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CESȦRO AND HȮLDER MEAN OF PRODUCT SUMMABILITY METHODS

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Abstract

In [6], the Cesȧro means and Cesȧro summability were discussed for sequences. In [7], the Holder means and Holder summability were discussed for sequences. In [5], the definition of product summability method $(D,k)(C,l)$ for functions was given and some of its properties were investigated. In [2], $(D,k)(C,\alpha,\beta)$ $(k > 0, \alpha > 0, \beta > -1)$ summability for functions are defined and some of its properties were investigated .

1. Introduction

Kuttner [1], introduced the summability method (D,α) for functions and investigated some of its properties . Pathak [5], defined the product summability method $(D,k)(C,l)$ for functions and investigated some of its properties . Mishra and Srivastava [4], introduced the summability method (C,α,β) for functions by generalizing (C,α) summability method . In this paper,we define Cesaro and Holder mean of product summability $(D,k)(C,\alpha,\beta)$ $(k > 0, \alpha > 0, \beta > -1)$ for functions and investigate some of its properties.

2. Some Relations and definitions

We would like to first introduce Summability method. Summability method is more general than that of ordinary convergence. If we are given a sequence (s_n), we can construct a generalized sequence (σ_n), the arithmetic mean of (s_n) by this sequence (s_n). If (σ_n) is convergent in ordinary sense for all $n > 0$, then we say that (s_n) is summable $(C, 1)$ to the sum s. This $(C, 1)$ is called Cesaro mean of first order.

If $s_n \to s \Rightarrow \sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots+s_n}{n+1} \to s$, ie if a sequence is convergent, it is summable by method of arithmetic mean. Also a series $1 - 1 + 1 + 1 +$ is not convergent , but is summable to the sum $\frac{1}{2}$. The space of summable sequences is larger than space of convergent sequences. If $\sigma_n \to s$ as $n \to \infty$, then we say that sequence (s_n) is summable by method of arithmetic mean.

For example : Consider the series

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \cdots \ldots \ldots \quad (1)$$

And let $\sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots+s_n}{n+1}$, It may happen that whereas (1) diverges , the quantities (the arithmetic mean of partial sum of series) converges to a definite limit as $n \to \infty$.

For example $1 - 1 + 1 - 1 + \cdots \ldots \ldots$ diverges, but in this case $s_0 = 1$, $s_1 = 1 - 1 = 0$, $s_2 = 1 - 1 + 1 = 1$, $s_3 = 0 \ldots \ldots \ldots \ldots \ldots \ldots$ $(s_n) = (1,0,1,0,1 \ldots \ldots)$. Since $s_n = \frac{1+(-1)^n}{2}$,

$$\sigma_n = \frac{s_0+s_1+\cdots\ldots\ldots+s_n}{n+1}$$

$$= \frac{1+(-1)^0}{2} + \frac{1+(-1)^1}{2} + \frac{1+(-1)^2}{2} + \cdots \ldots \ldots \ldots \ldots + \frac{1+(-1)^n}{2} / (n+1)$$

$$= \frac{(n+1)}{2} + \frac{1}{2}\{1 - 1 + 1 - \cdots \ldots \ldots \ldots (n+1) terms\} / (n+1)$$

$$= \frac{1}{2} + \frac{1+(-1)^n}{4(n+1)}$$

, If n is even then $\sigma_n = \frac{1}{2} + \frac{1}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$ and if n is odd then $\sigma_n = \frac{1}{2}$. So in either case $\lim_{n\to\infty} \sigma_n = \frac{1}{2}$, $\therefore\ s_n \notin C\ but\ s_n \epsilon S$. Therefore space of summable sequences is larger than thar of space of convergent sequences .

In mathematical analysis, **Cesàro summation** assigns values to some infinite sums that are not convergent in the usual sense, while coinciding with the standard sum if they are convergent. The Cesàro sum is defined as the limit of the arithmetic mean of the partial sums of the series.

Regularity of a Method: A summability method *M* is regular if it agrees with the actual limit on all convergent series .
Cesàro method is always regular.
Cesàro summation is named for the Italian analyst Ernesto Cesàro (1859–1906).
On the other hand, now let $a_n = n$ for $n \geq 1$. That is, $\{a_n\}$ is the sequence

$1, 2, 3, 4, \ldots.$ and let G now denote the series

$$\sum_{n=1}^{\infty} a_n = 1 + 2 + 3 + 4 + 5 + \cdots$$

Then the sequence of partial sums $\{s_n\}$ is

$$1, 3, 6, 10, \ldots,$$

and the evaluation of G diverges to infinity. The terms of the sequence of means of partial sums $\{t_n\}$ are here

$$\frac{1}{1}, \frac{4}{2}, \frac{10}{3}, \frac{20}{4}, \ldots.$$

Thus, this sequence diverges to infinity as well as G, and G is now not Cesàro summable. In fact, any series which diverges to (positive or negative) infinity the Cesàro method also leads to a sequence that diverges likewise, and hence such a series is not Cesàro summable.

(C, α) summation :
In 1890, Ernesto Cesàro stated a broader family of summation methods which have since been called (C, α) for non-negative integers α. The (C, 0) method is just ordinary summation, and (C, 1) is Cesàro summation as described above.

The higher-order methods can be described as follows: given a series Σa_n, define the quantities

$$A_n^{-1} = a_n; \quad A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}$$

(where the upper indices do not denote exponents) and define E_n^{α} to be A_n^{α} for the series $1 + 0 + 0 + 0 + \cdots$. Then the (C, α) sum of Σa_n is denoted by (C, α) - Σa_n and has the value

$$(C, \alpha) - \sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \frac{A_n^{\alpha}}{E_n^{\alpha}}$$

if it exists (Shawyer & Watson 1994, pp.16-17). This description represents an α-times iterated application of the initial summation method and can be restated as

$$(C,\alpha)-\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+\alpha}{j}} a_j$$

Even more generally, for $\alpha \in \mathbb{R} \setminus (-\mathbb{N})$, let A_n^α be implicitly given by the coefficients of the series

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} a_n x^n}{(1-x)^{1+\alpha}},$$

and E_n^α as above. In particular, E_n^α are the binomial coefficients of power $-1-\alpha$. Then the (C, α) sum of $\Sigma\, a_n$ is defined as above.

If Σa_n has a (C, α) sum, then it also has a (C, β) sum for every β>α, and the sums agree; furthermore we have $a_n = o(n^\alpha)$ if $\alpha > -1$.

Cesàro summability of an integral :

Let α ≥ 0. The integral $\int_0^\infty f(x)\,dx$ is Cesàro summable (C, α) if

$$\lim_{\lambda\to\infty} \int_0^\lambda \left(1-\frac{x}{\lambda}\right)^\alpha f(x)\,dx$$

exists and is finite (Titchmarsh 1948, §1.15). The value of this limit, should it exist, is the (C, α) sum of the integral. Analogously to the case of the sum of a series, if α=0, the result is convergence of the improper integral. In the case α=1, (C, 1) convergence is equivalent to the existence of the limit

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \int_0^\lambda \left\{\int_0^x f(y)\,dy\right\} dx$$

which is the limit of means of the partial integrals.

As is the case with series, if an integral is (C,α) summable for some value of α ≥ 0, then it is also (C, β) summable for all β > α, and the value of the resulting limit is the same.

Let $f(x)$ be any function which is Lebesgue integrable in (o, x) and that $f : [0, +\infty) \to R,$ and integrable in (0, x), for any finite $x > 0$ and which is bounded in some right hand neighbourhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty}\int_0^x$, $\int_0^x$ being a Lebesgue integral. For any $\alpha > 0$, we write $F_n(x)$ for the n^{th} integral,

$F_n(x) = \frac{1}{\Gamma(n)} \int_0^x (x-y)^{n-1} f(y)dy$,

$F_0(x) = f(x)$. The (C,α,β) transform of $f(t)$, which we denote by $\partial_{\alpha,\beta}(t)$ is given by

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{t^{\alpha+\beta}}\int_0^t (t-u)^{\alpha-1}u^{\beta}a(y)dy\,,\ (\alpha>0,\beta>-1) \qquad ,(2.1)$$

If, for t > 0 , the integral defining $\partial_{\alpha,\beta}(t)$ exists and if $\partial_{\alpha,\beta}(t) \to s$ as $t \to \infty$, we say that f(x) is summable (C,α,β) to s, and we write $\boldsymbol{f(x)} \to$ s (C, α, β) .

We write

$$g(t) = g^{(k)}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} f(x)dx \text{ , (k > 0)} \qquad (2.2) \quad \text{if this exists ,}$$

We also write $U_{k,\alpha,\beta}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} \partial_{\alpha,\beta}(x)dx$, (2.3) if this exists .

With the usual terminology, we say that the function $f(x)$ is summable ,

(I) (D, k) to the sum s, if g (t) tends to a limit s as t→ ∞ ,

(II) (D, k) (C,α,β) to s , if $U_{k,\alpha,\beta}(t)$tends to s as t→ ∞.When $\beta = 0$,$(D,k)(C,\alpha,\beta)$ and $(D,k)(C,\alpha)$ denote the same method . The case $\beta = 0$ is due to Pathak [5] . We know that for any fixed t > 0, k > 0, it is necessary and sufficient for the convergence of (2.3) that $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}\ dx$ should converge . (2.4)
If (2.4) converges , write for x > 0 , $F_{\alpha,\beta}(x) = \int_x^\infty \frac{\partial_{\alpha,\beta}(t)}{t^2}\ dt$.

3. Main Results

In this section ,we have following theorems.

Theorem 3.1 : If $\alpha > \gamma \geq 1,\ k > 0$ then $f(x) \to s\ (D,k)(C,\alpha-1,\beta)$,whenever $a(x) \to s(D,k)(C,\gamma-1,\beta)$.

Theorem 3.2 : Let $\alpha > \gamma \geq 0,\ \beta > -1$, and suppose that f(x) is summable (C,γ,β) to s and that $\int_1^\infty \frac{\partial_{\gamma,\beta}(x)}{x^2}\ dx$ converges . Then f(x) is summable $(D,k)(C,\alpha,\beta)$ to s .

Proof of theorem 3.1: To prove this theorem,we first note by lemma 4.2 [3] that, if $U_{k,\gamma,\beta}(x)$ is defined and so is $U_{k,\alpha,\beta}(x)$. Hence we deduce from lemma 4.2 [3] that if $\alpha > \gamma \geq 1$ and $\lim_{x\to\infty} U_{k,\gamma-1,\beta}(x) = 0$ then $\lim_{x\to\infty} U_{k,\alpha-1,\beta}(x) = 0$. Consequently, for $\alpha > \gamma \geq 1, f(x) \to s(D,k)(C,\alpha-1,\beta)$ whenever $f(x) \to s(D,k)(C,\gamma-1,\beta)$. This establishes the theorem.

Proof of theorem 3.2: We know that $F_n(x) = \frac{1}{\Gamma(n)} \int_0^x (x-y)^{n-1} f(y)dy$, $(\alpha > 0)$ and $F_0(x) = f(x)$. From the definition of

Or $\partial_{\alpha+p,\beta}(x) = \frac{\Gamma(\alpha+\beta+p+1)}{\Gamma(\alpha+p-\gamma)\Gamma(\gamma+\beta+1)} \frac{1}{x^{\alpha+p+\beta}} \int_0^x (x-t)^{\alpha+p-r-1} t^{\gamma+\beta} \partial_{\gamma,\beta}(t)dt$, here

$\frac{1}{x^{\alpha+p+\beta}} \partial_{\alpha+p,\beta}(x)$ is just $\frac{\Gamma(\alpha+\beta+p+1)}{\Gamma(\gamma+\beta+1)}$ times the $(\alpha+p-\gamma)th$ indefinite integral of

$t^{\gamma+\beta} \partial_{\gamma,\beta}(t)$. We also write $\phi(x) = \int_x^\infty \frac{\partial_{\gamma,\beta}(t)}{t^2} dt$

It is clear that, whenever $\int_1^\infty \frac{\partial_{\gamma,\beta}(x)}{x^2} dx$ converges, $\phi(x)$ is defined for $x > 0$ and that

$\phi(x) \to 0$ as $x \to \infty$. It follows from (3.21) that

$$\partial_{\alpha+1,\beta}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1-\gamma)\Gamma(\gamma+\beta+1)} \frac{1}{x^{\alpha+1+\beta}} \int_0^x (x-t)^{\alpha-\gamma} t^{\gamma+\beta} \partial_{\gamma,\beta}(t)dt \qquad (3.22)$$

$$= -C \frac{1}{x^{\alpha+1+\beta}} \int_0^x (x-t)^{\alpha-\gamma} t^{\gamma+2+\beta} d\phi(t).$$

$$= -C \frac{1}{x^{\alpha+1+\beta}} \left[t^{\alpha-\gamma+\gamma+\beta+2} \phi(t)\right]_0^x + C \frac{1}{x^{\alpha+1+\beta}} \int_0^x \{(x-t)^{\alpha-\gamma}(\gamma+\beta+$$

$$2) t^{\gamma+1+\beta} - (\alpha-\gamma)(x-t)^{\alpha-\gamma-1} t^{\gamma+\beta+2}\} \phi(t)\, dt. = 0(x)$$

And therefore for $p \geq 1$, $\partial_{\alpha+p,\beta}(x) = 0(x^p)$. (3.23)
It is enough to prove the theorem for the case in which r is an integer, and $s(x)$ is summable (C,γ,β) to zero. Note that $\partial_{r,\beta}(x) = 0(1)$ (3.24)
Since $s(x) \to 0(c,k) \Rightarrow s(x) \to 0(c,k')$ for $k' > k \geq 0$
Using (3.24) we get

$$\partial_{\alpha+\gamma,\beta}(x) = \frac{\Gamma(\alpha+\beta+\gamma+1)}{\Gamma(\alpha+r-\gamma)\Gamma(\gamma+\beta+1)} \frac{1}{x^{\alpha+r+\beta}} \int_0^x (x-t)^{\alpha+r-\gamma-1} t^{\gamma+\beta} \partial_{\gamma,\beta}(t)dt$$

$$= 0(1). \qquad (3.25)$$

We write $U_{k,\alpha,\beta}(y) = ky \int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}} \partial_{\alpha,\beta}(x)dx$

$$= ky\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)dx \qquad (3.26)$$

Using the result that a rth indefinite integral of $x^{\alpha+\beta}\partial_{\alpha,\beta}(x)$ is a constant times $x^{\alpha+r+\beta}\partial_{\alpha+r,\beta}(x)$ and integrating by parts (3.26) r times, we deduce that

$$U_{k,\alpha,\beta}(y) = ky\int_0^\infty \partial_{\alpha+r,\beta}(x)x^{\alpha+r+\beta}\left\{\frac{d^r}{dx^r}\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}\right)\right\}dx \qquad (3.27)$$

It is easily verified that the expression in curly brackets in (3.27) is 0 ($\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}$) uniformly in $x \geq 0, y \geq 0$.

We know that $g(y) = g^{(k)}(y) = kt\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}}f(x)dx,$

If $f(x) = 1$, then $ky\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}}dx = 1$

Hence by (3.27) we have

$$U_{k,\alpha,\beta}(y) = ky\int_0^\infty \partial_{\alpha+r,\beta}(x)x^{\alpha+r+\beta}\, 0\left\{\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}\right)\right\}$$

$$= ky\int_0^\infty o(1)O\left\{\left(\frac{x^{k-1}}{(x+y)^{k+1}}\right)\right\} = 0(1),\ as\ y \to \infty.$$

This completes the proof of theorem.

Examples : The function $sin\,\frac{k}{2}\,x + \sin(k-1)x$ is summable and theorem (3.1) holds for this function.

Applications :
The significance of the concept of summability has been strikingly demonstrated in various contexts, for example, in analytic continuation, quantum mechanics, probability theory, Fourier analysis, approximation theory, and fixed point theory. The methods of almost summability and statistical summability have become an active area of research in the recent years. The aim of this special issue is to focus on recent developments and achievements in summability theory such as sequences spaces and their geometry, statistical summability and statistical approximation, almost summability, fuzzy sequence spaces, matrix summability, compact matrix operators between sequence spaces and infinite systems of differential and integral equations in sequence spaces, and various applications. Summability theory fits within the broader mathematical topic grouping called analysis. It also has links to number theory. G. H. Hardy and J. E. Littlewood are perhaps the most well-known

summabilists. The classic explanation of summability, and still the best, is Hardy's book on Divergent Series.

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