Equitable, restrained and $k$-domination in intuitionistic fuzzy graphs

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Abstract

Graph theory is one of the most flourishing branches of modern mathematics and computer applications. The theory of domination has been the nucleus of research activity in graph theory in recent times. This is largely due to a variety of new parameters that can be developed from the basic definition of domination.

In this paper, the concept of equitable domination, $k$-domination, restrained domination, connected equitable domination, total $k$-domination and global restrained domination in intuitionistic fuzzy graphs (IFGs) have been introduced. The lower and upper bound for the equitable, restrained and $k$-domination numbers of an IFG are obtained. Further, some properties of the domination numbers $d_e, d_k, d_r$ and $d_{gr}$ with known parameters of $G$ are investigated. Also, domination parameters such as equitable domination, $k$-domination on direct, cartesian and strong products of two IFGs are analyzed.

AMS subject classification:

Keywords: Equitable domination, $k$-domination, restrained domination.

1. Introduction

Graph theory has numerous applications in modern science and technology. Domination in graph theory has many interesting applications in real world applications such as locating radar stations, nuclear power plants, communication networks and voting situations. Finding the minimal dominating set can be used to optimize time and distance while traveling, to optimize the performance of computer communication networks.

The study of dominating sets in graphs was started by Ore and Berge [4, 10] and the domination number was introduced by Cockayne and Hedetniemi [6]. Presently, science
and technology are featured with complex processes and phenomena for which complete and precise information is not always available. For such cases, mathematical models are developed to handle the types of systems containing elements of uncertainty.

The notion of fuzzy sets was introduced by Zadeh [18] as a method of representing uncertainty and vagueness. The first definition of fuzzy graphs was proposed by Kaufmann [8] from the fuzzy relations introduced by Zadeh. The concept of domination in fuzzy graphs introduced by A.Somasundaram and S.Somasundaram [15]. Intuitionistic fuzzy models give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. In 1984, Atanassov [1] introduced intuitionistic fuzzy sets as a generalization of fuzzy sets added a new component which determines the degree of non-membership. Intuitionistic fuzzy graph was introduced in [1]. In [11], the concept of domination, total domination, connected domination have been introduced. In this way, the authors got motivated to work further on the theory of domination and to introduce new concepts such as equitable dominating set, $k$-dominating set, restrained and global restrained dominating set which have since been introduced. This paper is organized as follows: section 2 contains basic notations and definitions required for this work. In section 3, the definition of equitable dominating set, $k$-dominating set, restrained and global restrained dominating set of an IFG is given. Also, analyzed some properties of the domination numbers $d_e, d_k, d_r$ and $d_{gr}$ with known parameters of $G$. In section 4, equitable domination, $k$-domination is studied on direct, strong and cartesian products of two IFGs. Section 5 concludes the paper. For other notations and terminologies not mentioned in the paper, the readers may read [7, 11, 16, 17].

2. Preliminaries

In this section, some basic definitions relating to IFGs are given. Throughout this paper, simple and undirected IFGs are taken into consideration.

**Definition 2.1.** [1] Let a set $E$ be fixed. An intuitionistic fuzzy set (IFS) $A$ in $E$ is an object of the form $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in E\}$, where the function $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ determine the degree of membership and the degree of non-membership of the element $x \in E$ respectively, and for every $x \in E$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

**Notations**

- $\mu_i, \gamma_i$ Degrees of membership and non-membership of the vertex $v_i$ of $G$
- $\mu_{ij}, \gamma_{ij}$ Degrees of membership and non-membership of the edge $e_{ij}$ of $G$
- $\mu_i^\infty, \gamma_i^\infty$ $\mu, \gamma$-strength of connectedness between the vertex $v_i$ and $v_j$ in $G$
- $\text{deg}_{\mu_i}, \text{deg}_{\gamma_i}$ $\mu, \gamma$-degree of a vertex $v_i$ in $G$
- $d_e(G)$ Equitable domination number of $G$
- $d_{ie}(G)$ Independent equitable domination number of $G$
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\( EN^i_v \)  
Equitable neighborhood of a vertex \( v_i \) in \( G \).

\( EN^i_\mu, EN^i_\gamma \)  
\( \mu, \gamma \)-equitable neighborhood of a vertex \( v_i \) in \( G \).

\( deg_{EN^i_\mu}, deg_{EN^i_\gamma} \)  
\( \mu, \gamma \)-equitable neighborhood degree of a vertex \( v_i \) in \( G \).

\( d_k(G) \)  
k-domination number of \( G \).

\( d_{ik}(G) \)  
k-independent domination number of \( G \).

\( d_r(G) \)  
Restrained domination number of \( G \).

\( d_{gr}(G) \)  
Global restrained domination number of \( G \).

\( d_{se}(G), d_{we}(G) \)  
Strong (weak) equitable domination number of \( G \).

\( d_{ce}(G) \)  
Connected equitable domination number of \( G \).

\( d_{tk}(G) \)  
Total k-domination number of \( G \).

\( d_{rc}(G) \)  
Restrained connected domination number of \( G \).

\textbf{Definition 2.2.} [12] Let \( X \) be a universal set and let \( V \) be an IFS over \( X \) in the form \( V = \{(v_i, \mu_i, \gamma_i) | v_i \in V \} \) such that \( 0 \leq \mu_i + \gamma_i \leq 1 \). \textit{Six types of cartesian products} of \( n \) elements of \( V \) over \( X \) are defined as

\[ v_1 \times v_2 \times v_3 \times \cdots \times v_n = \{(v_1, v_2, \ldots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \} \]

\[ v_1 \times_2 v_2 \times_2 v_3 \times_2 \cdots \times_2 v_n = \{(v_1, v_2, \ldots, v_n), \sum_{i=1}^{n} \mu_i - \sum_{i \neq j} \mu_i \mu_j \]

\[ + \sum_{i \neq j \neq k} \mu_i \mu_j \mu_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \mu_i \mu_j \mu_k \cdots \mu_n \]

\[ +(-1)^{n-1} \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i | v_1, v_2, \cdots, v_n \in V \}

\[ v_1 \times_3 v_2 \times_3 v_3 \times_3 \cdots \times_3 v_n = \{(v_1, v_2, \ldots, v_n), \prod_{i=1}^{n} \mu_i, \prod_{i=1}^{n} \gamma_i \}
\]

\[ + \sum_{i \neq j} \gamma_i \gamma_j \gamma_k - \cdots + (-1)^{n-2} \sum_{i \neq j \neq k \cdots \neq n} \gamma_i \gamma_j \gamma_k \cdots \gamma_n \]

\[ +(-1)^{n-1} \prod_{i=1}^{n} \gamma_i | v_1, v_2, \cdots, v_n \in V \}

\[ v_1 \times_4 v_2 \times_4 v_3 \times_4 \cdots \times_4 v_n = \{(v_1, v_2, \ldots, v_n), \min(\mu_1, \mu_2, \ldots, \mu_n), \]

\[ \max(\gamma_1, \gamma_2, \ldots, \gamma_n) | v_1, v_2, \cdots, v_n \in V \}

\[ v_1 \times_5 v_2 \times_5 v_3 \times_5 \cdots \times_5 v_n = \{(v_1, v_2, \ldots, v_n), \max(\mu_1, \mu_2, \ldots, \mu_n), \]

\[ \min(\gamma_1, \gamma_2, \cdots, \gamma_n) | v_1, v_2, \cdots, v_n \in V \}.

\[ v_1 \times_6 v_2 \times_6 v_3 \times_6 \cdots \times_6 v_n = \{(v_1, v_2, \cdots, v_n), \frac{\sum_{i=1}^{n} \mu_i \sum_{i=1}^{n} \gamma_i}{n} \}
\]

\[ | v_1, v_2, \cdots, v_n \in V |. \]

It must be noted that \( v_i \times_t v_j \) is an IFS, where \( t = 1, 2, 3, 4, 5, 6 \) such that the sum of
their degrees of membership and non-membership lies in $[0, 1]$.

**Definition 2.3.** [7] An intuitionistic fuzzy graph (IFG) is of the form $G = (V, E)$ where (i) $V = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_i : V \rightarrow [0, 1]$ and $\gamma_i : V \rightarrow [0, 1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and 

$$0 \leq \mu_i(v_i) + \gamma_i(v_i) \leq 1$$

for every $v_i \in V, i = 1, 2, \ldots, n$

(ii) $E \subseteq V \times V$ where $\mu_{ij} : V \times V \rightarrow [0, 1]$ and $\gamma_{ij} : V \times V \rightarrow [0, 1]$ are such that

$$\mu_{ij} \leq \mu_i \otimes \mu_j$$

$$\gamma_{ij} \leq \gamma_i \otimes \gamma_j$$

and

$$0 \leq \mu_{ij} + \gamma_{ij} \leq 1$$

where $\mu_{ij}$ and $\gamma_{ij}$ are the degrees of membership and non-membership of the edge $(v_i, v_j)$; the values $\mu_i \otimes \mu_j$ and $\gamma_i \otimes \gamma_j$ can be determined by one of the six cartesian products $\times_t, t = 1, 2, 3, 4, 5, 6$ for all $i$ and $j$ given in Definition 2.2.

**Definition 2.4.** [11] Let $G = (V, E)$ be an IFG, then the cardinality of a subset $S$ of $V$ is defined as 

$$|S| = \sum_{v_i \in S} \left( \frac{1}{2} + \mu_i - \gamma_i \right)$$

for all $v_i \in S$.

**Definition 2.5.** [11] The number of vertices in $G$ is called as order of an IFG, $G = (V, E)$, denoted by $o(G)$, and is defined as 

$$o(G) = \sum_{v_i \in V} \left( \frac{1}{2} + \mu_i - \gamma_i \right)$$

for all $v_i \in V$.

**Definition 2.6.** [11] If $v_i, v_j \in V \subseteq G$, the $\mu$-strength of connectedness between $v_i$ and $v_j$ is $\mu_{ij}^{\infty} = \sup\{\mu_{ij}^k \mid k = 1, 2, \ldots, n\}$ and $\gamma$-strength of connectedness between $v_i$ and $v_j$ is $\gamma_{ij}^{\infty} = \inf\{\gamma_{ij}^k \mid k = 1, 2, \ldots, n\}$.

If $v_i, v_j$ are connected by means of paths of length $k$ then $\mu_{ij}^k$ is defined as $\sup\{\mu_{i1} \land \mu_{12} \land \mu_{23} \ldots \land \mu_{k-1} \land v_j \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$ and $\gamma_{ij}^k$ is defined as $\inf\{\gamma_{i1} \lor \gamma_{12} \lor \gamma_{23} \ldots \lor \gamma_{k-1} \lor v_j \mid v_i, v_1, v_2 \ldots v_{k-1}, v_j \in V\}$.

**Definition 2.7.** [11] An edge $(v_i, v_j)$ is said to be a strong edge of an IFG $G = (V, E)$, if $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$.

**Definition 2.8.** [11] An IFG, $G = (V, E)$ is said to be connected IFG if there exists a path between every pair of vertices $v_i, v_j$ in $V$. Connected IFG is also defined using strength of connectedness as follows:

(i) $\mu_{ij}^{\infty} > 0$, and $\gamma_{ij}^{\infty} > 0$
(ii) $\mu^\infty_{ij} = 0$, and $\gamma^\infty_{ij} > 0$

(iii) $\mu^\infty_{ij} > 0$, and $\gamma^\infty_{ij} = 0$ for all $v_i, v_j \in V$.

**Definition 2.9.** [7] The complement of an IFG, $G = (V, E)$ is an IFG, $\overline{G} = (V, \overline{E})$, where

(i) $\overline{V} = V$,

(ii) $\overline{\mu_i} = \mu_i$ and $\overline{\gamma_i} = \gamma_i$, for all $i = 1, 2, \ldots, n$,

(iii) $\overline{\mu_{ij}} = \min(\mu_i, \mu_j) - \mu_{ij}$ and $\overline{\gamma_{ij}} = \max(\gamma_i, \gamma_j) - \gamma_{ij}$ for all $i, j = 1, 2, \ldots, n$.

**Definition 2.10.** [16] The cartesian product of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \Box G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle)$, $V_1 \cap V_2 = \phi$ where

(i) $V = \{v_iu_p \mid v_i \in V_1, u_p \in V_2\}$, $i = 1, 2, \ldots, m$, $p = 1, 2, \ldots, n$.

(ii) $E = \{(v_iu_p, v_ju_q) \mid \text{such that either one of the following is true:} \}$

- $(u_p, u_q) \in E_2$, when $i = j$
- $(v_i, v_j) \in E_1$, when $p = q$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of $G$, and is given by $\langle \mu_r, \gamma_r \rangle = \{\min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p)\}$ for all $v_r \in V$, $r = 1, 2, 3, \ldots, mn$.

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of $G$, and is given by

$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} 
\{\min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p)\} & \text{if } i = j, (u_p, u_q) \in E_2 \\
\{\min(\mu_p, \mu_i), \max(\gamma_p, \gamma_i)\} & \text{if } p = q, (v_i, v_j) \in E_1 \\
\{0, 0\} & \text{otherwise}
\end{cases}$$

**Definition 2.11.** [16] The direct product of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \otimes G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle)$, $V_1 \cap V_2 = \phi$ where

(i) $V = \{v_iu_p \mid v_i \in V_1, u_p \in V_2\}$, $i = 1, 2, \ldots, m$, $p = 1, 2, \ldots, n$.
(ii) $E = (v_i u_p, v_j u_q)$ if $i \neq j$, $p \neq q$, $(v_i, v_j) \in E_1$ and $(u_p, u_q) \in E_2$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of $G$, and is given by
$$\langle \mu_r, \gamma_r \rangle = \{ \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \}$$
for all $v_r \in V, r = 1, 2, 3, \ldots, mn$

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of $G$, and is given by
$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} 
\{ \min(\mu_{ij}, \mu_{pq}), \max(\gamma_{ij}, \gamma_{pq}) \} & \text{if } i \neq j, p \neq q, (v_i, v_j) \in E_1 \\
\{ \min(\mu_p, \mu_{ij}), \max(\gamma_p, \gamma_{ij}) \} & \text{if } p = q, (v_i, v_j) \in E_1 \\
\{ \min(\mu_{ij}, \mu_{pq}), \max(\gamma_{ij}, \gamma_{pq}) \} & \text{if } i \neq j, p \neq q, (v_i, v_j) \in E_1, (u_p, u_q) \in E_2 \\
\{0, 0\} & \text{otherwise}
\end{cases}$$

**Definition 2.12.** [16] The *strong product* of two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \boxtimes G_2$, is an IFG $G = (V, E, \langle \mu_r, \gamma_r \rangle, \langle \mu_{rs}, \gamma_{rs} \rangle)$, $V_1 \cap V_2 = \emptyset$ where

(i) $V = \{ v_i u_p \mid v_i \in V_1, u_p \in V_2 \}$, $i = 1, 2, \ldots, m$, $p = 1, 2, \ldots, n$

(ii) $E = (v_i u_p, v_j u_q)$, such that either one of the following is true:

- $(u_p, u_q) \in E_2$, when $i = j$, $p \neq q$
- $(v_i, v_j) \in E_1$, when $p = q$, $i \neq j$
- $(v_i, v_j) \in E_1$ and $(u_p, u_q) \in E_2$, when $i \neq j, p \neq q$

(iii) $\langle \mu_r, \gamma_r \rangle$ denote the degrees of membership and non-membership of vertices of $G$, and is given by
$$\langle \mu_r, \gamma_r \rangle = \{ \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \}$$
for all $v_r \in V, r = 1, 2, 3, \ldots, mn$

(iv) $\langle \mu_{rs}, \gamma_{rs} \rangle$ denote the degrees of membership and non-membership of edges of $G$, and is given by
$$\langle \mu_{rs}, \gamma_{rs} \rangle = \begin{cases} 
\{ \min(\mu_i, \mu_p), \max(\gamma_i, \gamma_p) \} & \text{if } i = j, (u_p, u_q) \in E_2 \\
\{ \min(\mu_p, \mu_{ij}), \max(\gamma_p, \gamma_{ij}) \} & \text{if } p = q, (v_i, v_j) \in E_1 \\
\{ \min(\mu_{ij}, \mu_{pq}), \max(\gamma_{ij}, \gamma_{pq}) \} & \text{if } i \neq j, p \neq q, (v_i, v_j) \in E_1, (u_p, u_q) \in E_2
\end{cases}$$
3. **Equitable, k-domination and restrained domination in IFG**

In this section, the concepts such as equitable dominating set, equitable independent set, k-dominating set restrained dominating set and global restrained dominating set of an IFGs are given. Also, we analyzed some properties of the domination numbers $d_e, d_k, d_r$ and $d_{gr}$ with known parameters of $G$.

**Definition 3.1.** Let $G = (V, E)$ be an IFG. The degree of a vertex $v_i$ in $G$, is defined to be sum of the weights of the strong edges incident at $v_i$. The $\mu$-degree of a vertex $v_i$ in $G$, denoted by $\text{deg}_\mu(v_i)$, is defined as $\text{deg}_\mu(v_i) = \sum_{i \neq j} \mu_{ij}, j = 1, 2, \ldots, n$. The $\gamma$-degree of a vertex $v_i$ in $G$, denoted by $\text{deg}_\gamma(v_i)$, is defined as $\text{deg}_\gamma(v_i) = \sum_{i \neq j} \gamma_{ij}, j = 1, 2, \ldots, n$. The degree of a vertex $v_i$ in $G$, denoted by $\text{deg}(v_i)$ takes the form $\text{deg}(v_i) = (\text{deg}_\mu(v_i), \text{deg}_\gamma(v_i))$. The minimum degree of $G$ is $\alpha(G) = (\alpha_\mu(v_i), \alpha_\gamma(v_i))$, where $\alpha_\mu(v_i) = \min(\text{deg}_\mu(v_i) | v_i \in V)$ and $\alpha_\gamma(v_i) = \min(\text{deg}_\gamma(v_i) | v_i \in V)$. The maximum degree of $G$ is $\Delta_1(G) = (\Delta_1\mu(v_i), \Delta_1\gamma(v_i))$, where $\Delta_1\mu(v_i) = \max(\text{deg}_\mu(v_i) | v_i \in V)$ and $\Delta_1\gamma(v_i) = \max(\text{deg}_\gamma(v_i) | v_i \in V)$.

**Note 3.2.** In an IFG, it is not always possible for a vertex to have both minimum $\mu$-degree and minimum $\gamma$-degree and also maximum $\mu$-degree and minimum $\gamma$-degree.

**Definition 3.3.** Let $G = (V, E)$ be an IFG. A subset $S$ of $V$ is called an **equitable dominating set** in $G$ if for every $v_i \in (V - S)$, there exists $v_j \in D$ such that $|\text{deg}_\mu(v_i) - \text{deg}_\mu(v_j)| \leq 1, \mu_{ij} \geq \mu_{\infty}$ and $|\text{deg}_\gamma(v_i) - \text{deg}_\gamma(v_j)| \leq 1, \gamma_{ij} \geq \gamma_{\infty}$. '-' refers to set difference.

**Definition 3.4.** The minimum cardinality of an equitable dominating set is called the **equitable domination number** of $G$, and is denoted by $d_e(G)$.

**Definition 3.5.** An IFG $G = (V, E)$ is said to be **degree equitable IFG**, for every $v_i \in V$ there exists $(v_i, v_j) \in E$, $|\text{deg}_\mu(v_i) - \text{deg}_\mu(v_j)| \leq 1, \mu_{ij} \geq \mu_{\infty}$ and $|\text{deg}_\gamma(v_i) - \text{deg}_\gamma(v_j)| \leq 1, \gamma_{ij} \geq \gamma_{\infty}$. $\mu_{\infty}$ and $\gamma_{\infty}$ are constants.

**Definition 3.6.** An equitable dominating set $S$ of an IFG is said to be a **minimal equitable dominating set** if no proper subset of $S$ is a equitable dominating set.

**Definition 3.7.** A subset $S$ of $V$ is said to be **independent equitable set** of $G$, $|\text{deg}_\mu(v_i) - \text{deg}_\mu(v_j)| > 1, \mu_{ij} < \mu_{\infty}$ and $|\text{deg}_\gamma(v_i) - \text{deg}_\gamma(v_j)| > 1, \gamma_{ij} < \gamma_{\infty}$ for all $v_i, v_j \in S$.

**Definition 3.8.** The minimum cardinality of a independent equitable dominating set is called the **independent equitable domination number** of $G$, and is denoted by $d_{ie}(G)$.

**Definition 3.9.** The **$\mu$-equitable neighborhood** of a vertex $v_i$ in $V$, denoted by $EN_i^\mu$, is defined as
EN\textsubscript{i,µ} = \{v_j \in V | v_j \in N(v_i), |deg_{µ_i} - deg_{µ_j}| \leq 1, µ_{ij} \geq µ_{ij}^\infty \}.

The γ-equitable neighborhood of a vertex \( v_i \) in \( V \), denoted by \( EN_i^γ \) is defined as
\[ EN_i^γ = \{v_j \in V | v_j \in N(v_i), |deg_{γ_i} - deg_{γ_j}| \leq 1, γ_{ij} \geq γ_{ij}^\infty \} \].

The equitable neighborhood of a vertex \( v_i \) in \( V \), denoted by \( EN_i \), is defined as
\[ EN_i = (EN_i^µ, EN_i^γ) \].

**Definition 3.10.** A vertex \( vi \) of an equitable IFG is said to be an equitable isolated vertex if a vertex \( v_j \in V \) be such that \( |deg_{µ_i} - deg_{µ_j}| > 1, µ_{ij} < µ_{ij}^\infty \) and \( |deg_{γ_i} - deg_{γ_j}| > 1, γ_{ij} < γ_{ij}^\infty \) for all \( v_j \in V - \{v_i\} \). i.e. \( EN_i = \phi \).

**Definition 3.11.** The equitable neighborhood degree of a vertex is defined as
\[ deg_{EN_i} = (deg_{EN_i^µ}, deg_{EN_i^γ}) \] where
\[ deg_{EN_i^µ} = \sum_{v_j \in N(v_i)} µ_{ij} \text{ and } deg_{EN_i^γ} = \sum_{v_j \in N(v_i)} γ_{ij} \]
The minimum equitable neighborhood degree is defined as
\[ δ_{EN_i} = (δ_{EN_i^µ}, δ_{EN_i^γ}) \] where
\[ δ_{EN_i^µ} = \min \{deg_{EN_i^µ}, v_i \in V\} \text{ and } δ_{EN_i^γ} = \min \{deg_{EN_i^γ}, v_i \in V\} \].
The maximum equitable neighborhood degree is defined as
\[ Δ_{EN_i} = (Δ_{EN_i^µ}, Δ_{EN_i^γ}) \] where
\[ Δ_{EN_i^µ} = \max \{deg_{EN_i^µ}, v_i \in V\} \text{ and } Δ_{EN_i^γ} = \max \{deg_{EN_i^γ}, v_i \in V\} \].

**Definition 3.12.** Let \( G = (V, E) \) be an IFG. Then \( D \subseteq V \) is said to be a strong (weak) equitable dominating set of \( G \) if every vertex \( v_j \in (V - D) \) is strongly (weakly) dominated by any vertex \( v_i \) in \( D \).

**Definition 3.13.** The minimum cardinality of a strong(weak)equitable dominating set is called the strong (weak)equitable domination number of \( G \), and is denoted by \( d_{se}(G) \) and \( d_{we}(G) \).

**Definition 3.14.** Let \( G \) be a connected IFG. A subset \( V' \) of \( V \) is called a connected equitable dominating set of \( G \), if

(i) For every \( v_j \in (V - V') \), there exists \( v_i \in V' \) such that \( (v_i, v_j) \in E \), \( |deg_{µ_i} - deg_{µ_j}| \leq 1, µ_{ij} \geq µ_{ij}^\infty \) and \( |deg_{γ_i} - deg_{γ_j}| \leq 1, γ_{ij} \geq γ_{ij}^\infty \).

(ii) The sub graph \( H = (V', E') \) of \( G = (V, E) \) induced by \( V' \) is connected.

The minimum cardinality of a connected equitable dominating set is called the connected equitable domination number of \( G \), and is denoted by \( d_{ce}(G) \).

**Definition 3.15.** Let \( k \geq 1 \) be an integer. A set \( D \subseteq V \) of an IFG is a \( k \)-dominating set if for every \( v_i \in (V - D) \) there exists a path \( (v_i, v_j) \), which contains at least \( k \)-strong edges for \( v_j \in D \).
The minimum cardinality of a \( k \)-dominating set is called the \( k \)-domination number of \( G \), and is denoted by \( d_k(G) \).

**Definition 3.16.** Let \( G = (V, E) \) be an IFG without isolated vertices. A subset \( D \) of \( V \) is a total \( k \)-dominating set if for every vertex \( v_i \in V \), there exists at least \( k \)-strong edges \((v_j, v_i) \in E(G)\) for \( v_j \in D, v_i \neq v_j \). The minimum cardinality of a total \( k \)-dominating set is called the total \( k \)-domination number of \( G \), and is denoted by \( d_{tk}(G) \).

**Definition 3.17.** A set \( D \subseteq V \) of an IFG is a \( k \)-independent dominating set if for every \( v_i \in (V - D) \) there exists a path \((v_i, v_j)\), which contains at least \( k + 1 \) strong edges for \( v_j \in D \).

The minimum cardinality of a \( k \)-independent dominating set is called the \( k \)-independent domination number of \( G \), and is denoted by \( d_{ik}(G) \).

**Definition 3.18.** Let \( G = (V, E) \) be an IFG. A subset \( S \subseteq V \) of an IFG is said to be a restrained dominating set of \( G \), if every vertex in \( V - S \) dominates a vertex in \( S \) and also a vertex in \( V - S \). The minimum cardinality of a restrained dominating set is called the restrained domination number of \( G \), and is denoted by \( d_r(G) \).

**Definition 3.19.** Let \( G = (V, E) \) be an IFG. A subset \( S \subseteq V \) is said to be a restrained global dominating set of \( G \), if it is a restrained dominating set of both \( G \) and \( \overline{G} \). The minimum cardinality of a restrained global dominating set is called the global restrained domination number of \( G \), and is denoted by \( d_{gr}(G) \).

**Definition 3.20.** Let \( G \) be a connected IFG. A subset \( V' \) of \( V \) is called a connected restrained dominating set of \( G \), if

(i) For every \( v_j \in (V - V') \), there exists \( v_i \in V' \) such that \( \mu_{ij} \geq \mu_{ij}^\infty \) and \( \gamma_{ij} \geq \gamma_{ij}^\infty \).

Also there exists \( v_k \in (V - V') \) such that \( \mu_{kj} \geq \mu_{kj}^\infty \) and \( \gamma_{kj} \geq \gamma_{kj}^\infty \).

(ii) The sub graph \( H = (V', E') \) of \( G = (V, E) \) induced by \( V' \) is connected.

**Definition 3.21.** The minimum cardinality of a restrained connected dominating set is called the restrained connected domination number of \( G \), and is denoted by \( d_{rc}(G) \).

**Example 3.22.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3, v_4\} \),
\( E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3), (v_2, v_3), (v_1, v_4)\} \), as given in Figure 1.

![Figure 1: Equitable dominating set of G.](image-url)
Here, \{v_3\} is a minimal equitable dominating set of \( G \). The equitable domination number is 0.35.

**Example 3.23.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3, v_4, v_5\} \), \( E = \{(v_1, v_2), (v_2, v_4), (v_3, v_4), (v_1, v_3), (v_2, v_3), (v_1, v_4), (v_5, v_4)\} \), as given in Figure 2.

![Figure 2: Independent equitable dominating set of \( G \).](image)

Here, \( \{v_1, v_5\} \) is a minimal independent equitable dominating set of \( G \). The independent equitable domination number is 1.25.

**Example 3.24.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3\} \), \( E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3)\} \), as given in Figure 3 is an degree equitable IFG.

![Figure 3: Degree Equitable IFG.](image)

**Example 3.25.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3, v_4\} \), \( E = \{(v_1, v_2), (v_2, v_4), (v_1, v_3), (v_4, v_3), (v_1, v_4)\} \), as given in Figure 4.

![Figure 4: 3-dominating set.](image)
Here, \( \{v_2\} \) is a minimal 3-dominating set of \( G \). The 3-domination number is 0.55.

**Example 3.26.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \), \( E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_4, v_3), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8), (v_8, v_5)\} \), as given in Figure 5.

![Figure 5: k-total dominating set.](image)

Here, \( \{v_3, v_4, v_7\} \) is an minimal 2-independent dominating set of \( G \). The 2-independent domination number is \( d_{ik} \) is 1.5. \( \{v_2, v_4, v_6\} \) is an minimal 2-total dominating set of \( G \). The 2-total domination number is 1.35.

**Example 3.27.** Consider an IFG, \( G = (V, E) \), with \( V = \{v_1, v_2, v_3, v_4, v_5\} \), \( E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3), (v_2, v_4), (v_4, v_5), (v_5, v_3)\} \), as given in Figure 6.

![Figure 6: Restrained dominating set.](image)

Here, \( \{v_2, v_5\} \) is a minimal restrained dominating set of \( G \). The restrained domination number is 0.95. \( \{v_3, v_5\} \) is a minimal global restrained dominating set of \( G \). The global restrained domination number is 1.15.

**Theorem 3.28.** Any equitable dominating set \( D \) is a minimal equitable dominating set of an IFG \( G \) if and only if for each \( v_i \in D \) one of the following conditions is satisfied:
(i) $v_i$ is an isolated vertex in $D$ for some $v_j \in D$, $\mu_{ij} = 0$ and $\gamma_{ij} = 0$

(ii) $EN_i^v \cap (V - D) \neq \emptyset$

(iii) There exists a vertex $v_k \in (V - D)$ such that $EN(v_k) \cap D = \{v_i\}$.

Proof. Let $D$ be a minimal equitable dominating set of an IFG $G$. Then for every vertex $v_i \in D$, $D - \{v_i\}$ is not an equitable dominating set. This means that some vertex $v_k$ in $(V - D) \cup \{v_i\}$ is not dominated by any vertex in $D - \{v_k\}$. Then there arises two cases:

Case (i) : If $v_k = v_i$, then $v_i$ is not adjacent to any vertex $v_j$ in $D$ such that $\mu_{ij} = 0$ and $\gamma_{ij} = 0$. $v_i$ is an isolated vertex in $D$. The neighborhood of each vertex in $D$ is a strong neighbor in $(V - D)$. Then $EN_i^v \cap (V - D) \neq \emptyset$.

Case (ii): If $v_k \in (V - D)$, then $v_k$ is not dominated by any vertex in $D - \{v_i\}$, but it is dominated by some vertex $v_i$ in $D$ then $v_j$ is adjacent to only vertex $v_k$ in $V - D$. Thus $EN_k^v \cap D = \{v_i\}$.

Conversely, Let $D$ be an equitable dominating set and for each vertex $v_i \in D$, one of the above conditions holds. Assume that $D$ is not a minimal equitable dominating set. Then, there exists a vertex $v_i \in D$ such that $D - \{v_i\}$ is an equitable dominating set. Hence, $v_i$ is adjacent to atleast one vertex in $D - \{v_i\}$, and therefore the condition (i) does not hold. Also if $D - \{v_i\}$ is a dominating set, then every vertex of $(V - D)$ is adjacent to at least one vertex in $D - \{v_i\}$, which implies that conditions (ii) and (iii) do not hold, which is a contradiction. □

Theorem 3.29. Let $G = (V, E)$ be an IFG of order $o(G)$, then

(i) $d_e(G) \leq d_{se}(G) \leq o(G) - \Delta_{EN^\mu}(G)$

(ii) $d_e(G) \leq d_{we}(G) \leq o(G) - \delta_{EN^\mu}(G)$.

Proof. Every strong equitable dominating set is an equitable dominating set of $G$, $d_e(G) \leq d_{se}(G)$ and every weak equitable dominating set is a equitable dominating set of $G$, $d_e(G) \leq d_{we}(G)$. Let $v_i, v_j \in V$. If $deg_{EN^\mu} = \Delta_{EN^\mu}(G)$ and $deg_{EN^\mu} = \delta_{EN^\mu}(G)$.

Clearly $V - EN_i^v$ is a strong equitable dominating set and $V - EN_i^v$ is a weak equitable dominating set. Therefore, $d_{se}(G) \leq |V - EN_i^v|$ and $d_{we}(G) \leq |V - EN_i^v|$, i.e

$d_{se}(G) \leq o(G) - \Delta_{EN^\mu}(G)$ and $d_{we}(G) \leq o(G) - \delta_{EN^\mu}(G)$. □

Theorem 3.30. Any connected equitable dominating set of an IFG $G$ is an equitable dominating set.
Proof. Let $S$ be a connected equitable dominating set of an IFG $G$. For every $v_i \in (V - S)$, there exists a vertex $v_j \in S$, such that $(v_i v_j) \in E$, $|\deg_{\mu_i} - \deg_{\mu_j}| \leq 1$, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $|\deg_{\gamma_i} - \deg_{\gamma_j}| \leq 1$, $\gamma_{ij} \geq \gamma_{ij}^{\infty}$ and the subgraph $S$ is connected. Thus, $S$ is an equitable dominating set of an IFG $G$.

**Theorem 3.31.** Let $G = (V, E)$ be an IFG without isolated vertices. Let $D$ be a minimal equitable dominating set of $G$. Then $(V - D)$ is an equitable dominating set of $G$.

Proof. Let $v_i$ be any vertex in $D$. For any two vertices $v_i, v_j \in V$, there exists an edge $(v_i, v_j) \in E$ such that $\mu_{ij} > 0$ and $\gamma_{ij} > 0$. There is an vertex $v_k \in E N_i$. By Theorem 3.1, $v_k \in (V - D)$. Thus every element of $D$ is dominated by some element of $(V - D)$. Then $(V - D)$ is an equitable dominating set of $G$.

**Theorem 3.32.** Let $G = (V, E)$ be a connected IFG and $H$ is a spanning subgraph of $H$. Then $d_r(G) \geq d_r(H)$.

Proof. Let $G = (V, E)$ be a connected IFG. and let $H = (V', E')$ is the spanning subgraph of $H$. $S$ is the minimum equitable dominating set of $H$. Then $S$ is also equitable dominating set in $V(H) - S$. That is, $S$ is an equitable dominating set in $G$. Thus $d_r(G) \geq d_r(H)$.

**Theorem 3.33.** For any complete IFG, $d_r(G) = \min \left\{ \sum_{v_i \in V} \left( \frac{1 + \mu_i - \gamma_i}{2} \right) \right\}$ for all $v_i \in G$.

Proof. Let $G = (V, E)$ be a complete IFG. For all $(v_i, v_j) \in G$, $\mu_{ij} = \min(\mu_i, \mu_j)$ and $\gamma_{ij} = \max(\gamma_i, \gamma_j)$. Every edge in $G$ is an strong edge. That is, $\mu_{ij} \geq \mu_{ij}^{\infty}$ and $\gamma_{ij} \geq \gamma_{ij}^{\infty}$. So every vertex dominates all other vertices. Let $D \subseteq V$ be a minimal restrained dominating set of $G$. Then every vertex in $(V - D)$ dominates a vertex in $D$ and another vertex in $(V - D)$. Thus , $d_r(G) = \min \left\{ \sum_{v_i \in V} \left( \frac{1 + \mu_i - \gamma_i}{2} \right) \right\}$ for all $v_i \in G$.

**Note 3.34.** For a complete bipartite IFG $K_{V_1, V_2}$ with $|V_1| \geq 2$ and $|V_2| \geq 2$, $d_r(G) = \min_{v_i \in V_1} \left( \frac{1 + \mu_i - \gamma_i}{2} \right) + \min_{v_j \in V_2} \left( \frac{1 + \mu_i - \gamma_i}{2} \right)$

**Theorem 3.35.** For an IFG $G = (V, E)$, $|V| = n$ then, $d_{gr}(G) = o(G)$ if and only if $G = K_V$ or $K_V^C$, where $o(G)$ is the cardinality of all the vertices in $G$.

Proof. To prove $d_{gr}(G) = o(G)$. Assume that $G = K_V$ or $K_V^C$. To prove $d_{gr}(G) = o(G)$. For every edge $(v_i, v_j)$ in $K_V \mu_{ij} = \min(\mu_i, \mu_j)$ and $\gamma_{ij} = \max(\gamma_i, \gamma_j)$. Then $K_V^C$ contains no edges. All the vertices in $K_V^C$ is an isolated vertices. For all $v_i \in K_V^C$
\(\mu_{ij} = 0, \gamma_{ij} = 0\). Each vertex dominates itself. Let \(\{v_1, v_2, v_3, \ldots, v_i, v_n\}\) be the vertices of \(G\). Suppose \(\{v_i, v_{i+1}, \ldots, v_n\}\) is the restrained dominating set of \(G\), but it is not the dominating set of \(K_V^C\). So the only dominating set of both \(K_V\) and \(K_V^C\) is the set containing all the vertices in \(G\). The global restrained dominating set of \(K_V\) or \(K_V^C\) contains \(n\) vertices. Hence, \(d_{gr}(G) = o(G)\).

Conversely, suppose \(d_{gr}(G) = o(G)\). To prove \(G = K_V\) or \(K_V^C\), suppose \(G \neq K_V\) or \(G \neq K_V^C\), then \(G\) has atleast one vertex \(v_i\) which is not dominated by \(v_j\). \(V - \{v_i\}\) is a global restrained dominating set containing \(n - 1\) vertices. This implies that \(d_{gr}(G) \neq o(G)\) which is a contradiction. Hence \(d_{gr}(G) = o(G)\). \(\blacksquare\)

**Note 3.36.** For a complete bipartite IFG \(G = K_{V_1, V_2}\),

(i) \(d_{gr}(G) = o(G)\), if \(|V_1| \leq 2\) and \(|V_2| \leq 2\)

(ii) \(d_{gr}(G) = \min_{v_i \in V_1} \left(\frac{1 + \mu_i - \gamma_i}{2}\right) + \min_{v_i \in V_2} \left(\frac{1 + \mu_i - \gamma_i}{2}\right)\), if \(|V_1| > 2\) and \(|V_2| > 2\)

(iii) \(d_{gr}(G) = \min_{v_i \in V_1} \left(\frac{1 + \mu_i - \gamma_i}{2}\right) + \min_{v_i \in V_2} \left(\frac{1 + \mu_i - \gamma_i}{2}\right)\), if \(|V_1| = 2\) and \(|V_2| > 2\).

**Theorem 3.37.** Let \(G = (V, E)\) be an IFG, then the following hold:

(i) If \(G\) be an intuitionistic fuzzy cycle and \(|V| \geq 6\), then \(d_r(G) = d_{gr}(G)\).

(ii) If \(G\) be a path and \(|V| \geq 9\), then \(d_r(G) = d_{gr}(G)\).

**Note 3.38.** For \(|V| < 6\) and \(|V| < 8\), the result is verified with suitable examples.

**Example 3.39.** Consider an IFG, \(G = (V, E)\), with \(V = \{v_1, v_2, v_3, v_4, v_5, v_6\}\),

\(E = \{(v_1, v_2), (v_2, v_4), (v_4, v_5), (v_5, v_6), (v_3, v_1)\}\), as given in Figure 7.

![Figure 7: Intuitionistic fuzzy cycle G.](image)

Here, \(D = \{v_1, v_4, v_5\}\) is restrained dominating set of \(G\) it also dominating set of \(\overline{G}\). Then, \(d_r(G) = d_{gr}(G) = 1.36\).
Example 3.40. Consider an IFG, $G = (V, E)$, with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8), (v_8, v_9), (v_9, v_{10})\}$, as given in Figure 8.

Here, $D = \{v_1, v_4, v_7, v_{10}\}$ is restrained dominating set of $G$ it also dominating set of $\overline{G}$. Then $d_r(G) = d_{gr}(G) = 1.85$.

Theorem 3.41. Let $G = (V, E)$ be an IFG with $o(G)$. If every vertex $v_i \in V$ dominates at least two vertices in $V$ and for vertex $v_j$ of maximum $\mu$-degree $\Delta_{\mu}$, $N(v_j) \neq \phi$, then $d_r(G) \leq o(G) - \Delta_{\mu}$.

Proof. Let $v_j$ be a vertex of maximum $\mu$-degree $\Delta_{\mu}$. If $\Delta_{\mu} = o(G) - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$ for a vertex $v_i \in V$. Then $\{v_i\}$ is a restrained dominating set of $G$ and so, $d_r(G) \leq \left(\frac{1 + \mu_i - \gamma_i}{2}\right) = o(G) - \Delta_{\mu}$. Hence, $\Delta_{\mu} < o(G) - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$. Let $D = V - N(v_j)$. Let $|D| = o(G) - \Delta_{\mu} - \left(\frac{1 + \mu_i - \gamma_i}{2}\right)$. Since $N(v_j)$ is without isolated vertex, then $D \cup \{v_i\}$ is a restrained dominating set of $G$. ■

Therefore, $d_r(G) \leq o(G) - \Delta_{\mu}$.

Theorem 3.42. Let $G = (V, E)$ is an intuitionistic fuzzy cycle and $|V| \geq 6$, then $d_r(\overline{G}) \leq d_r(G)$.

Proof. Let $S$ be a restrained dominating set of $G$. Then $(V - S)$ has at least two components. Let $\{v_i, v_{i+1}\}$ and $\{v_j, v_{j+1}\}$ be two arbitrary components of $(V - S)$. Then, $v_i$ dominates $v_{i+2}$, in $\overline{G}$. Therefore, $d_r(\overline{G}) \leq d_r(G)$. ■

4. Equitable domination and $k$-domination in product of intuitionistic fuzzy graphs

In this section, equitable domination and $k$-domination are defined for cartesian, direct and strong products of two intuitionistic fuzzy graphs.
Theorem 4.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs with $V_1 \cap V_2 = \emptyset$. Then the strong product $G = G_1 \boxtimes G_2$ remains connected even after removal of all weak edges in it.

Proof. Let $G = G_1 \boxtimes G_2$ be the strong product of two IFGs $G_1$ and $G_2$. Let $e = (v_i u_p, v_j u_q)$ be a weak edge in $G$ such that $\mu_{ip,jq} < \mu_{ip,jq}^\infty$ and $\gamma_{ip,jq} < \gamma_{ip,jq}^\infty$. To prove $G' = G - e$ is connected. Assume that, $G'$ is a disconnected IFG. The edge $e = (v_i u_p, v_j u_q)$ disconnect the graph into more than one components. This implies that, there is no path between $v_i u_p$ and $v_j u_q$ except the edge $e = (v_i u_p, v_j u_q)$ in $G'$. Then $\mu_{ip,jq} \geq \mu_{ip,jq}^\infty$ and $\gamma_{ip,jq} \geq \gamma_{ip,jq}^\infty$, which is a contraction. Then $G'$ is connected. ■

Theorem 4.2. If a vertex $v_i$ dominates $v_j$ in $G_1$ and the vertex $u_p$ dominates $u_q$ in $G_2$, then the vertex $v_i u_p$ does not dominate the vertex $v_j u_q$ in $G_1 \boxtimes G_2$.

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs. If $v_i$ dominates $v_j$ in $G_1$, there exists an strong edge $(v_i, v_j)$ in $G_1$ such that $\mu_{ij} \geq \mu_{ij}^\infty$ and $\gamma_{ij} \geq \gamma_{ij}^\infty$. Similarly, if $u_p$ dominates $u_q$ in $G_2$, there exists a strong edge $(u_p, u_q)$ in $G_2$ such that $\mu_{pq} \geq \mu_{pq}^\infty$ and $\gamma_{pq} \geq \gamma_{pq}^\infty$. By Definition 2.10, there is no edge between $v_i u_p$ and $v_j u_q$ in $G_1 \boxtimes G_2$. That is, $\mu_{ip,jq} = 0$ and $\gamma_{ip,jq} = 0$. Then $v_i u_p$ does not dominate $v_j u_q$ in $G_1 \boxtimes G_2$. ■

Note 4.3. If $v_i$ dominates $v_j$ in $G_1$ and $u_p$ dominates $u_q$ in $G_2$, then the vertex $v_i u_p$ dominates the vertex $v_j u_q$ in $G_1 \boxtimes G_2$.

Theorem 4.4. For any two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $d_{te}(G_1 \boxtimes G_2) \leq \min(|D_1 \times V_2|, |D_2 \times V_1|)$.

Proof. Let $D_1 \subseteq V_1$ and $D_2 \subseteq V_2$ be the minimum equitable total dominating sets of $G_1$ and $G_2$ respectively. Suppose $D = D_1 \times D_2$ be the minimum equitable total dominating set of $G_1 \boxtimes G_2$. Let $v_i u_p$ be an arbitrary vertex of $G_1 \boxtimes G_2$. Then, there exists $v_j \in D_1$ such that $|\text{deg}_{ui} - \text{deg}_{\mu ij}| \leq 1$, $\mu_{ij} \geq \mu_{ij}^\infty$ and $|\text{deg}_{\gamma_1} - \text{deg}_{\gamma_1'}| \leq 1$, $\gamma_{ij} \geq \gamma_{ij}^\infty$. Also, $u_q \in D_2$, $|\text{deg}_{\mu p} - \text{deg}_{\mu q}| \leq 1$, $\mu_{pq} \geq \mu_{pq}^\infty$ and $|\text{deg}_{\gamma_2} - \text{deg}_{\gamma_2'}| \leq 1$, $\gamma_{pq} \geq \gamma_{pq}^\infty$. That is, $v_j$ dominates $v_i$ and $u_q$ dominates $u_p$. Thus, $|\text{deg}_{\mu ip} - \text{deg}_{\gamma_1'}| \leq 1$, $\mu_{ip,jq} \geq \mu_{ip,jq}^\infty$ and $|\text{deg}_{\gamma p} - \text{deg}_{\gamma q}| \leq 1$, $\gamma_{ip,jq} \geq \gamma_{ip,jq}^\infty$. $v_i u_p$ dominates $v_i u_p$ in $G_1 \boxtimes D_2$. Then, $D$ is an equitable dominating set of $G_1 \boxtimes G_2$. Therefore $d_{te}(G_1 \boxtimes G_2) \leq \min(|D_1 \times V_2|, |D_2 \times V_1|)$. ■

Theorem 4.5. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two complete IFGs. Then $G = G_1 \boxtimes G_2$ is also a complete IFG.

Example 4.6. Consider the IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{v_1, v_2\}, E_1 = \{(v_1, v_2)\}$ and $V_2 = \{u_1, u_2\}, E_1 = \{(u_1, u_2)\}$. The graph of $G = G_1 \boxtimes G_2$ is displayed in Figure 9.
The graph of \( G_1 \otimes G_2 \) is a complete IFG. The converse of the theorem is not true in general.

**Theorem 4.7.** Let \( D_1 \) and \( D_2 \) be \( k \)-dominating sets of connected IFGs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) respectively. Then

(i) \( G_1 \otimes G_2 \) is connected

(ii) If \( D_1 \) is connected, then \( D_1 \times V_2 \) is a connected \( k \)-dominating set of \( G_1 \otimes G_2 \).

(iii) If \( D_2 \) is connected, then \( V_1 \times D_2 \) is a connected \( k \)-dominating set of \( G_1 \otimes G_2 \).

**Proof.** To prove \( G_1 \otimes G_2 \) is connected, it is enough to prove that for any two arbitrary distinct vertices \( v_{i_1}u_p, v_{j_1}u_q \) in \( G_1 \otimes G_2 \) such that \( \mu_{i_1p,j_1q} > 0 \) and \( \gamma_{i_1p,j_1q} > 0 \).

**Case (i).** \( v_{i_1} = v_{j_1}, \) \( G_2 \) is a connected IFG. Then, there exists a path \( p = u_{1}, u_2, \ldots, u_p \) such that \( (\mu_{pq}, \gamma_{pq}) > 0 \) for any two vertices \( u_p, u_q \) of path \( p \). This implies that, \( \mu_{i_1p,j_1q} = \mu_{i_1p} \wedge \mu_{pq} > 0 \) and \( \gamma_{i_1p,j_1q} = \gamma_{i_1p} \vee \gamma_{pq} > 0 \) and hence \( p' = v_{i_1}u_p, v_{i_1}u_1, v_{i_1}u_2 \cdots v_{i_1}u_q \) is the path between \( v_{i_1}u_p \) and \( v_{i_1}u_q \) in \( G_1 \otimes G_2 \).

**Case (ii).** \( u_p = u_q, \) \( G_1 \) is a connected IFG. Then, there exists a path \( q = v_1, v_2, \ldots, v_i \) such that \( (\mu_{ij}, \gamma_{ij}) > 0 \) for any two vertices \( v_i, v_j \) of path \( q \). This implies that, \( \mu_{i_1p,j_1q} = \mu_{i_1p} \wedge \mu_{ij} > 0 \) and \( \gamma_{i_1p,j_1q} = \gamma_{i_1p} \vee \gamma_{ij} > 0 \) and hence \( q' = v_{i_1}u_p, v_{i_1}u_p, v_{i_1}u_p \cdots v_{i_1}u_p \) is the path between \( v_{i_1}u_p \) and \( v_{i_1}u_p \) in \( G_1 \otimes G_2 \).

**Case (iii).** \( v_{i_1} \neq v_{j_1}, u_p \neq u_q. \) By case (i), there exists a path between \( v_{i_1}u_p \) and \( v_{i_1}u_q \) in \( G_1 \otimes G_2 \) and by case (ii), there exist a path between \( v_{i_1}u_p \) and \( v_{j_1}u_p \) in \( G_1 \otimes G_2 \). The union of these two disjoint paths is a path between \( v_{i_1}u_q, v_{j_1}u_p \) in \( G_1 \otimes G_2 \).
Therefore, $D_1 \times V_2$ and $V_1 \times D_2$ are dominating sets and the proof of connectivity of $D_1 \times V_2$ and $V_1 \times D_2$ is similar. $\blacksquare$

**Theorem 4.8.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two IFGs without isolated vertices and let $D_1$ be a total $k$-dominating set of $G_1$. Then $G_1 \Box G_2$ has no isolated dominating vertices and $D_1 \times V_2$ is a total $k$-dominating set of $G_1 \Box G_2$.

**Proof.** Let $D_1 \subseteq V_1$ be a total $k$-dominating set of $G_1$. To prove $D_1 \times V_2$ is a total $k$-dominating set of $G_1 \Box G_2$, first prove that any vertex in $G_1 \Box G_2$ is not an isolated vertex. Let $v_iu_p$ be an arbitrary vertex in $G_1 \Box G_2$. Then, there exists a vertex $v_j$ in $D_1$ such that $v_i \in N(v_j)$.

\[
\mu_{ip,jp} = \mu_{ij} \land \mu_p \\
= \mu_i \land \mu_j \land \mu_p \\
= \mu_{ip} \land \mu_{jp}
\]

\[
\gamma_{ip,jp} = \gamma_{ij} \lor \gamma_p \\
= \gamma_i \lor \gamma_j \lor \gamma_p \\
= \gamma_{ip} \lor \gamma_{jp}
\]

This implies that, $v_iu_p \in N(v_ju_p)$. Therefore, $v_iu_p$ is not an isolated vertex and $v_ju_p \in D_1 \times V_2$ is a total $k$-dominating set of $G_1 \Box G_2$. $\blacksquare$

**Theorem 4.9.** Let $D_1$ and $D_2$ be $k$-dominating set of an IFGs $G_1$ and $G_2$ respectively.

(1) $D_1 \times V_2$ is an independent $k$-dominating set of $G_1 \Box G_2$ if and only if $D_1$ is $k$-independent and

(i) $\mu_{ij} < \mu_p$ and $\gamma_{ij} < \gamma_p$ for $v_i, v_j \in D_1, u_p \in V_2$

(ii) $\mu_{pq} < \mu_i$ and $\gamma_{pq} < \gamma_i$ for $v_i \in D_1, u_p, u_q \in V_2$

(iii) $\mu_{pq} < \mu_p \land \mu_q$ and $\gamma_{pq} < \gamma_p \land \gamma_q$ for $u_p, u_q \in V_2$

(2) $V_1 \times D_2$ is an independent $k$-dominating set of $G_1 \Box G_2$ if and only if $D_2$ is $k$-independent and

(i) $\mu_{ij} < \mu_p$ and $\gamma_{ij} < \gamma_p$ for $v_i, v_j \in V_1, u_p \in V_2$

(ii) $\mu_{ij} < \mu_i \land \mu_j$ and $\gamma_{ij} < \gamma_i \land \gamma_j$ for $v_i, v_j \in V_1,$

(iii) $\mu_{pq} < \mu_i$ for $u_p, u_q \in V_2.$
Proof. To prove that every two distinct vertices $v_i u_p, v_j u_q$ in $D_1 \times V_2$ are not adjacent. If $v_i = v_j$ then

$$
\mu_{ip,jp} = \mu_i \land \mu_{pq} < \mu_i \land \mu_p \land \mu_q < \mu_{ip} \land \mu_{iq}
$$

$$
\gamma_{ip,jp} = \gamma_i \lor \gamma_{pq} < \gamma_i \lor \gamma_p \lor \gamma_q < \gamma_{ip} \lor \gamma_{iq}
$$

If $u_p = u_q$, the result is obtained by independence of $v_i, v_j$ of $D_1$. If $v_i \neq v_j, u_p \neq u_q$, by Definition 2.10, $\mu_{ip,jq} = 0$ and $\gamma_{ip,jq} = 0$. Hence, $v_i u_p, v_j u_q$ are not adjacent in $G_1 \Box G_2$. Conversely, suppose 1(iii) is false. That is, $u_p, u_q \in V_2$ such that $\mu_{pq} < \mu_p \land \mu_q$ and $\gamma_{pq} < \gamma_p \land \gamma_q$. If $v_i$ is any vertex in $D_1$, then

$$
\mu_{ip,jp} = \mu_i \land \mu_{pq} = \mu_i \land \mu_p \land \mu_q = \mu_{ip} \land \mu_{iq}.
$$

$$
\gamma_{ip,jp} = \gamma_{ij} \lor \gamma_p = \gamma_i \lor \gamma_j \lor \gamma_p = \gamma_{ip} \lor \gamma_{jp}.
$$

This implies that, $D_1 \times V_2$ is not independent, which is a contradiction. Hence 1(iii) is true. That is, $\mu_{pq} < \mu_p \land \mu_q$ and $\gamma_{pq} < \gamma_p \land \gamma_q$. The proof of 2 is similar to the proof of 1.

5. Conclusion

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems. Intuitionistic fuzzy models also give more precision, flexibility, and compatibility to the system like fuzzy models. Domination in IFGs have many applications in the real world situations such as location problems, communication network, pattern clustering, routings. The concept equitable dominating set, independent equitable dominating set, $k$-dominating set and $k$-total dominating of an IFG are defined with suitable examples. Also, the lower and upper bound for the strong and weak equitable domination is established.

For an intuitionistic fuzzy cycle and path, the global restrained domination is obtained. Also, domination parameters like equitable domination and $k$- domination on direct, cartesian and strong products of two IFGs are analysed. Further, the authors proposed to apply these concepts in clustering techniques.
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