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Traveling Wave Solutions for the Nonlinear Boussinesq Water Wave Equation

Jaharuddin

Department of Mathematics,
Bogor Agricultural University,
Jl. Meranti, Kampus IPB Dramaga,
Bogor 16880, Indonesia.

Abstract

The Boussinesq equation is one of the nonlinear evolution equations which describes the model of shallow water waves. By using the modified F-expansion method, we obatained some exact solutions. Some exact solutions expressed by hyperbolic function and exponential function are obtained. The results show that the modified F-expansion method is straightforward and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics and engineering sciences.

AMS subject classification: 35C08, 35K99, 35P05.
Keywords: F-expansion method, traveling wave solutions, Boussinesq equation.

1. Introduction

The Boussinesq equation appears in studying the transverse motion and nonlinearity in acoustic waves on elastic rods with circular cross section. This equation arises in other physical applications such as nonlinear lattice waves [3], iron sound waves in plasma [7], and in vibrations in a nonlinear string [10]. One of the fundamental problems for these equation is to obtain their traveling waves solution as well as solitary wave solution. Traveling wave solutions represents an important type of solutions for nonlinear partial differential equations as many nonlinear PDE have been found to have a variety of traveling wave solutions. It is well-known that the investigation of the exact solutions of nonlinear PDE's plays an important role in the study of nonlinear physical phenomena. The sixth-order Boussinesq equation is one the nonlinear PDE's to be discussed. Many researchers studied the sixth-order Boussinesq equation to construct analytical solutions by using different methods. For instance, H. Naher, F.A. Abdullah, and M.A. Akbar [6] implemented the exp-function method to investigate this equation for obtaining traveling wave solutions. In [4], M. Hosseini, H. Abdollahzadeh, and M. Abdollahzadeh employed the sin-cosine method to construct exact solutions of the same equation. A general approach to construct exact solution to the Boussinesq equation is given by Clarkson [2] has deduced conservation laws. Wazwaz [8] obtained the single soliton solutions of the sixth-order Boussinesq equation using the tanh method and multiple soliton solutions using Hirota bilinear method. Wang and Li [9] presented a powerful method which is called the F-expansion method. The F-expansion method is an effective and direct algebraic method for finding the exact solutions of nonlinear evolution equation, many nonlinear equations have been successfully solved. The F-expansion method gives a unified formation to construct various traveling wave solutions and provides a guideline to classify the various types of traveling wave solutions. Later, the further develop methods named the modified F-expansion. Using the new method, exact solutions of many nonlinear evolution equations are successfully obtained.

The purpose of this paper is to implement the modified F-expansion method to search traveling wave solutions for the sixth-order Boussinesq equation. In former literature, the traveling wave solutions to sixth-order the Bossinesq equation have not been studied by this method. The paper is organized as follows. In Section 2, the modified F-expansion method is introduced briefly. Section 3 is devoted to applying this method to the sixth-order nonlinear Boussinesq water wave equation. The last section is a short summary and discussion.

2. Description of the method

Based on [1], the main procedures of modified F-expansion method are as follows. Consider the nonlinear partial differential equation in the form:

$$G\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \ldots\right) = 0 \tag{2.1}$$

where $u(x,t)$ is a traveling wave solution of equation (2.1), G is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. We use the transformation

$$\xi = \alpha(x - ct), \tag{2.2}$$

where c and α are constants. Using (2.2) to transfer the nonlinear partial differential equation (2.1) to nonlinear ordinary differential equation for $U = U(\xi)$

$$H\left(U, \frac{dU}{d\xi}, \frac{d^2U}{d\xi^2}, \frac{d^3U}{d\xi^3}, \ldots\right) = 0. \tag{2.3}$$

Suppose that the solution of equation (2.3) can be written as follows:

$$U(\xi) = A_0 + \sum_{k=1}^{N} \left(A_k (F(\xi))^k + B_k (F(\xi))^{-k}\right) \tag{2.4}$$

where $A_0,\ A_k,\ B_k\ (k = 1, 2, \dots N)$ are constants to be determined later. The positive integer N can be determined by considering the homogenous balance between the highest order derivatives and the nonlinear terms appearing in equation (2.3), and $F(\xi)$ satisfies the first-order linear ordinary differential equation:

$$\frac{dF}{d\xi} = h_0 + h_1 F + h_2 F^2, \tag{2.5}$$

where h_0, h_1, h_2 are the arbitrary constants to be determined later. The special condition chosen for the h_0, h_1, and h_2, we can obtain the solution of equation (2.5). For example, when $h_1 = 0$, and $h_2 = -h_0$, the solution of equation (2.5) is

$$F(\xi) = \tanh(h_0\xi).$$

Next, when $h_2 = 0$, and $h_1 = -h_0$, the solution of equation (2.5) is

$$F(\xi) = \frac{1}{h_1}\exp(h_1\xi) + 1.$$

Substituting (2.4) into (2.3) and using (2.5), we obtain a polynomial in $(F(\xi))^k$ $(k = 0, \pm1, \pm2, \dots)$. Equating each coefficient of the resulting polynomial to zero yields a set of algebraic equations for c, α, h_0, h_1, h_2, and $A_0,\ A_k,\ B_k\ (k = 1, 2, \dots N)$. Assuming that the constants $A_0,\ A_k,\ B_k\ (k = 1, 2, \dots N)$ can be obtained by solving the algebraic equations and then substituting there constants and the solution of (2.5), depending on the special conditions chosen for the h_0, h_1, and h_2 we can obtain the explicit solutions of equation (2.1) immediately. Obviously, different types of exact solutions of the equation (2.5) may result in different types of exact traveling wave solutions for equation (2.1).

3. Application of the method

In this section, we apply the modified F-expansion method to construct exact traveling wave solutions for the sixth-order Boussinesq equation. Let us consider the equation

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 15u\frac{\partial^4 u}{\partial x^4} - 30\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} - 15\left(\frac{\partial^2 u}{\partial x^2}\right)^2 \\ -45u^2\frac{\partial^2 u}{\partial x^2} - 90u\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial^6 u}{\partial x^6} = 0.\end{aligned} \tag{3.1}$$

The equation (3.1) was introduced by J.V. Boussinesq (1842–1929) to illustrate the propagation of long waves on the surface of water with small amplitude [5], where $u(x, t)$ is the elevation of the free surface of the fluid. Making a transformation $u(x, t) = U(\xi)$ with $\xi = \alpha(x - ct)$, equation (3.1) can be reduced to the following ordinary differential equation:

$$(c^2 - 1)\frac{\partial^2 U}{\partial \xi^2} - 15\alpha^2 u\frac{\partial^4 u}{\partial \xi^4} - 30\alpha^2\frac{\partial u}{\partial \xi}\frac{\partial^3 u}{\partial \xi^3} - 15\alpha^2\left(\frac{\partial^2 u}{\partial \xi^2}\right)^2 \tag{3.2}$$

$$-45u^2\frac{\partial^2 u}{\partial \xi^2} - 90u\left(\frac{\partial u}{\partial \xi}\right)^2 - \alpha^4\frac{\partial^6 u}{\partial \xi^6} = 0,$$

where c is wave velocity which moves a long the direction of horizontal axis and α is the wave number. By integrating equation (3.2) twice with respect to the variable ξ and assuming a zero constants of integration, we have

$$(c^2-1)U - 15\alpha^2 U\frac{\partial^2 u}{\partial \xi^2} - 15U^3 - \alpha^4\frac{\partial^4 U}{\partial \xi^4} = 0. \tag{3.3}$$

The homogenous balance between U^3 and $\frac{\partial^4 U}{\partial \xi^4}$ in equation (3.3) implies $n = 2$. Suppose that solution of equation (3.3) is of the following form

$$U(\xi) = A_0 + A_1F(\xi) + A_2F^2(\xi) + \frac{B_1}{F(\xi)} + \frac{B_2}{F^2(\xi)} \tag{3.4}$$

where A_0, A_1, A_2, B_1 and B_2 are constants to be determined. Substituting equation (3.4) along with (2.5) into (3.3) and then setting all the coefficients of $F^k(\xi)$, $(k = -6, -5, \ldots 5, 6)$ of the resulting systems to zero, we can obtain a system of algebraic equations for c, α, h_0, h_1, h_2, A_0, A_1, A_2, B_1 and B_2 as follows

$$\begin{aligned} F^6 &: (-90a^2A_2^2h_2^2 - 15A_2^3 - 120a^4A_2h_2^4) = 0,\\ F^k &: f_k(c, \alpha, h_0, h_1, h_2, A_0, A_1, A_2, B_1, B_2) = 0,\\ F^{-6} &: (-90a^2B_2^2h_0^2 - 15B_2^3 - 120a^4B_2h_0^4) = 0. \end{aligned}$$

where f_k $(k = 0, \pm1, \pm2, \pm3, \pm4, \pm5,)$ are nonlinear function. Choose $h_1 = 0$ and $h_2 = -h_0$, solving the obtained algebraic equations by use of Mathematica 10.1, we obtain

$$A_0 = 2(\alpha h_0)^2, A_1 = 0, A_2 = -2(\alpha h_0)^2, B_1 = 0, B_2 = 0, \tag{3.5}$$

where $c = \sqrt{1 + 16(\alpha h_0)^4}$ and

$$A_0 = (\alpha h_0)^2\left(1 - \frac{1}{15}\sqrt{105}\right), A_1 = 0, A_2 = -2(\alpha h_0)^2, \tag{3.6}$$

$$B_1 = 0, B_2 = 0,$$

where $c = \sqrt{1 + 2\alpha^2h_0^4(11 + \sqrt{105})}$.

Substituting equation (3.5) and equation (3.6) into equation (3.4), we obtain two exact solutions for equation (3.1) in the form

$$u(x,t) = 2(\alpha h_0)^2\left(1 - \tanh^2(h_0\alpha(x - \sqrt{1 + 16(\alpha h_0)^4}t))\right) \tag{3.7}$$

and

$$u(x,t) = (\alpha h_0)^2\left(1 - \frac{1}{15}\sqrt{105}\right)$$

$$-2(\alpha h_0)^2 \tanh^2\left(h_0\alpha\left(x-\sqrt{1+2\alpha^2h_0^4(11+\sqrt{105})}t\right)\right). \tag{3.8}$$

Next, choose $h_2 = 0$ and $h_1 = -h_0$, solving the obtained algebraic equations by use of Mathematica 10.1, we obtain

$$A_0 = 0,\ A_1 = 0,\ A_2 = 0,\ B_1 = 2(\alpha h_0)^2,\ B_2 = -2(\alpha h_0)^2, \tag{3.9}$$

where $c = \sqrt{1+(\alpha h_0)^4}$ and

$$A_0 = (\alpha h_0)^2\left(-\frac{1}{4}+\frac{1}{60}\sqrt{105}\right),\ A_1 = 0,\ A_2 = 0, \tag{3.10}$$

$$B_1 = 2(\alpha h_0)^2,\ B_2 = -2(\alpha h_0)^2,$$

where $c = \sqrt{1+\frac{1}{8}\alpha^4h_0^4(11-\sqrt{105})}$.

Substituting equation (3.5) and equation (3.6) into equation (3.4), we obtain two exact solutions for equation (3.1) in the form

$$\begin{aligned} u(x,t) &= \frac{2(\alpha h_0)^2}{1-\frac{1}{h_0}\exp\left(-h_0\alpha(x-\sqrt{1+(\alpha h_0)^4}t)\right)} \\ &\quad - \frac{2(\alpha h_0)^2}{\left(1-\frac{1}{h_0}\exp\left(-h_0\alpha(x-\sqrt{1+(\alpha h_0)^4}t)\right)\right)^2} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} u(x,t) &= (\alpha h_0)^2\left(-\frac{1}{4}+\frac{1}{60}\sqrt{105}\right) \\ &\quad + \frac{2(\alpha h_0)^2}{1-\frac{1}{h_0}\exp\left(-h_0\alpha(x-\sqrt{1+\frac{1}{8}\alpha^4h_0^4(11-\sqrt{105})}t)\right)} \\ &\quad - \frac{2(\alpha h_0)^2}{\left(1-\frac{1}{h_0}\exp\left(-h_0\alpha(x-\sqrt{1+\frac{1}{8}\alpha^4h_0^4(11-\sqrt{105})}t)\right)\right)^2} \end{aligned} \tag{3.12}$$

The correctness of the solution is verified by substituting them into original equation (3.1). Our solutions in (3.7), (3.8), (3.11), and (3.12) are coincided with the published results for a special case which are gained by Hosseini et.al. [4]. For direct-viewing analysis, we provide the figures of $u(x,t)$ in equation (3.7) and (3.11), where we choose $\alpha = 1$ and $h_0 = -1$.

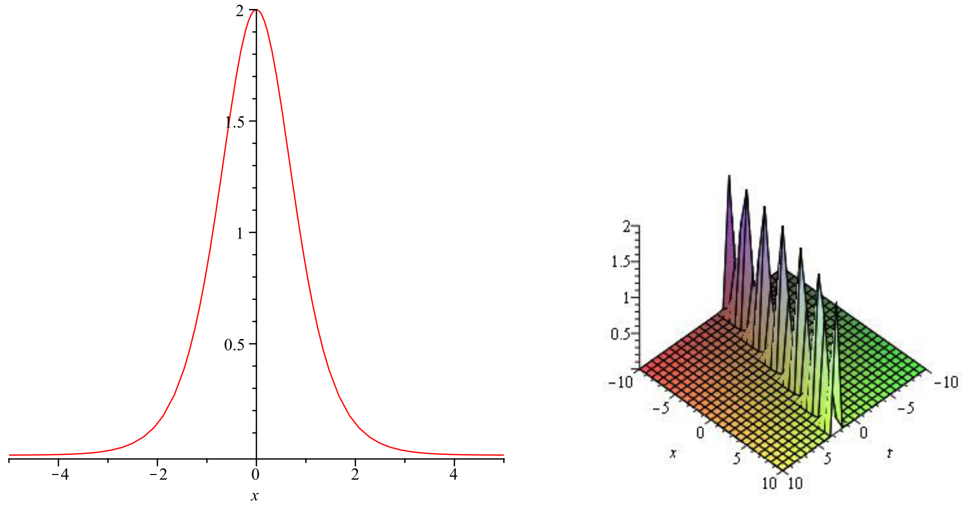

Figure 1: The graph of function (3.7) which was derived from equation (3.1) (right), and its profile graph for t_0.

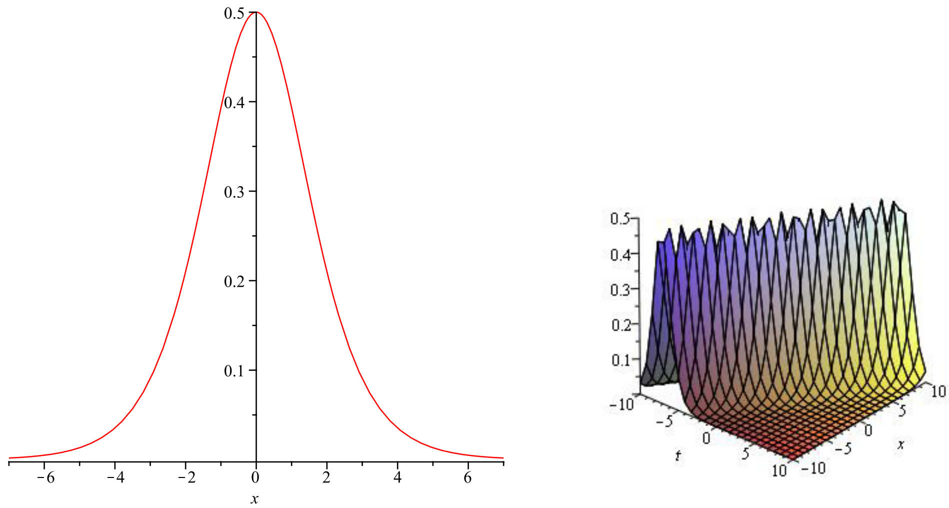

Figure 2: The graph of function (3.11) which was derived from equation (3.1) (right), and its profile graph for t_0.

4. Conclusions

In this paper, modified F-expansion method has been successfully applied to find the some exact traveling wave solutions of the sixth-order Boussinesq equation. The method is more effective and simple than other methods and a number of solutions can be obtained at the same time. The obtained solutions are presented through the hyperbolic functions and the exponential functions. The obtained results are verified by putting them back into the original equation. Some of our solutions are in good agreement with already published results for a special case. This study shows that the modified F-expansion method is quite efficient and practically well suited for use in finding exact solutions for the nonlinear evolution equation in mathematical physics and engineering sciences.

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