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On convergence theorems for new classes of multivalued hemicontractive-type mappings

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Abstract

It is our purpose in this paper to prove weak and strong convergence theorem in Hilbert space via multivalued version of noor iteration process for a new classes of multivalued demicontractive-type and hemicontractive-type mappings which are related to the class of multivalued pseudocontractive-type mappings studied by [8]. Our results extend several corresponding results appear in current literature.

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1. Introduction

Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. A mapping $T : K \to K$ is said to be

(i) lipschitz, if
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \text{for all } x, y \in K;$$

(ii) nonexpansive, if $L = 1$;

(iii) strongly pseudocontractive [1], if for some $\alpha \in (0, 1)$,
$$\langle Tx - Ty, x - y\rangle \leq \alpha\|x - y\|^2, \quad \text{for all } x, y \in K;$$

(iv) pseudocontractive [1], if $\alpha = 1$.

Note that inequality

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \ \ \text{for all } x, y \in K$$

can be equivalently written as

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \ \ \text{for all } x, y \in K$$

where I denotes the identity operator.

In 1967, Browder and Petryshyn [1] introduced the class of pseudocontractive mappings in Hilbert space. The class of pseudocontractive mapping is more general than nonexpansive mapping. It is well known that every nonexpansive mapping is a continuous pseudocontractive mapping but the converse need not be true (see e.g. [3]). In last five decades various researchers has attracted towards to develop different iterative techniques to approximate the fixed points for strongly pseudocontractive mappings in Hilbert spaces (see e.g. [2, 7, 11, 12, 20, 21, 22, 23]). Recently, Cheng et al. [4] introduced a three-step iteration to obtain convergence theorems for a countable family of uniformly closed uniformly Lipschitzian pseudocontractive mapping in Hilbert space and further the results were extended by Deng [5] to relaxing uniformly assumption on the involved Lipschitzian and closed mapping.

On the other hand, the fixed point problems for multivalued mappings are more difficult than those of a single valued mappings and play very important role in applied sciences and economics. The study of the fixed points for multivalued contraction and nonexpansive mapping using the Hausdorff metric was proved by Nadler [14] and Markin [13].

Let X be a real normed linear space. A subset K of X is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$\|x - k\| = \inf\{\|x - y\| : y \in K\} = d(x, K).$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. For a nonempty set X, we shall denote the family of all nonempty proximinal subsets of X by $P(X)$, the family of all nonempty closed and bounded subsets of X by $CB(X)$.

Let H denote the Hausdorff metric induced by the metric d on X, that is, for every $A, B \in CB(X)$,

$$H(A, B) \ = \ max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}.$$

Let $T : D(T) \subseteq X \to 2^X$ be a multivalued mapping on X. A point $x \in D(T)$ is called a fixed point of T, if $x \in Tx$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of T. A point $x \in D(T)$ is called a strict fixed point of T if $Tx = \{x\}$.

A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is called L-Lipschitzian if there exists $L \ge 0$ such that for all $x, y \in D(T)$,

$$H(Tx, Tp) \le L\|x - y\| \tag{1.1}$$

In (1.1) if $L \in [0, 1)$, T is said to be a contraction, while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \phi$ and for all $p \in F(T)$,

$$H(Tx, Tp) \leq \|x - p\| \tag{1.2}$$

Clearly, every nonexpansive mapping with nonempty fixed point set is quasi nonexpansive. The existence of fixed point for multivalued nonexpansive mapping in uniformly convex Banach space was proved by Lim [10]. Recently, many authors have applied different iterative algorithms for finding the fixed point of multivalued nonexpansive mapping using Hausdorff metric spaces (see e.g. [9, 15, 16, 17, 18]).

In 2013, Isiogugu [7] define a new class of multivalued mappings as follows.

Definition 1.1. [7] Let X be a normed space. A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is said to be k-strictly pseudocontractive-type in the sense of Browder and Petryshyn [1] if there exists $k \in [0, 1)$ such that given any $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $\|u - v\| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2 \tag{1.3}$$

If $k = 1$ in (1.3), T is said to be pseudocontractive-type mapping and it is called nonexpansive-type if $k = 0$. Clearly, every multivalued nonexpansive mapping is nonexpansive-type mapping.

From the definitions, it is clear that every multivalued nonexpansive-type mapping is k-strictly pseudocontractive-type and every k-strictly pseudocontractive-type mapping is pseudocontractive-type. In [7], it is proved that the class of nonexpansive-type mappings is properly contained in the class of k-strictly pseudocontractive-type mappings and that the class of k-strictly pseudocontractive-type mappings is properly contained in the class of pseudocontractive-type mappings.

In 2014, Isiogugu et al. [8] define a new classes of multivalued demicontractive-type and hemicontractive-type mappings which are related to the class of multivalued pseudocontractive-type mappings as follows and proved weak and strong convergence theorems in Hilbert space (see Theorem 3.1 and 3.2 of [8]).

Definition 1.2. [8] Let X be a real normed space. A mapping $T : D(T) \subseteq X \to 2^X$ is said to be demicontractive in the sense of Hicks and Kubicek [6] if $F(T) \neq \phi$ and for all $p \in F(T)$, $x \in D(T)$ there exists $k \in [0, 1)$ such that

$$H^2(Tx, Tp) \leq \|x - p\|^2 + kd^2(x, Tx), \tag{1.4}$$

where $H^2(Tx, Tp) = [H(Tx, Tp)]^2$ and $d^2(x, p) = [d(x, p)]^2$. If $k = 1$ in (1.4) then T is called a hemicontractive mapping.

It is our purpose in this paper to prove weak and strong convergence theorem in Hilbert space via multivalued version of noor iteration process for a new classes of multivalued demicontractive-type and hemicontractive-type mappings which are related to the class of multivalued pseudocontractive-type mappings stutied by [8]. Our results extend several corresponding results appear in current literature.

2. Preliminaries

For our main results we need the following lemmas and definitions.

Lemma 2.1. [18] Let K be a normed space. $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then the following are equivalent:

(1) $x \in Tx$;

(2) $P_T x = \{x\}$;

(3) $x \in F(P_T)$.
Moreover, $F(T) = F(P_T)$.

Lemma 2.2. [19] Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0, 1]$ the following inequality holds:

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$$

Lemma 2.3. [8] Let X be a normed space. Suppose T is a multivalued mapping such that $F(T) \neq \phi$ and P_T is pseudocontractive-type mapping, then P_T is hemicontractive.

Lemma 2.4. [8] Let X be a normed space. Suppose $T : D(T) \subseteq X \to P(X)$ be a multivalued pseudocontractive-type with a nonempty fixed point set. Let $Tp = \{p\}$ for all $p \in F(T)$; then for any $x \in D(T)$, $p \in F(T)$ and $u \in Tx$ with $\|u - x\| = d(x, Tx)$ we have

$$H^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2 = \|x - p\|^2 + d^2(x, Tx).$$

Lemma 2.5. [8] Let E be a metric space. If $A, B \in P(E)$ and $a \in A$, then it is a simple consequence of the Hausdorff metric H that there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) \tag{2.1}$$

3. Main Results

Recall that a multivalued mapping $T : K \to P(K)$ is said to satisfy condition (I) (see [17]), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))), \quad \text{for all } x, y \in K.$$

Theorem 3.1. Let K be a nonempty closed and convex subset of a Hilbert space X and suppose that $T : X \to P(K)$ is a Lipschitzian hemicontractive mapping from K into the family of proximinal subset of K and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose T satisfies the condition (I). Then the noor iteration process define by

$$\begin{cases} x_1 & \in K, \\ x_{n+1} & = (1-\alpha_n)x_n + \alpha_n v_n, \\ y_n & = (1-\beta_n)x_n + \beta_n w_n, \\ z_n & = (1-\gamma_n)x_n + \gamma_n u_n, \end{cases} \tag{3.1}$$

where $u_n \in Tx_n$, $v_n \in Ty_n$, $w_n \in Tz_n$ satisfying the conditions of Lemma (2.5) and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences satisfying (i) $0 \le \alpha_n \le \beta_n \le \gamma_n < 1$; (ii) $\liminf_{n\to\infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n\ge 1} \gamma_n \le \gamma$ with $2\gamma + L^2\gamma^2 + L^2\gamma + L^4\gamma^2 < 1$. Then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Suppose $p \in F(T)$. Using (3.1) and Lemma (2.2), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|(1-\gamma_n)x_n + \gamma_n u_n - p\|^2 \\ &\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|u_n - p\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\ &\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n H^2(Tx_n, Tp) - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\ &\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n[\|x_n - p\|^2 + d^2(x_n, Tx_n)] \\ &\quad - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 + \gamma_n\|x_n - u_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\ &\le \|x_n - p\|^2 + \gamma_n^2\|x_n - u_n\|^2 \end{aligned} \tag{3.2}$$

Again, using (3.1) and Lemma (2.2), we have

$$\begin{aligned} \|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n w_n - p\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n H^2(Tz_n, Tp) - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n[\|z_n - p\|^2 + d^2(z_n, Tz_n)] \\ &\quad - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + \beta_n d^2(z_n, Tz_n) \\ &\quad - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \beta_n\gamma_n^2\|x_n - u_n\|^2 + \beta_n d^2(z_n, Tz_n) \\ &\quad - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \|x_n - p\|^2 + \beta_n\gamma_n^2\|x_n - u_n\|^2 + \beta_n d^2(z_n, Tz_n) \\ &\quad - \beta_n(1-\beta_n)\|x_n - w_n\|^2. \end{aligned} \tag{3.3}$$

But

$$\begin{aligned}
d^2(z_n, Tz_n) &\le \|z_n - w_n\|^2 \\
&\le \|(1-\gamma_n)x_n + \gamma_n u_n - w_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + \gamma_n\|u_n - w_n\|^2 \\
&\quad - \gamma(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + \gamma_n H^2(Tx_n, Tz_n) - \gamma(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + \gamma_n L^2\|x_n - z_n\|^2 - \gamma(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + \gamma_n L^2\|x_n - (1-\gamma_n)x_n - \gamma_n u_n\|^2 \\
&\quad - \gamma(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + \gamma_n^3 L^2\|x_n - u_n\|^2 - \gamma(1-\gamma_n)\|x_n - u_n\|^2 \\
&\le (1-\gamma_n)\|x_n - w_n\|^2 + (\gamma_n^3 L^2 - \gamma_n + \gamma_n^2)\|x_n - u_n\|^2 \qquad (3.4)
\end{aligned}$$

Using (3.3) and (3.4), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|x_n - p\|^2 + \beta_n\gamma_n^2\|x_n - u_n\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\quad + \beta_n(1-\gamma_n)\|x_n - w_n\|^2 + \beta_n(\gamma_n^3 L^2 - \gamma_n + \gamma_n^2)\|x_n - u_n\|^2 \\
&\le \|x_n - p\|^2 + (\beta_n\gamma_n^2 + \beta_n\gamma_n^3 L^2 - \beta_n\gamma_n + \beta_n\gamma_n^2)\|x_n - u_n\|^2 \\
&\quad + \beta_n(\beta_n - \gamma_n)\|x_n - w_n\|^2 \\
&\le \|x_n - p\|^2 - \beta_n\gamma_n(1 - 2\gamma_n + L^2\gamma_n^2)\|x_n - u_n\|^2 \\
&\quad - \beta_n(\gamma_n - \beta_n)\|x_n - w_n\|^2. \qquad (3.5)
\end{aligned}$$

From (3.1) and Lemma (2.2), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n v_n - p\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp) - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \alpha_n d^2(y_n, Ty_n) \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2. \qquad (3.6)
\end{aligned}$$

But

$$\begin{aligned}
d^2(y_n, Ty_n) &\le \|y_n - v_n\|^2 \\
&\le \|(1-\beta_n)x_n + \beta_n w_n - v_n\|^2 \\
&\le (1-\beta_n)x_n\|x_n - v_n\|^2 + \beta_n\|w_n - v_n\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\le (1-\beta_n)\|x_n - v_n\|^2 + \beta_n H^2(Tz_n, Ty_n) - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\le (1-\beta_n)\|x_n - v_n\|^2 + \beta_n L^2\|z_n - y_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - y_n\|^2 &= \|(1-\gamma_n)x_n + \gamma_n u_n - y_n\|^2 \\
&\leq (1-\gamma_n)\|x_n - y_n\|^2 + \gamma_n\|u_n - y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\leq (1-\gamma_n)\|x_n - y_n\|^2 + \gamma_n\|u_n - y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\leq (1-\gamma_n)\|x_n - (1-\beta_n)x_n - \beta_n w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n\|u_n - y_n\|^2 \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n\|u_n - (1-\beta_n)x_n - \beta_n w_n\|^2 \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n[(1-\beta_n)\|x_n - u_n\|^2 \\
&\quad + \beta_n\|u_n - w_n\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2] \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n\beta_n H^2(Tx_n, Tz_n) - \gamma_n\beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n\beta_n L^2\|x_n - z_n\|^2 - \gamma_n\beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n\beta_n L^2\|x_n - (1-\gamma_n)x_n - \gamma_n u_n\|^2 - \gamma_n\beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\leq (1-\gamma_n)\beta_n^2\|x_n - w_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n(1-\beta_n)\|x_n - u_n\|^2 \\
&\quad + \gamma_n^3\beta_n L^2\|x_n - u_n\|^2 - \gamma_n\beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\leq (\beta_n^2 - \gamma_n\beta_n)\|x_n - w_n\|^2 + (\gamma_n^2 - \gamma_n\beta_n + \gamma_n^3\beta_n L^2)\|x_n - u_n\|^2. \quad (3.8)
\end{aligned}$$

From (3.7) and (3.8), we have

$$\begin{aligned}
d^2(y_n, Ty_n) &\leq (1-\beta_n)\|x_n - v_n\|^2 + \beta_n L^2(\beta_n^2 - \gamma_n\beta_n\|x_n - w_n\|^2 \\
&\quad + \beta_n L^2(\gamma_n^2 - \gamma_n\beta_n + \gamma_n^3\beta_n L^2)\|x_n - u_n\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\
&\leq (1-\beta_n)\|x_n - v_n\|^2 + (\beta_n^3 L^2 - \gamma_n\beta_n^2 L^2 - \beta_n + \beta_n^2)\|x_n - w_n\|^2 \\
&\quad + \beta_n L^2(\gamma_n^2 - \gamma_n\beta_n + \gamma_n^3\beta_n L^2)\|x_n - u_n\|^2 \qquad (3.9)
\end{aligned}$$

Using (3.5) and (3.9) in (3.6), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n[\|x_n-p\|^2\\ &\quad -\beta_n\gamma_n(1-2\gamma_n-L^2\gamma_n^2)\|x_n-u_n\|^2\\ &\quad -\beta_n(\gamma_n-\beta_n)\|x_n-w_n\|^2]+\alpha_n[(1-\beta_n)\|x_n-v_n\|^2\\ &\quad +(\beta_n^3L^2-\gamma_n\beta_n^2L^2-\beta_n+\beta_n^2)\|x_n-w_n\|^2\\ &\quad +\beta_nL^2(\gamma_n^2-\gamma_n\beta_n+\gamma_n^3\beta_nL^2)\|x_n-u_n\|^2]\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-v_n\|^2\\ &\leq \|x_n-p\|^2+\alpha_n(\alpha_n-\beta_n)\|x_n-v_n\|^2\\ &\quad +\alpha_n\beta_n[2\beta_n-\gamma_n-1)+\beta_nL^2(\beta_n-\gamma_n)]\|x_n-w_n\|^2\\ &\quad +\alpha_n\beta_n\gamma_n[(2\gamma_n+L^2\gamma_n^2-1)\\ &\quad +L^2(\gamma_n-\beta_n+L^2\gamma_n^2)]\|x_n-u_n\|^2 \end{aligned} \tag{3.10}$$

Since form condition (iii), we have that

$$2\gamma+L^2\gamma^2-1+L^2\gamma+L^4\gamma^2<0$$

This implies

$$(2\gamma_n+L^2\gamma_n^2-1)+L^2(\gamma_n+L^2\gamma^2)<0$$

$$(2\gamma_n+L^2\gamma_n^2-1)+L^2(\gamma_n-\beta_n+L^2\gamma^2)<0$$

$$\alpha_n\beta_n\gamma_n[(2\gamma_n+L^2\gamma_n^2-1)+L^2(\gamma_n-\beta_n+L^2\gamma^2)]<0$$

Form condition (i)

$$\alpha_n-\beta_n\leq 0$$

$$2\beta_n-\gamma_n-1\leq 0$$

Also

$$\beta_n-\gamma_n\leq 0$$

So, inequality (3.10) implies that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2-\alpha_n\beta_n\gamma_n[(1-2\gamma_n-L^2\gamma_n^2)\\ &\quad -L^2(\gamma_n-\beta_n+L^2\gamma_n^2)]\|x_n-u_n\|^2\end{aligned} \tag{3.11}$$

Then

$$\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2 \tag{3.12}$$

It is obvious that $\lim_{n\to\infty}\|x_n-p\|$ exists, then $\{\|x_n-p\|\}$ is bounded. This implies that $\{x_n\}$, $\{u_n\}$, $\{z_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{v_n\}$ are also bounded.

From (3.11) and using conditions (i), (ii) and (iii), we get

$$\begin{aligned}
&\sum \alpha^3[(1-2\gamma - L^2\gamma^2) - L^2(\gamma + L^2\gamma^2)]\|x_n - u_n\|^2 \\
&\le \sum \alpha_n\beta_n\gamma_n[(1-2\gamma_n - L^2\gamma_n^2) \\
&- \ L^2(\gamma_n + L^2\gamma_n^2)]\|x_n - u_n\|^2 \\
&\le \sum \alpha_n\beta_n\gamma_n[(1-2\gamma_n - L^2\gamma_n^2) \\
&- L^2(\gamma_n - \beta_n + L^2\gamma_n^2)]\|x_n - u_n\|^2 \\
&\le \|x_{n+1} - p\|^2 - \|x_n - p\|^2 < \infty
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \tag{3.13}$$

Since $u_n \in Tx_n$, we have $d(x_n, Tx_n) \le \|x_n - u_n\| \to 0$ as $n \to \infty$. ■

Theorem 3.2. Let K, X and T be as in Theorem (3.1) with $T(p) = \{p\}$ for all $p \in F(T)$. Suppose that T satisfies condition (I), then the sequence $\{x_n\}$ defined by (3.1) converges strongly to fixed point of T.

Proof. By Theorem (3.1), we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

Since T satisfies condition (I), therefore

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n)$$

Since f is nondecreasing function with $f(0) = 0$, therefore we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| \le \dfrac{1}{2^k}$ for some $\{p_k\} \subseteq F(T)$.

From (3.12),

$$\|x_{n_{k+1}} - p\| \le \|x_{n_k} - p\|$$

We now prove that $\{p_k\}$ is a Cauchy sequence in $F(T)$. We have

$$\begin{aligned}
\|p_{k+1} - p_k\| &\le \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\
&\le \frac{1}{2^{k+1}} + \frac{1}{2^k} \\
&= \frac{1}{2^{k-1}}.
\end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence and converges to some $q \in K$ because K is closed. Now,

$$\|x_{n_k} - q\| \leq \|x_n - p_k\| + \|p_k - q\|.$$

Hence $x_{n_k} \to q$ as $k \to \infty$. We have

$$\begin{aligned} d(q, Tq) &\leq \|q - p_k\| + \|p_k - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq \|q - p_k\| + \|p_k - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + L\|x_{n_k} - q\| \end{aligned}$$

Hence $q \in Tq$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim \|x_n - q\|$ exists we see that x_n converges strongly to $q \in F(T)$. ■

Corollary 3.3. Let K be a nonempty closed and convex subset of a real Hilbert space X. Suppose that $T : K \to P(K)$ is an L-Lipschitzian pseudocontractive-type mapping from K into the family of all proximinal subsets of K such that $F(T) \neq \phi$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition (I). Then the noor sequence $\{x_n\}$ defined in (3.1) converges strongly to $p \in F(T)$.

Proof. The proof follows easily from Lemma (2.4), Lemma (2.5) and Theorem (3.1). ■

Corollary 3.4. Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K such that $F(T) \neq \phi$. Suppose P_T is an L-Lipschitzian hemicontractive mapping. If T satisfies condition (I). Then the noor sequence $\{x_n\}$ defined in (3.1) converges strongly to $p \in F(T)$.

Proof. The proof follows easily from Lemma (2.1) and Theorem (3.1). ■

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