v_j u_q$ in $G_1 \Box G_2$ such that $\mu_{ip,jq} > 0$ and $\gamma_{ip,jq} > 0$.

Case (i). $v_i = v_j$. G_2 is a connected IFG. Then, there exists a path $p = u_1, u_2, \cdots, u_p$ such that $(\mu_{pq}, \gamma_{pq}) > 0$ for any two vertices u_p, u_q of path p. This implies that, $\mu_{ip,iq} = \mu_i \wedge \mu_{pq} > 0$ and $\gamma_{ip,iq} = \gamma_i \vee \gamma_{pq} > 0$ and hence $p' = v_i u_p, v_i u_1, v_i u_2 \cdots v_i u_q$ is the path between $v_i u_p$ and $v_i u_q$ in $G_1 \Box G_2$.

Case (ii). $u_p = u_q$. G_1 is a connected IFG. Then, there exists a path $q = v_1, v_2, \cdots, v_i$ such that $(\mu_{ij}, \gamma_{ij}) > 0$ for any two vertices v_i, v_j of path q. This implies that, $\mu_{ip,jq} = \mu_p \wedge \mu_{ij} > 0$ and $\gamma_{ip,jq} = \gamma_p \vee \gamma_{ij} > 0$ and hence $q' = v_1 u_p, v_2 u_p, v_3 u_p \cdots v_i u_p$ is the path between $v_i u_p$ and $v_j u_p$ in $G_1 \Box G_2$.

Case (iii).$v_i \neq v_j, u_p \neq u_q$. By case (i), there exists a path between $v_i u_p$ and $v_i u_q$in $G_1 \Box G_2$ and by case (ii), there exist a path between $v_i u_p$ and $v_j u_p$in $G_1 \Box G_2$. The union of these two disjoint paths is a path between $v_i u_q, v_j u_p$ in $G_1 \Box G_2$.

Therefore, $D_1 \times V_2$ and $V_1 \times D_2$ are dominating sets and the proof of connectivity of $D_1 \times V_2$ and $V_1 \times D_2$ is similar. ■

Theorem 4.8. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs without isolated vertices and let D_1 be a total k-dominating set of G_1. Then $G_1 \square G_2$ has no isolated vertices and $D_1 \times V_2$ is a total k-dominating set of $G_1 \square G_2$.

Proof. Let $D_1 \subseteq V_1$ be a total k-dominating set of G_1. To prove $D_1 \times V_2$ is a total k-dominating set of $G_1 \square G_2$, first prove that any vertex in $G_1 \square G_2$ is not an isolated vertex. Let $v_i u_p$ be an arbitrary vertex in $G_1 \square G_2$. Then, there exists a vertex v_j in D_1 such that $v_i \in N(v_j)$.

$$\begin{aligned} \mu_{ip,jp} &= \mu_{ij} \wedge \mu_p \\ &= \mu_i \wedge \mu_j \wedge \mu_p \\ &= \mu_{ip} \wedge \mu_{jp} \end{aligned}$$

$$\begin{aligned} \gamma_{ip,jp} &= \gamma_{ij} \vee \gamma_p \\ &= \gamma_i \vee \gamma_j \vee \gamma_p \\ &= \gamma_{ip} \vee \gamma_{jp} \end{aligned}$$

This implies that, $v_i u_p \in N(v_j u_p)$ Therefore, $v_i u_p$ is not an isolated vertex and $v_j u_p \in D_1 \times V_2$ is a total k-dominating set of $G_1 \square G_2$. ■

Theorem 4.9. Let D_1 and D_2 be k-dominating set of an IFGs G_1 and G_2 respectively.

(1) $D_1 \times V_2$ is an independent k-dominating set of $G_1 \square G_2$ if and only if D_1 is k-independent and

  (i) $\mu_{ij} < \mu_p$ and $\gamma_{ij} < \gamma_p$ for $v_i, v_j \in D_1, u_p \in V_2$

  (ii) $\mu_{pq} < \mu_i$ and $\gamma_{pq} < \gamma_i$ for $v_i \in D_1, u_p, u_q \in V_2$

  (iii) $\mu_{pq} < \mu_p \wedge \mu_q$ and $\gamma_{pq} < \gamma_p \wedge \gamma_q$ for $u_p, u_q \in V_2$

(2) $V_1 \times D_2$ is an independent k- dominating set of $G_1 \square G_2$ if and only if D_2 is k-independent and

  (i) $\mu_{ij} < \mu_p$ and $\gamma_{ij} < \gamma_p$ for $v_i, v_j \in V_1, u_p \in V_2$

  (ii) $\mu_{ij} < \mu_i \wedge \mu_j$ and $\gamma_{ij} < \gamma_i \wedge \gamma_j$ $v_i, v_j \in V_1$,

  (iii) $\mu_{pq} < \mu_i$ for $u_p, u_q \in V_2$.

Proof. To prove that every two distinct vertices v_iu_p,v_ju_q in $D_1 \times V_2$ are not adjacent. If $v_i = v_j$ then

$$\begin{aligned}\mu_{ip,jp} &= \mu_i \wedge \mu_{pq}\\ &< \mu_i \wedge \mu_p \wedge \mu_q\\ &< \mu_{ip} \wedge \mu_{iq}\end{aligned}$$

$$\begin{aligned}\gamma_{ip,jp} &= \gamma_i \vee \gamma_{pq}\\ &< \gamma_i \vee \gamma_p \vee \gamma_q\\ &< \gamma_{ip} \vee \gamma_{iq}\end{aligned}$$

If $u_p = u_q$, the result is obtained by independence of v_i, v_j of D_1. If $v_i \neq v_j$, $u_p \neq u_q$, by Definition 2.10, $\mu_{ip,jq} = 0$ and $\gamma_{ip,jq} = 0$. Hence, v_iu_p,v_ju_q are not adjacent in $G_1 \Box G_2$. Conversely, suppose $1(iii)$ is false. That is, $u_p, u_q \in V_2$ such that $\mu_{pq} < \mu_p \wedge \mu_q$ and $\gamma_{pq} < \gamma_p \wedge \gamma_q$. If v_i is any vertex in D_1, then

$$\begin{aligned}\mu_{ip,jp} &= \mu_i \wedge \mu_{pq}\\ &= \mu_i \wedge \mu_p \wedge \mu_q\\ &= \mu_{ip} \wedge \mu_{iq}\end{aligned}$$

$$\begin{aligned}\gamma_{ip,jp} &= \gamma_{ij} \vee \gamma_p\\ &= \gamma_i \vee \gamma_j \wedge \gamma_p\\ &= \gamma_{ip} \vee \gamma_{jp}\end{aligned}$$

This implies that, $D_1 \times V_2$ is not independent, which is a contradiction. Hence $1(iii)$ is true. That is, $\mu_{pq} < \mu_p \wedge \mu_q$ and $\gamma_{pq} < \gamma_p \wedge \gamma_q$. The proof of 2 is similar to the proof of 1. ■

5. Conclusion

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems. Intuitionistic fuzzy models also give more precision, flexibility, and compatibility to the system like fuzzy models. Domination in IFGs have many applications in the real world situations such as location problems, communication network, pattern clustering, routings. The concept equitable dominating set, independent equitable dominating set, k-dominating set and k-total dominating of an IFG are defined with suitable examples. Also, the lower and upper bound for the strong and weak equitable domination is established.

For an intuitionistic fuzzy cycle and path, the global restrained domination is obtained. Also, domination parameters like equitable domination and k- domination on direct, cartesian and strong products of two IFGs are analysed. Further, the authors proposed to apply these concepts in clustering techniques.

Acknowledgement

The author G. Thamizhendhi would like to thank the University Grants Commission, Hyderabad, India for its financial support to Minor Research Project No.F MRP-4956 (SERO/UGC) dated 17th March 2014.

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