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Functions fixed by a weighted Berezin transform

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Abstract

If f is $\mathcal{M}$-harmonic and integrable with respect to a weighted radial measure ν_α over the unit ball B_n of $\mathbb{C}^n$, then $\int_{B_n} (f \circ \psi)\, d\nu_\alpha = f(\psi(0))$ for every $\psi \in \mathrm{Aut}(B_n)$. Equivalently f is fixed by the weighted Berezin transform; $T_\alpha f = f$. In this paper, we find the necessary and sufficient condition such that the equation $T_\alpha f = f$ has only $\mathcal{M}$-harmonic solutions in $L^1(B_n, \nu_\alpha)$.

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1. Introduction

Let B_n be the unit ball of $\mathbb{C}^n$ with norm $|z| = \langle z, z\rangle^{1/2}$ where $\langle\, ,\, \rangle$ is the Hermitian inner product, and let ν be the Lebesgue measure on $\mathbb{C}^n$ normalized to $\nu(B_n) = 1$. When $\alpha > -1$, we define a finite measute ν_α on B_n by $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, where $c_\alpha = \dfrac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$ is a normalizing constant so that $\nu_\alpha(B_n) = 1$.

For such α and $f \in L^1(B_n, \nu_\alpha)$, the weighted Berezin transform $T_\alpha f$ on B_n is defined by,

$$(T_\alpha f)(z) = \int_{B_n} f\big(\varphi_z(w)\big)\, d\nu_\alpha(w) \quad \text{for } z \in B_n,$$

where $\varphi_a \in \mathrm{Aut}(B_n)$ is the canonical automorphism given by

$$\varphi_a(z) = \frac{a - Pz - (1 - |a|^2)^{1/2} Qz}{1 - \langle z, a\rangle}$$

where P is the projection into the space spanned by $a \in B_n$ and $Q_z = z - Pz$. Equivalently we can write

$$(T_\alpha f)(z) = \int_{B_n} f(w) \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2n+2+2\alpha}} \, d\nu_\alpha(w). \tag{1.1}$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B_n)$ by $(\tilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0)$. The $\mathcal{M}$-harmonic functions in B_n are those for which $\tilde{\Delta} f = 0$.

If a function $f \in L^1(B_n, \nu_\alpha)$ is $\mathcal{M}$-harmonic, then $f \circ \psi$ is also $\mathcal{M}$-harmonic for every $\psi \in \mathrm{Aut}(B_n)$. Thus for $\alpha > -1$, f satisfies an invariant mean value property

$$\int_{B_n} (f \circ \psi) \, d\nu_\alpha = f(\psi(0)) \quad \text{for every } \psi \in \mathrm{Aut}(B_n),$$

which is, by an elementary calculation, equivalent to saying that $(T_\alpha f)(z) = f(z)$ for every $z \in B_n$.

Conversely, Furstenberg [5, 6] proved that $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $\mathcal{M}$-harmonic. Indeed, his result says much more is true: On any dimensional symmetric domain, a bounded function which is invariant under a weighted Berezin transform is harmonic with respect to the intrinsic metric. Later in the ’90s, Benyamini and Weit [2], Engils [3] provided different proofs of Furstenberg’s result in the unit disc in $\mathbb{C}$ using a commutative Banach algebra. In their papers, they used the spectral theorem of Katznelson and Tzafriri [10] and theory of the Gelfund transform on the commutative Banach algebra $L^1_R(D, \tau)$ (also see [7], [8]).

Then in 1993, Ahern, Flores and Rudin [1] used entirely analytic mathods to prove that on B_n, an integrable function f with $T_0 f = f$ has to be $\mathcal{M}$-harmonic if and only if $n \le 11$. They also introduced the equation

$$g(x) = (1-x)^{n+1} \int_0^1 \frac{n+tx}{(1+tx)^{n+2}} g(t) t^{n-1} \, dt. \tag{1.2}$$

The equation (1.2) has only constant solutions in $L^1([0,1])$ if and only if $T_0 f = f$ has only $\mathcal{M}$-harmonic solutions in $L^1(B_n, \nu)$. And then they suggested the following open problem:
At which point between 11 and 12 does the above mentioned uniqueness fail?.

In his 1995 Ph.D thesis, Yi [12] introduced the critical point $\rho = 11 + \epsilon_0$, $0 < \epsilon_0 < 1$ such that the equation (1.2) has only constant solutions in $L^1([0,1])$ if and only if $n \le \rho$. And then Jevtic [9] introduced the equation

$$g(x) = (1-x)^\gamma \frac{\gamma}{2} \int_0^1 \frac{1+tx}{(1+tx)^{\gamma+1}} g(t) t^{\gamma/2-1} \, dt, \tag{1.3}$$

which arised naturally in the study of the the invariant-mean value properties of hyperbolically-harmonic functions. The main result of [9] is that the constants are the only solution of (2.2) if and only if $2 \le \gamma \le 12 + \epsilon_0$.

Here, in this paper we study functions on B_n which are invariant under the weighted Berezin transform. Indeed we use the $\rho = 11 + \epsilon_0$ introduced in [12] to prove the main result of this paper.

Theorem 1.1. For $\alpha > -1$, the equation $T_\alpha f = f$ has only $\mathcal{M}-$ harmonic solutions in $L^1(B_n, \nu_\alpha)$ if and only if $n + 2\alpha \leq 11 + \epsilon_0$.

In section 2, we prove Theorem 1.1 by using ideas and methods similar to those in [1].

2. Proof of theorem 1.1

In this section, we use and extend the idea of [1] to prove our Theorem 1.1.

Proof [Proof of Theorem 1.1]. We'll first prove the following statement.

Claim (i): $T_\alpha f = f$ has non-zero solutions satisfying $f \in L^1(\nu_\alpha)$ and $\tilde{\Delta} f = \mu f$ for some $\mu \neq 0$ if and only if $n + 2\alpha > 11 + \epsilon_0$.

To prove the claim, we let

$$X_\mu = \{f \in C(B_n) : \tilde{\Delta} f = \mu f\}$$

and let $\mu = 4\beta(n - \beta)$. Then by the same calculation as Proposition 3.2 of [1], we get $L^1(\nu_\alpha) \cap X_\mu \neq \{0\}$ if and only if $-(\alpha + 1) < \Re\beta < n + (\alpha + 1)$. Now we denote

$$\Sigma_\alpha = \{\beta \in \mathbb{C} : -(\alpha + 1) < \Re\beta < n + (\alpha + 1)\}.$$

If $f \in L^1(\nu_\alpha) \cap X_\mu$, then by 4.2.4 of [1]

$$\int_S f(\varphi_z(r\zeta))d\sigma(\zeta) = f(z) \int_S \frac{(1 - r^2)^\beta}{|1 - \langle r, \zeta \rangle|^{2\beta}} d\sigma(\zeta).$$

Now multiplying by $2nc_\alpha r^{2n-1}(1 - r^2)^\alpha$ and integrate over $0 \leq r < 1$, we get

$$\int_{B_n} f\big(\varphi_z(w)\big)\, d\nu_\alpha(w) = f(z) \int_{B_n} c_\alpha (1 - |w|^2)^\alpha \int_S \frac{(1 - |w|^2)^\beta}{|1 - \langle w, \zeta \rangle|^{2\beta}} d\sigma(\zeta) d\nu(w). \tag{2.1}$$

Then, using the formula $(1-x)^{-\beta} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)} x^k$ for $|x| < 1$ to get

$$\begin{aligned}
\int_S \frac{1}{|1-\langle w,\zeta\rangle|^{2\beta}} d\sigma(\zeta) &= \int_S \left| \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)} \langle w,\zeta\rangle^k \right|^2 d\sigma(\zeta) \\
&= \sum_{k=0}^{\infty} \left| \frac{\Gamma(k+\beta)}{k!\Gamma(\beta)} \right|^2 \int_S |\langle w,\zeta\rangle|^2 k d\sigma(\zeta) \\
&= \frac{\Gamma(n)}{\Gamma^2(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\beta)}{\Gamma(k+1)\Gamma(k+n)} |w|^{2k}. \qquad (2.2)
\end{aligned}$$

Thus the integral in the right hand side of (2.1) is

$$\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{B_n} (1-|w|^2)^{\alpha+\beta} \frac{\Gamma(n)}{\Gamma^2(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\beta)}{\Gamma(k+1)\Gamma(k+n)} |w|^{2k} d\nu(w).$$

Integrating in polar coordinates and then putting $x = r^2$, it becomes

$$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma^2(\beta)} \int_0^1 \sum_{k=0}^{\infty} \frac{\Gamma^2(k+\beta)x^k}{k!(k+n-1)!} x^{n-1}(1-x)^{\alpha+\beta} dx.$$

By using [4, p.78, 2.4 (2)] with $a = b = \beta$ and $s = n, c = \alpha+\beta+n+1, 0 < t < 1$, we get

$$F(\beta,\beta,\alpha+\beta+n+1;t) = \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(n)\Gamma(\alpha+\beta+1)} \int_0^1 F(\beta,\beta,n;tx)x^{n-1}(1-x)^{\alpha+\beta} dx$$

Let $r \to 1$ and compare to the above integral (2.1), then using the formula

$$F(\beta,\beta,\alpha+\beta+n+1;1) = \frac{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha-\beta+n+1)}{\Gamma^2(\alpha+n+1)},$$

we get

$$T_\alpha f(z) = \frac{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\alpha+1)} f(z).$$

Now we'll prove Claim (i) by showing that there is no $\beta \in \Sigma_\alpha \backslash \{0, n\}$ satisfying

$$\frac{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\alpha+1)} = 1$$

if and only if $n + 2\alpha \le 11 + \epsilon_0$. Here let us denote $\Psi_{n,\alpha}$, Φ_γ for $\alpha > -1$ and $\gamma \ge 1$,

$$\Psi_{n,\alpha}(\beta) = \frac{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\alpha+1)}$$

and

$$\Phi_\gamma(\beta) = \frac{\Gamma(\beta+1)\Gamma(\gamma+1-\beta)}{\Gamma(\gamma+1)}.$$

From [12, Theorem 4.5] and [9, Theorem 5.7], the function $\Phi_\gamma - 1$ has no zeros in

$$\{\beta \in \mathbb{C} : -1 < \Re\beta < \gamma + 1\}\backslash\{0, \gamma\} \text{ if and only if } \gamma \leq \rho,$$

where

$$\rho = \inf \{\gamma > 0 : \Phi_\gamma(-1 - it) \text{ is real and } > 1 \text{ for some } t > 0\}.$$

Here $\rho = 11 + \epsilon_0$, where $0.25 < \epsilon_0 < 0.69$. Now, by an easy calculation, we get

$$\Psi_{n,\alpha}(-1-\alpha-it) = \Phi_{n+2\alpha}(-1-it)\,\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)\Gamma(\alpha+1)}.$$

Therefore, the condition of n and α for which the curve $C(t) = \Psi_{n,\alpha}(-1-\alpha-it)$ crosses the real axis to the right of the point 1 is that $n + 2\alpha \leq \rho = 11 + \epsilon_0$. Thus $\Psi_{n,\alpha}(\beta) - 1$ has no zeros in $\Sigma_\alpha\backslash\{0, n\}$ if and only if $n + 2\alpha > 11 + \epsilon_0$ and this proves the Claim (i).

To complete the proof of the theorem, we suppose that $n + 2\alpha \leq 11 + \epsilon_0$ and follow the methods of [1]. Let

$$M_\alpha = \{f \in L^1(\nu_\alpha) : T_\alpha f = f\}$$

and let $\tilde{\Delta}_\alpha$ be the restriction of $\tilde{\Delta}$ to M_α. Then $\tilde{\Delta}_\alpha$ is a bounded linear operator on the Banach space M_α and $G_{n,\alpha}(\tilde{\Delta}_\alpha)$ is an identity operator on M_α, where

$$G_{n,\alpha}(z) = \prod_{j=1}^{\infty}\left(1 - \frac{z}{4(j+\alpha)(n+j+\alpha)}\right)$$

is an entire function satisfying $G_{n,\alpha}(0) = 0$ and $G'_{n,\alpha}(0) \neq 0$. Now if we define

$$H_{n,\alpha}(z) = \frac{G_{n,\alpha}(z) - 1}{z},$$

then $H_{n,\alpha}$ is an entire function with $H_{n,\alpha}(0) \neq 0$. Thus, for every $f \in M_\alpha$, we have $H_{n,\alpha}(\tilde{\Delta}_\alpha)\tilde{\Delta}_\alpha f = 0$. Since the point spectrum of $\tilde{\Delta}_\alpha$ is $\sigma_p(\tilde{\Delta}_\alpha) = \{0\}$, by the spectral mapping theorem we have $0 \notin H_{n,\alpha}(\sigma_p(\tilde{\Delta}_\alpha)) = \sigma_p(H_{n,\alpha}(\tilde{\Delta}_\alpha))$, which means that $H_{n,\alpha}(\tilde{\Delta}_\alpha)$ is $1-1$. Since $\tilde{\Delta}_\alpha$ belongs to the null-space of $H_{n,\alpha}(\tilde{\Delta}_\alpha)$, we get $\tilde{\Delta}_\alpha = 0$. Therefore, every $f \in M_\alpha$ is $\mathcal{M}$-harmonic and this completes the proof of Theorem 1.1. ∎

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