Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 1 (2015), pp. 95–104

http://www.ripublication.com/gjpam.htm

On strongly left McCoy rings

Sang Jo Yun[1]

Department of Mathematics Education,
Daegu University, Kyeongsan-City,
Kyeongbuk, 712-714, Korea.

Sincheol Lee, Sung Ju Ryu, Yunho Shin, and Hyo Jin Sung

Department of Mathematics,
Pusan National University,
Pusan 609-735, Korea.

Abstract

Hong et al. show that the class of strongly right McCoy rings contains both reversible rings and right duo rings. We prove in this note that both reversible rings and left duo rings are strongly left McCoy, focusing on the shapes of annihilators.

AMS subject classification: 16U80, 16D25, 16S36.
Keywords: Strongly left McCoy ring, reversible ring, left duo ring, Abelian ring.

1. Introduction

Throughout this note every ring is associative with identity unless otherwise stated. We recall first the concept of McCoy ring (resp., strongly McCoy ring) was introduced by [17] (resp., [8]). We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Denote the n by n full matrix ring over R by $Mat_n(R)$ and the n by n upper (resp. lower) triangular matrix ring over R by $U_n(R)$ (resp. $L_n(R)$). Note $Mat_n(R)[x] \cong Mat_n(R[x])$ and we will use this freely. Use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}_n$ denotes the ring of integers modulo n. The set of all

[1]Corresponding author.

nilpotent elements in R is written by $N(R)$, and $N_*(R)$ denotes the prime radical of R. Let $\mathbb{N}$ denote the set of nonnegative integers.

The *right* (resp. *left*) *annihilator* of S in R is denoted by $r_R(S)$ (resp. $l_R(S)$) , and by $r_R(a)$ (resp. $l_R(a)$) when $S = \{a\}$. An element c of a ring R is called *right regular* if $r_R(c) = 0$, *left regular* if $l_R(c) = 0$, and *regular* if $r_R(c) = 0 = l_R(c)$. A ring is called a *domain* if every nonzero element is regular. It is well-known that for a division ring D an element of the n by n full matrix ring is regular if and only if it is invertible.

McCoy [15] obtained the following in 1957. If R is a commutative ring, then

$$f(x)g(x) = 0 \text{ implies } f(x)c = 0 \text{ for some nonzero } c \in R,$$

where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Based on this result, Nielsen [17] called a ring R (possibly non-commutative) *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some nonzero $c \in R$, where $f(x)$, $g(x)$ are nonzero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then the ring is called a *McCoy* ring.

Following Cohn [4], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson and Camillo [1], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible. Reversible rings are McCoy by Nielsen [17, Theorem 2]. A ring R is called *reduced* if $N(R) = 0$. The class of McCoy rings is easily shown to contain both reduced rings and commutative rings. We use these facts freely.

Due to Bell [2], a right (or left) ideal I of a ring R is said to have the *insertion-of-factors-property* (simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So it is natural to call a ring R an IFP ring if the zero ideal of R has the IFP. Shin [18] used the term SI for the IFP, while Narbonne [16] called IFP rings *semi-commutative.* Subrings of IFP rings are also IFP obviously. Shin showed that the equivalences for IFP rings in [18, Lemma 1.2]. Reversible rings are clearly IFP. A ring R is called *Abelian* if every idempotent of R is central. It is easily checked that IFP rings are Abelian and $N(R) = N_*(R)$. It is well-known that the implications above are irreversible. The concepts of IFP and right McCoy are independent of each other by help of [17, section 3].

According to Lambek [13], a ring R is called *symmetric* if $rst = 0$ implies $rts = 0$ for all $r, s, t \in R$; while Anderson-Camillo [1] took the the term ZC_3 for this notion. By [13, Proposition 1], a ring R is symmetric if and only if $r_1r_2\cdots r_n = 0$, with n any positive integer, implies $r_{\sigma(1)}r_{\sigma(2)}\cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$; while Anderson-Camillo obtained this result independently by [1, Theorem I.1]. Commutative rings are clearly symmetric, but the converse need not be true by [1, Example I.5] or [9, Theorem 2.3]. Symmetric rings are clearly reversible, but the converse need not hold by [1, Example II.5] or [14, Examples 5 and 7]. Reduced rings are symmetric by [14, Theorem I.3], but the converse need not hold since there are many non-reduced commutative (so symmetric) rings.

Hong et al. consider the following conditions in [8]. For the case of $f(x)g(x) = 0$

with $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{m} b_j x^j$ in $R[x]$, we have

$$b_t(a_{t_1} \cdots a_{t_h}) \neq 0 \text{ and } f(x)b_t(a_{t_1} \cdots a_{t_h}) = 0 \quad (*)$$

or

$$(a_{t_1} \cdots a_{t_h})b_t \neq 0 \text{ and } f(x)(a_{t_1} \cdots a_{t_h})b_t = 0 \quad (**)$$

for some $\{a_{t_1}, \ldots, a_{t_h}\} \subseteq \{a_0, \ldots, a_n\}$ with $h \leq m$ (if any) and $b_t \in \{b_0, \ldots, b_m\}$. Hong et al. prove that reversible rings satisfy the condition (*) and symmetric rings satisfy the codntion (**) in [8, Theorem 1.6 and Proposition 1.7].

Following Hong et al. [8], a ring R is called *strongly right McCoy* provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the right ideal of R generated by the coefficients of $g(x)$, where $f(x)$ and $g(x)$ are nonzero polynomials in $R[x]$. Strongly right McCoy rings are clearly right McCoy, but the converse need not hold by [8, Example 1.9]. It is obvious that rings satisfying the condition (*) are strongly right McCoy (the converse need not hold by Note in [8, page 1818]). Next, R is called *strongly left McCoy* provided that $f(x)g(x) = 0$ implies $rg(x) = 0$ for some nonzero r in the left ideal of R generated by the coefficients of $f(x)$. If a ring is both strongly left and strongly right McCoy then the ring is called a *strongly McCoy* ring. Reversible rings are strongly right McCoy by [8, Theorem 1.6] or the proof of Nielsen [17, Theorem 2].

Feller [5] called a ring *right duo* if each right ideal is two-sided. The left duo ring is defined similarly. It is easily checked that left or right duo rings are Abelian. Any right duo ring is strongly right McCoy by [8, Theorem 1.11] or the proof of [3, Theorem 8.2], but there are strongly right McCoy rings which are neither reversible nor right duo, by help of [8, Example 1.10 or Theorem 2.2].

2. On strongly left McCoy rings

In this section, we prove that reversible rings and left duo rings are both strongly left McCoy. The proofs are done by applying the methods in Hong et al. [8], McCoy [15], and Nielsen [17]. However the arguments are complicated, so we provide them in this note. To do it, we consider first the following conditions which are related to reversible rings and symmetric rings. For the case of $f(x)g(x) = 0$ with $0 \neq f(x) = \sum_{i=0}^{m} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$, we have

$$(b_{t_h} \cdots b_{t_1})a_t \neq 0 \text{ and } (b_{t_h} \cdots b_{t_1})a_t g(x) = 0 \quad (\dagger)$$

or

$$a_t(b_{t_h} \cdots b_{t_1}) \neq 0 \text{ and } a_t(b_{t_h} \cdots b_{t_1})g(x) = 0 \quad (\ddagger)$$

for some $\{b_{t_1}, \ldots, b_{t_h}\} \subseteq \{b_0, \ldots, b_n\}$ with $h \leq n$ (if any) and $a_t \in \{a_0, \ldots, a_m\}$.

One may compare the following with [3, lemma 5.4].

Lemma 2.1. Let R be an IFP ring and $m,\ n \in \mathbb{N}$. For $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, if there exists $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ with $f(x)g(x) = 0$, then $f(x)b_0^{m+1} = 0$ and $f(x)b_n^{m+1} = 0$.

Proof. We apply the proof of [3, Lemma 5.4]. First note that by setting $f^*(x) = x^m f(x^{-1})$ and $g^*(x) = x^n g(x^{-1})$, we have just reversed the coefficients on the polynomials and obtain the equation $f^*(x)g^*(x) = 0$. To finish the lemma it suffices to prove $f(x)b_0^{m+1} = 0$, because we then obtain $f^*(x)b_n^{m+1} = 0$ by symmetry, and this last equations is equivalent to $f(x)b_n^{m+1} = 0$.

Clearly $a_0 b_0 = 0$, Assume by induction that $a_l b_0^{l+1} = 0$ for all $l < i$. Looking at the degree i coefficient of the equation $f(x)g(x) = 0$ yields $\sum_{j=0}^{i} a_{i-j} b_j = 0$. Multiplying on the right by b_0^i, we have

$$0 = \sum_{j=0}^{i} a_{i-j} b_j b_0^i = a_i b_0^{i+1} + \sum_{j=1}^{i} a_{i-j} b_j b_0^i.$$

But, by the IFP property and the induction hypothesis, we get

$$\sum_{j=1}^{i} a_{i-j} b_0^i = a_{i-1} b_0^i + a_{i-2} b_0 b_0^{i-1} + \cdots + a_0 b_0^{i-1} b_0 = 0,$$

entailing

$$a_i b_0^{i+1} = 0.$$

Therefore we are done by induction. ■

This lemma helps us to obtain the following.

Theorem 2.2.

(1) Let R be a reversible ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^n$ be nonzero polynomials over R with $f(x)g(x) = 0$. Then there exists $r = b_s^{l_s} \cdots b_1^{l_1} b_0^{l_0} \in R$ ($s \leq n$ and $l_k \geq 0$ for $k = 0, \ldots, s$) with $rf(x) \neq 0$ and $ra_i b_j = 0$ for all i and j.

(2) Reversible rings satisfy the condition (†).

Proof. We apply the proof of [8, Theorem 1.6] to the left case. (1) We can assume $a_0, a_m, b_0, b_n \in R\backslash\{0\}$. If $deg\ g(x) = 0$ then $a_i g(x) = 0$ for all $i \in \{0, \dots, m\}$. So we let $deg\ g(x) \geq 1$. If $f(x)b_0 = 0$ then $b_0 f(x) = 0$ since R is reversible.

Suppose $f(x)b_0 \neq 0$. Then it is possible to choose $l_0 \geq 0$ such that $f(x)b_0^{l_0+1} = 0$ and $f(x)b_0^{l_0} \neq 0$ by Lemma 2.1. Thus

$$b_0^{l_0} f(x) b_0 = 0 \text{ (i.e., } b_0^{l_0} a_i b_0 = 0 \text{ for all } i) \text{ and } b_0^{l_0} f(x) \neq 0$$

since R is reversible.

Let here

$$f_1(x) = b_0^{l_0} f(x).$$

Then we get $f_1(x)g(x) = 0$ from $f(x)g(x) = 0$, and

$$\begin{aligned} 0 &= f_1(x)g(x) \\ &= b_0^{l_0} f(x)(b_0 + b_1 x + \cdots + b_n x^n) = b_0^{l_0} f(x)(b_1 x + \cdots + b_n x^n). \end{aligned}$$

We consider next whether $f_1(x)b_1 = 0$ or not. Repeating the preceding process, there exists $l_1 \geq 0$ such that

$$b_1^{l_1} f_1(x) b_1 = 0 \text{ (i.e., } b_1^{l_1} b_0^{l_0} a_i b_1 = 0 \text{ for all } i) \text{ and } b_1^{l_1} f_1(x) \neq 0$$

since R is reversible.

Continue this computation. Then, after a finite number of steps, we finally obtain $s \leq n$ and $l_s \geq 0$ such that

$$(b_s^{l_s} \cdots b_1^{l_1} b_0^{l_0}) f(x) \neq 0$$

and

$$(b_s^{l_s} \cdots b_1^{l_1} b_0^{l_0}) f(x) b_t = 0 \text{ for all } t \in \{s, \dots, n\}.$$

These results yield

$$(b_s^{l_s} \cdots b_1^{l_1} b_0^{l_0}) f(x) b_j = 0 \text{ for all } j \in \{0, 1, \dots, n,$$

using the fact that reversible rings are IFP. Therefore, letting

$$r = b_s^{l_s} \cdots b_1^{l_1} b_0^{l_0},$$

we obtain that $r a_i b_j = 0$ for all i and j, noting $r \neq 0$.

(2) Let R be a reversible ring. We apply the proof of [15, Theorem]. Let $fg = 0$ for $0 \neq f = \sum_{i=0}^{m} a_i x^m$ and $0 \neq g = \sum_{j=0}^{n} b_j x^n$ in $R[x]$. It can be assumed that $a_n \neq 0$, $b_m \neq 0$.

If $a_n g = 0$ then we are done. So suppose $a_n g \neq 0$. Then $f b_j \neq 0$ for some $j \in \{0, 1, \dots, n\}$. Let p be the largest integer such that $f b_p \neq 0$. Then we get

$a_m b_p = 0$ from $fg = 0$. Since R is reversible, $b_p a_m = 0$ and $b_p f \neq 0$; hence $b_p f$ is a nonzero polynomial whose degree is less than m. But we also have $(b_p f)g = 0$.

We work next on $b_p f$ in place of g. Continue this computation. Then, after a finite number of steps, we finally obtain

$$b_{t_1}, \ldots, b_{t_h} \text{ and } a_t$$

with

$$t_1 = p,\ h \leq n,\ \{b_{t_1}, \ldots, b_{t_h}\} \subseteq \{b_0, \ldots, b_n\},\ \text{and } t \in \{0, 1, \ldots, m\}$$

such that

$$0 \neq (b_{t_h} \cdots b_{t_1})a_t$$

is the leading coefficient of $(b_{t_h} \cdots b_{t_1})f$ and

$$[(b_{t_h} \cdots b_{t_1})a_t]g = 0.$$

In the procedure of the computation, we also use the reversibility of R as in the case of $b_p f$ to get that

$$\deg\,(b_{t_h} \cdots b_{t_1})f = \deg\,(b_{t_{h-1}} \cdots b_{t_1})f b_{t_h}$$

for $k \in \{2, \ldots, h\}$. ■

Reversible rings are strongly left McCoy by Theorem 2.2, so reversible rings are strongly McCoy by combining this result with [8, Theorem 1.6].

Theorem 2.3. Symmetric rings satisfy the conditions (†) and (‡).

Proof. Symmetric rings are reversible, and so Theorem 2.2 holds in this situation. Moreover, by using the symmetry, we can obtain the condition (‡) from the computation in the proof of Theorem 2.2. ■

Recall that a ring is called *left duo* if every left ideal is two-sided. Note that left duo rings are IFP.

Theorem 2.4. Let R be a left duo ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials over R with $f(x)g(x) = 0$. Then there exists $r \in R$ with $rf(x) \neq 0$ and $ra_i b_j = 0$ for all i and j.

Proof. We apply the proof of [8, Theorem 1.11]. Let R be a left duo ring and suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ are nonzero polynomials in $R[x]$ satisfying $f(x)g(x) = 0$. We can assume $a_0, a_m, b_0, b_n \in R\backslash\{0\}$. We will prove that there exists $r \in R$ such that $rf(x) \neq 0$ and $ra_i g(x) = 0$ for all $i \in \{0, \ldots, m\}$.

If *deg* $g(x) = 0$ then $a_i g(x) = 0$ for all $i \in \{0, \dots, m\}$.

So let *deg* $g(x) \geq 1$. Suppose $f(x)b_0 \neq 0$. Let i be minimal so that $a_i b_0 \neq 0$. By help of Lemma 2.1, there exists an integer $t > 0$ such that

$$a_i b_0^t \neq 0 \text{ and } a_i b_0^{t+1} = 0,$$

noting that left duo rings are IFP.

Since R is left duo, there exists $r_1 \in R$ with $a_i b_0^t = r_1 a_i$. Letting $f_1(x) = r_1 f(x)$, we get that

$$f_1(x)g(x) = 0 \text{ and } f_1(x) \neq 0$$

because $r_1 a_i \neq 0$.

Note $f_1(x)b_0 = r_1 a_h b_0 x^h + \cdots + r_1 a_n b_0 x^m$ with $a_h b_0 \neq 0$ and $h \geq i + 1$, provided that

$$r_1 a_k b_0 = 0 \text{ for all } k = 0, \dots, i$$

because $r_1 a_i b_0 = a_i b_0^t b_0 = a_i b_0^{t+1} = 0$ and $a_s b_0 = 0$ for all $s \in \{0, \dots, i-1\}$.

Next we consider whether $f_1(x)b_0 = 0$ or not. If not we repeat the preceding process on $f_1(x)$, obtaining $r_2 \in R$ such that

$$r_2 f_1(x) \neq 0,$$

$$r_2 f_1(x)b_0 = (r_2 r_1) f(x) b_0 = r_2 r_1 a_l b_0 x^l + \cdots + r_2 r_1 a_m b_0 x^n$$

with $l \geq h + 1$, and

$$(r_2 r_1) a_k b_0 = 0 \text{ for all } k = 0, \dots, h.$$

So, after a finite number of steps, we finally obtain $r_1, \dots, r_s \in R$ such that

$$(r_s \cdots r_1) f(x) \neq 0, \ (r_s \cdots r_1) f(x) b_0 = 0,$$

and

$$(r_s \cdots r_1) a_k b_0 = 0 \text{ for all } k = 0, \dots, m.$$

Consequently, for any case of $f(x)b_0 = 0$ and $f(x)b_0 \neq 0$, there exists $t \in R$ such that $t a_i b_0 = 0$ for all $i = 0, \dots, m$; hence we get

$$0 = t f(x) g(x) = t f(x)(b_0 + b_1 x + \cdots + b_n x^n) = t f(x)(b_1 x + \cdots + b_n x^n).$$

Next, applying this process on $t f(x) b_1$, we get $v \in R$, depending on b_1, such that

$$v t a_i b_1 = 0 \text{ for all } i = 0, \dots, n; \text{ and } v t f(x) \neq 0.$$

Since $t a_i b_0 = 0$ for all $i = 0, \dots, n$, it follows that

$$v t a_i b_j = 0 \text{ for all } i \text{ and } j$$

with $i = 0, \dots, m$ and $j = 0, 1$.

Continue this computation. Then, after a finite number of steps, we finally obtain $w \in R$ such that

$$(w \cdots vt)f(x) \neq 0 \text{ and } (w \cdots vt)a_i b_j = 0$$

for all i, j with $i = 0, \dots, m$ and $j = 0, \dots, n$. This completes the proof, by letting $r = w \cdots vt$. ■

Left duo rings are strongly left McCoy by Theorem 2.4. However there exist strongly left McCoy rings which satisfy neither the condition (†) nor (‡) by applying [8, Example 1.10].

Lemma 2.5.

(1) A ring R, satisfying the condition (†), is Abelian.

(2) A ring R, satisfying the condition (‡), is Abelian.

Proof. We apply the proof of [8, Proposition 1.8].

(1) Let R be a ring satisfying the condition (†) and $e^2 = e \in R$. Assume on the contrary that there is $r \in R$ with $er(1-e) \neq 0$.

Consider $f(x) = e + er(1-e)x$ and $g(x) = (1-e) - er(1-e)x$ in $R[x]$. Then $f(x)g(x) = 0$. Note that

$$a_t g(x) \neq 0 \text{ and } b_j a_t = 0$$

for all coefficients b_j of $g(x)$ and a_t of $f(x)$. But, since R satisfies the condition (†), this induces a contradiction, forcing $er(1-e) = 0$. This implies $er = ere$ for all $r \in R$.

Exchanging the roles of e and $1-e$, we also get $re = ere$ for all $r \in R$. Thus R is Abelian.

(2) Let R be a ring satisfying the condition (‡) and $e^2 = e \in R$. Assume on the contrary that there is $r \in R$ with $er(1-e) \neq 0$.

Consider $f(x) = e + er(1-e)x$ and $g(x) = (1-e) - er(1-e)x$ in $R[x]$. Then $f(x)g(x) = 0$ and

$$e\{(1-e), er(1-e)\} = \{er(1-e)\}$$

and

$$(er(1-e))\{(1-e), er(1-e)\} = \{er(1-e)\} \neq 0.$$

But since R satisfies the condition (‡), we have

$$0 = er(1-e)g(x) = er(1-e),$$

a contradiction. This forces $er(1-e) = 0$, so $er = ere$ for all $r \in R$.

Exchanging the roles of e and $1-e$, we also get $re = ere$ for all $r \in R$. Thus R is Abelian. ■

However there exist strongly left McCoy rings which are not Abelina by help of [8, Example 1.10].

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ with $a = aba$. All conditions mentioned above are equivalent for regular rings.

Proposition 2.6. Let R be a regular ring. Then the following conditions are equivalent:

(1) R is reduced;

(2) R is symmetric;

(3) R is reversible;

(4) R satisfies the condition (†);

(5) R satisfies the condition (‡);

(6) R is strongly right (left) McCoy;

(7) R is right (left) McCoy;

(8) R is right (left) duo;

(9) R is IFP;

(10) R is Abelian.

Proof. The proof is done by Lemma 2.5, [6, Theorem 3.2], [8, Proposition 1.13], and [10, Proposition 2.14]. ■

References

[1] D.D. Anderson, V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), 2847–2852.

[2] H.E. Bell, *Near-rings in which each element is a power of itself, Bull*, Austral. Math. Soc. **2** (1970), 363–368.

[3] V. Camillo, P.P. Nielsen, (2008). *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), 599–615.

[4] P.M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), 641–648.

[5] E.H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.

[6] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.

[7] K.R. Goodearl, R.B. Warfield, JR., An Introduction to Noncommutative Noetherian Rings, *Cambridge University Press*, Cambridge-New York-Port Chester-Melbourne-Sydney (1989).

[8] C.Y. Hong, Y.C. Jeon, N.K. Kim, Y. Lee, *The McCoy Condition on Noncommutative Rings*, Comm. Algebra **39** (2011), 1809–1825.

[9] C. Huh, H.K. Kim, N.K. Kim, Y. Lee, *Basic examples and extensions of symmetric rings*, J. Pure Appl. Algebra **202** (2005), 154–167.

[10] B.-O. Kim, T.K. Kwak, Y. Lee, *On constant zero-divisors of linear polynomials*, Comm. Algebra, **43** (2015), 1098–1117.

[11] N.K. Kim, Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), 477–488.

[12] N.K. Kim, Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), 207–223.

[13] J. Lambek, *On the representation of modules by sheves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

[14] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002), 311–318.

[15] N.H. McCoy, *Annihilators in polynomial rings*, Amer. Math. Monthly **64** (1957), 28–29.

[16] L. Motais de Narbonne, *Anneaux semi-commutatifs et unis riels anneaux dont les id aux principaux sont idempotents*, Proceedings of the 106th National Congress of Learned Societies (Perpignan, 1981), Bib. Nat., Paris (1982), 71–73.

[17] P.P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), 134–141.

[18] G.Y. Shin, *Prime ideals and sheaf representation of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.