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Trigonometrically-fitted explicit four-stage fourth-order Runge–Kutta–Nyström method for the solution of initial value problems with oscillatory behavior

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Abstract

An explicit trigonometrically-fitted Runge–Kutta–Nyström (ETFRKN) method is constructed in this paper based on Simos technique, which exactly integrates initial-value problems whose solutions are linear combinations of functions of the form e^{iwx} and e^{-iwx} or equivalently $\sin(wx)$ and $\cos(wx)$ with $w > 0$ the principal frequency of the problem. The numerical results show the efficacy of the new method in comparison with other existing methods.

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1. Introduction

In the last decades, methods for the numerical solution of the initial value problems

$$y'' = f(x, y),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1.1}$$

whose solution shows a pronounced oscillatory behavior has attracted the interest of many researchers. Such problems occur in several fields of applied sciences such as: molecular dynamics, celestial mechanics, theoretical physics, physical chemistry and electronics. Bettis in [1] proposed the first Runge–Kutta (RK) methods with 3 and 4 stages for the solution of Ordinary Differential Equations (ODEs) with periodic solutions. More recently, monovasilis in [2], Franco in [3], Ambrosio in [4], Ramos in [5] proposed Runge–Kutta–Nyström (RKN) methods for the solutions of second order ordinary differential equations. Similarly, Simos in [6], Coleman in [7] constructed an explicit RK method which solves certain first order initial value problems with oscillatory solutions. In the same way, Berghe in [8] proposed some exponentially-fitted RK (EFRK) methods that integrates exactly first order systems whose solutions can be represented as the linear combination of some functions. Motivated by the work of Simos in [9], we construct a new accurate fourth order four stage explicit trigonometrically-fitted RKN method. The constructed method can exactly integrate the test equation $y^{''} = -w^2 y$ and the numerical results show the efficiency of the new method. The remaining part of this paper is designed as follows: Section 2 deals with the derivation of the proposed method. In section 3 we analyze the local truncation error of the new method. In section 4 we present the numerical results and the last section deals with the conclusion.

2. Derivation of the proposed method

In this section, we will derive four-stage fourth order explicit trigonometrically-fitted RKN method using Simos technique.

Consider the four-stage explicit RKN method given by:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{4} b_i f(x_n + c_i h, Y_i), \tag{2.1}$$

$$y'_{n+1} = y'_n + h \sum_{i=1}^{4} d_i f(x_n + cih, Y_i), \tag{2.2}$$

$$Y_i = y_n + c_i hy'_n + h^2 \sum_{j=1}^{3} a_{ij} f(x_n + cih, Y_j). \tag{2.3}$$

or in Butcher Tableau:

In this study, the four-stage fourth order dispersive of order eight RKN method will be considered as given in [10]. The coefficients of the method are given in Table 1 below:

0				
c_2	a_{21}			
c_3	a_{31}	a_{32}		
c_4	a_{41}	a_{42}	a_{43}	
	b_1	b_2	b_3	b_4
	d_1	d_2	d_3	d_4

Table 1: The RKN4(4,8,5)M Method [10]

0				
$\frac{1}{4}$	$\frac{1}{32}$			
$\frac{7}{10}$	$\frac{19}{600}$	$\frac{16}{75}$		
1	$\frac{32}{315}$	$\frac{58}{315}$	$\frac{3}{14}$	
	$\frac{1}{21}$	$\frac{28}{81}$	$\frac{50}{567}$	$\frac{1}{54}$
	$\frac{1}{14}$	$\frac{32}{81}$	$\frac{250}{567}$	$\frac{5}{54}$

Applying an explicit Runge-Kutta-Nyström method (2.1)–(2.3) to $y^{''} = -w^2 y$, we obtained:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{i=1}^{4} b_i(-w^2 Y_i), \tag{2.4}$$

and

$$y^{'}_{n+1} = y'_n + h \sum_{i=1}^{4} d_i(-w^2 Y_i). \tag{2.5}$$

where

$$Y_1 = y_n + c_1 h y_n^{'}, \tag{2.6}$$

$$Y_2 = y_n + c_2 h y_n^{'} - h^2 a_{21} w^2 Y_1, \tag{2.7}$$

$$Y_3 = y_n + c_3 h y_n^{'} + h^2(-a_{31} w^2 Y_1 - a_{32} w^2 Y_2), \tag{2.8}$$

$$Y_4 = y_n + c_4 h y_n^{'} + h^2(-a_{41} w^2 Y_1 - a_{42} w^2 Y_2 - a_{43} w^2 Y_3). \tag{2.9}$$

Now, let $y_n = e^{Iwx}$, computing the value of y_{n+1}, y_n' and y_{n+1}' and substituting in the equations (2.4)–(2.9) and by using $e^{Iv} = \cos(v) + I\sin(v)$ and comparing the real and imaginary part, we get the following system of equations:

$$\cos(v) = 1 - v^2 \sum_{i=1}^{4} b_i (1 - v^2 \sum_{j=1}^{3} a_{ij} Y_j e^{-Iwx}), \tag{2.10}$$

$$\sin(v) = v - v^2 \sum_{i=1}^{4} b_i c_i v, \tag{2.11}$$

$$\sin(v) = v \sum_{i=1}^{4} d_i (1 - v^2 \sum_{j=1}^{3} a_{ij} Y_j e^{-Iwx}), \tag{2.12}$$

$$\cos(v) = 1 - v^2 \sum_{i=1}^{4} d_i c_i. \tag{2.13}$$

where $v = wh$.

Solving (2.10)–(2.13) using the coefficients of the method in Table 1 for a_{31}, b_2, c_2, c_3, we obtain the solution as given in (2.14):

$$
\begin{aligned}
a_{31} &= \frac{272160\sin(v) - 2097v^5 + 36v^7 + 41560v^3 - 272160v}{600v^3(9v^2 - 200)}, \\
b_2 &= \frac{16(2v^4 - 37v^2 + 162v\sin(v) - 810 + 810\cos(v))}{405v^2(v^2 - 32)}, \\
c_2 &= \frac{3(\cos(v) + 4)(15v^3 + v^3\sin(v)^2 + 5\cos(v)\sin(v)v^2 - 480v - 32v\sin(v)^2)}{16(\sin(v)^2 + 15)v(v^2 + 6v\sin(v) + 30\cos(v) - 30)} \\
&\quad - \frac{3(160\cos(v)\sin(v) - 20\sin(v)v^2 + 640\sin(v))}{16(\sin(v)^2 + 15)v(v^2 + 6v\sin(v) + 30\cos(v) - 30)},
\end{aligned}
$$

$$
\begin{aligned}
c_3 \quad = \quad & \frac{3(90720v\sin(v)^3 + 21280v^2\sin(v)^2 - 699v^6\sin(v)^2 + 453600\sin(v)^2\cos(v))}{100(\sin(v)^2+15)(v^2+6v\sin(v)+30\cos(v)-30)v^2(9v^2-200)} \\
& + \frac{3(12v^8\sin(v)^2 - 2721600\sin(v)^2 + 11200v^4\sin(v)^2 + 13980v^5\sin(v) - 190400\sin(v)v^3)}{100(\sin(v)^2+15)(v^2+6v\sin(v)+30\cos(v)-30)v^2(9v^2-200)} \\
& + \frac{3(263200v\sin(v) - 3495\cos(v)v^5\sin(v) + 47600\cos(v)\sin(v)v^3 - 240v^7\sin(v))}{100(\sin(v)^2+15)(v^2+6v\sin(v)+30\cos(v)-30)v^2(9v^2-200)} \\
& + \frac{3(274400\cos(v)\sin(v)v + 60\cos(v)v^7\sin(v) - 1400v^4\cos(v) + 173600v^4 + 4536000)}{100(\sin(v)^2+15)(v^2+6v\sin(v)+30\cos(v)-30)v^2(9v^2-200)} \\
& - \frac{3(10485v^6 + 180v^8 - 823200v^2 - 4536000\cos(v) + 285600v^2\cos(v))(\cos(v)+4)}{100(\sin(v)^2+15)(v^2+6v\sin(v)+30\cos(v)-30)v^2(9v^2-200)}. \qquad (2.14)
\end{aligned}
$$

Which lead to our new method ETFRKN4(4,8,5)M.

The corresponding Taylor series expansion of the solution is given in (2.15).

$$
\begin{aligned}
a_{31} = & \frac{19}{600} + \frac{3}{20000}v^4 + \frac{1}{2000000}v^6 + \frac{349}{4400000000}v^8 \\
& + \frac{109999}{34320000000000}v^{10} + \frac{1618447}{11088000000000000}v^{12} \\
& + \frac{292360553}{44553600000000000000}v^{14} + \cdots, \\
b_2 = & \frac{28}{81} - \frac{1}{3600}v^4 + \frac{1}{161280}v^6 - \frac{19}{232243200}v^8 + \frac{449}{1226244096000}v^{10} \\
& - \frac{32869}{3570822807552000}v^{12} - \frac{312811}{17139949476249600000}v^{14} + \cdots, \\
c_2 = & \frac{1}{4} - \frac{1}{26880}v^4 + \frac{19}{7741440}v^6 - \frac{1019}{13624934400}v^8 + \frac{481093}{119027426918400}v^{10} \\
& - \frac{8198117}{57133164920832000}v^{12} + \frac{647998097}{93241325150797824000}v^{14} + \cdots, \\
c_3 = & \frac{7}{10} + \frac{47}{60000}v^4 - \frac{1759}{54000000}v^6 + \frac{335651}{1663200000000}v^8 - \frac{2163024121}{51891840000000000}v^{10} \\
& + \frac{1913285557}{124540416000000000000}v^{12} + \cdots \qquad (2.15)
\end{aligned}
$$

As $v \to 0$, the coefficients a_{31}, b_2, c_2 and c_3 of the new method (ETFRKN4(4,8,5)M) reduces to the coefficients of the original method (RKN4(4,8,5)M). That is to say $a_{31}(0), b_2(0), c_2(0)$ and $c_3(0)$ are identical to a_{31}, b_2, c_2 and c_3 of RKN4(4,8,5)M method.

3. Algebraic Order and Error Analysis

In this section, we are going to compute the local truncation error(LTE) of the new method and verify its algebraic order. We start by computing the Taylor series expansions of the theoretical solution $y(x_n+h)$, of its derivative $y^{'}(x_n+h)$, the approximate solution y_{n+1} and of its derivative $y^{'}_{n+1}$. Then, we compute the LTE of y and that of its derivative as given below:

$$\begin{aligned} LTE &= y_{n+1} - y(x_n+h), \\ LTE_{der} &= y^{'}_{n+1} - y^{'}(x_n+h). \end{aligned} \tag{3.1}$$

The LTE and LTE_{der} of the method ETFRKN4(4,8,5)M are given below:

$$\begin{aligned} LTE = &-\frac{h^5}{1440}(3y^{'}y^{''}f_{yy} + 3y^{'}f_{yxx} + (y^{'})^3 f_{yyy} + 3y^{''}f_{xy} \\ &+ 3(y^{'})^2 f_{xyy} + f_{xxx}) + O(h^6), \\ LTE_{der} = &\frac{h^5}{120}(6(y^{'})^2 y^{''} f_{yyy} + 5(y^{'})^2 f_{yy} f_y + 12y^{'}y^{''} f_{xyy} \\ &+ 6y^{'} f_{xy} f_y + 4y^{'} f_{yy} f_x + 4(y^{'})^3 f_{xyyy} \\ &+ 6y^{''} f_{yxx} + (f_y)^2 y^{''} + 4y^{'} f_{xxxy} + f_y f_{xx} + 4 f_{xy} f_x \\ &+ 6(y^{'})^2 f_{xxyy} + (y^{'})^4 f_{yyyy} \\ &+ 3(y^{''})^2 f_{yy} + f_{xxxx}) + O(h^6). \end{aligned} \tag{3.2}$$

From equation (3.2), we can see that the order of ETFRKN4(4,8,5)M is 4 because all of the coefficients up to h^4 vanished.

4. Problems Tested and Numerical Results

In this section, we will apply the new method to some second-order ordinary differential equation problems. The following RKN and RK method are used for the numerical comparisons.

- ETFRKN4(4,8,5)M: The new method derived in this paper,
- RKN4G: The fourth-order RKN method obtained by Garcia et al in [16],
- RKN4-4: An explicit RKN method given by Xinyuan Wu et al in [12],

Table 2: Numerical results for problem 1.

h	**Methods**	**T = 100**	**T = 1000**	**T = 10000**
0.025	RKN4(4,8,5)M	6.972504(−6)	6.991171(−5)	6.999528(−4)
	ETFRKN4(4,8,5)M	4.589344(−11)	5.240339(−9)	6.214849(−7)
	RKN4G	2.060889(−3)	2.060542(−3)	2.041889(−1)
	RKN4(4,8,5)S	1.268534(−6)	8.952827(−6)	8.771342(−5)
	RKN4-4	4.126979(−3)	4.117858(−2)	4.011603(−1)
	RK4	8.723902(−2)	8.701857(−1)	7.692638(+0)
0.05	RKN4(4,8,5)M	2.244837(−4)	2.255854(−3)	2.281038(−2)
	ETFRKN4(4,8,5)M	2.903877(−11)	2.974295(−9)	1.733919(−7)
	RKN4G	3.281656(−2)	3.202012(−1)	1.631807(+0)
	RKN 4(4,8,5)S	4.113511(−5)	3.805704(−4)	3.781592(−3)
	RKN4-4	6.576666(−2)	6.081756(−1)	1.556586(+0)
	RK4	1.349974(+0)	9.652030(+0)	1.125925(+0)
0.075	RKN4(4,8,5)M	1.722928(−3)	1.746987(−2)	1.888961(−1)
	ETFRKN4(4,8,5)M	1.719080(−11)	1.955921(−9)	1.158436(−7)
	RKN4G	1.628263(−1)	1.232240(+0)	1.491109(+0)
	RKN 4(4,8,5)S	4.189894(−4)	4.147285(−3)	4.076388(−2)
	RKN4-4	3.194580(−1)	1.405130(+0)	1.405130(+0)
	RK4	5.671354(+0)	1.001198(+1)	1.001198(+1)
0.1	RKN4(4,8,5)M	7.486879(−3)	7.699014(−2)	1.088929(+0)
	ETFRKN 4(4,8,5)M	5.150516(−10)	4.419555(−9)	1.313724(−7)
	RKN4G	4.835378(−1)	1.381059(+0)	1.381059(+0)
	RKN4(4,8,5)S	2.468325(−3)	2.503959(−2)	2.264669(−1)
	RKN4-4	8.242491(+0)	9.207004(+0)	9.207001(+0)
	RK4	9.159293(−1)	1.292106(+0)	1.292106(+0)

- RKN4(4,8,5)M: The RKN method with dispersion order eight and dissipation order five obtained by Senu in [10],
- RKN4(4,8,5)S: The RKN method with dispersion order eight and dissipation order five obtained by Senu in [10] and
- RK4: The classical RK method given by Butcher in [11].

Problem 1 Ahmad et al in [13]

$$y'' = -64y,\ y(0) = 1,\ y'(0) = -2$$

The exact solution is

$$y(x) = -\frac{1}{4}\sin(8x) + \cos(8x).$$

Table 3: Numerical results for problem 2

h	**Methods**	**T = 100**	**T = 1000**	**T = 10000**
0.025	RKN4(4,8,5)M	3.639342(-5)	3.660083(-4)	3.671962(-3)
	ETFRKN4(4,8,5)M	2.922861(-9)	1.118849(-8)	1.068178(-6)
	RKN4G	8.600175(-3)	8.609385(-2)	8.230410(-1)
	RKN4(4,8,5)S	1.201895(-5)	5.484566(-5)	4.939859(-4)
	RKN4-4	1.719217(-2)	1.712476(-1)	1.495620(+0)
	RK4	4.582085(-1)	4.424328(+0)	2.167420(+1)
0.05	RKN4(4,8,5)M	1.172498(-3)	1.191337(-2)	1.237884(-1)
	ETFRKN4(4,8,5)M	4.41177(-8)	4.814676(-8)	2.687882(-7)
	RKN4G	1.373487(-1)	2.136231(+0)	2.136231(+0)
	RKN4(5,8,5)S	3.525533(-4)	2.479343(-3)	2.366650(-2)
	RKN4-4	2.694122(-1)	1.846476(+0)	2.023791(+0)
	RK4	6.561813(+0)	1.812164(+1)	1.812164(+1)
0.075	RKN4(4,8,5)M	9.143561(-3)	9.486361(-2)	1.293305(+0)
	ETFRKN4(4,8,5)M	2.451447(-7)	2.472938(-7)	4.375056(-7)
	RKN4G	6.408312(-1)	1.929449(+0)	1.929449(+0)
	RKN 4(4,8,5)S	3.498339(-3)	2.909897(-2)	2.619785(-1)
	RKN4-4	1.123727(+0)	1.806809(+0)	1.806809(+0)
	RK4	1.594005(+1)	1.607897(+1)	1.607897(+1)
0.1	RKN4(4,8,5)M	3.925749(-2)	4.576288(-1)	2.206548(+1)
	ETFRKN 4(4,8,5)M	8.652444(-7)	1.081217(-6)	3.116065(-6)
	RKN4G	1.496559(+0)	1.773207(+0)	1.773207(+0)
	RKN4(4,8,5)S	2.114117(-2)	1.808480(-1)	1.0775279(+0)
	RKN4-4	1.654061(+0)	1.654061(+0)	1.654061(+0)
	RK4	1.492280(+1)	1.492280(+1)	1.492280(+1)

Problem 2 Anastassi and Kosti in [14]

$$y^{''} = -100y + 99\sin(x),\ y(0) = 1,\ y^{'}(0) = 11$$

The exact solution is

$$y(x) = \sin(10x) + \cos(10x) + \sin(x).$$

Problem 3 Moo et al in [17]

$$y_1^{''} = -y_1 + 0.001\cos(x),\ \ y_1(0) = 1,\ \ y_1^{'} = 0,$$
$$y_2^{''} = -y_2 + 0.001\sin(x),\ \ y_2(0) = 0,\ \ y_2^{'} = 0.9995$$

The exact solution is

$$y_1(x) = \cos(x) + 0.0005\, x \cos(x)$$
$$y_2(x) = \sin(x) - 0.0005\, x \sin(x).$$

Table 4: Numerical results for problem 3

h	**Methods**	**T = 100**	**T = 1000**	**T = 10000**
0.025	RKN4(4,8,5)M	3.417711(-11)	9.723635(-10)	1.067510(-7)
	ETFRKN4(4,8,5)M	7.858603(-12)	8.986945(-10)	1.062072(-7)
	RKN4G	6.034852(-8)	3.464664(-6)	1.613212(-5)
	RKN4(4,8,5)S	7.84666(-10)	1.575359(-9)	1.069329(-7)
	RKN4-4	1.206630(-7)	1.254851(-6)	3.255984(-5)
	RK4	4.520694(-7)	4.601692(-6)	4.591613(-5)
0.05	RKN4(4,8,5)M	1.081503(-9)	1.163141(-8)	1.171338(-7)
	ETFRKN4(4,8,5)M	5.258016(-12)	5.02382(-10)	2.977504(-8)
	RKN4G	9.654375(-7)	5.517615(-5)	2.627705(-4)
	RKN4(4,8,5)S	1.261104(-8)	1.381034(-8)	2.867273(-8)
	RKN4-4	1.930491(-6)	2.009134(-5)	5.256414(-4)
	RK4	7.232536(-6)	7.364302(-5)	7.361172(-4)
0.075	RKN4(4,8,5)M	6.317846(-9)	6.792010(-8)	1.781351(-6)
	ETFRKN4(4,8,5)M	1.089411(-9)	1.120594(-8)	1.211640(-7)
	RKN4G	4.886862(-6)	2.794100(-4)	1.330306(-3)
	RKN 4(4,8,5)S	6.369031(-8)	7.033042(-8)	2.461517(-7)
	RKN4-4	9.771535(-6)	1.017168(-4)	2.661288(-3)
	RK4	2.608892(-5)	2.711084(-4)	7.097847(-3)
0.1	RKN4(4,8,5)M	3.460976(-8)	3.721067(-7)	3.739522(-6)
	ETFRKN4(4,8,5)M	1.875833(-12)	2.308980(-10)	2.656270(-8)
	RKN4G	1.544134(-5)	8.831508(-4)	4.205253(-3)
	RKN4(4,8,5)S	2.043120(-7)	2.333710(-7)	5.823003(-7)
	RKN4-4	3.087523(-5)	3.214332(-4)	8.413109(-3)
	RK4	1.157343(-4)	1.1778225(-3)	1.177426(-2)

Problem 4 Senu et al in [15]

$$y'' = -y + x,\ y(0) = 1,\ y'(0) = 2$$

The exact solution is

$$y(x) = \cos(x) + \sin(x) + x.$$

The accuracy strategy used is finding log_{10} of the maximum global error,

$$Accuracy = log_{10}\max \|y(x_n) - y_n\|, \tag{4.1}$$

where $x_n = x_0 + nh$, $n = 1, 2, 3, \ldots (T - x_0)/h$. In this paper, we denote T as the interval used for the integration.

The numerical results are shown in Tables 2-5.

Table 5: Numerical results for problem 4

h	**Methods**	**T = 100**	**T = 1000**	**T = 10000**
0.025	RKN4(4,8,5)M	3.201817(-11)	6.383933(-10)	2.989178(-7)
	ETFRKN4(4,8,5)M	1.880414(-11)	5.408549(-10)	2.979411(-7)
	RKN4G	8.653461(-8)	3.466190(-6)	8.524657(-6)
	RKN4(4,8,5)S	7.805048(-10)	8.739462(-10)	2.91846(-7)
	RKN4-4	1.708399(-7)	1.721004(-6)	1.715370(-5)
	RK4	3.219511(-7)	3.347823(-6)	8.732349(-5)
0.05	RKN4(4,8,5)M	8.483979(-10)	9.229263(-9)	2.664778(-7)
	ETFRKN4(4,8,5)M	2.112705(-10)	1.812355(-9)	6.845243(-8)
	RKN4G	1.384249(-6)	5.523357(-5)	1.380617(-4)
	RKN4(4,8,5)S	1.253343(-8)	1.324117(-8)	7.672772(-8)
	RKN4-4	2.734130(-6)	2.754721(-5)	2.761332(-4)
	RK4	5.151875(-6)	5.357128(-5)	1.401884(-3)
0.075	RKN4(4,8,5)M	8.212211(-9)	8.827681(-8)	8.875049(-7)
	ETFRKN4(4,8,5)M	3.424816(-12)	3.370815(-10)	1.977423(-8)
	RKN4G	7.006304(-6)	2.795714(-4)	6.991754(-4)
	RKN 4(4,8,5)S	6.421809(-8)	7.074550(-8)	1.506032(-7)
	RKN4-4	1.384465(-5)	1.394256(-4)	1.398006(-3)
	RK4	3.663125(-5)	3.727062(-4)	3.725955(-3)
0.1	RKN4(4,8,5)M	2.641282(-8)	2.819148(-7)	7.435934(-6)
	ETFRKN4(4,8,5)M	3.433840(-9)	3.478845(-8)	3.733085(-7)
	RKN4G	2.215882(-5)	8.838602(-4)	2.209373(-3)
	RKN4(4,8,5)S	2.022193(-7)	2.302874(-7)	9.318692(-7)
	RKN4-4	4.377299(-5)	4.409937(-4)	4.417562(-3)
	RK4	8.246080(-5)	8.565301(-4)	2.243462(-2)

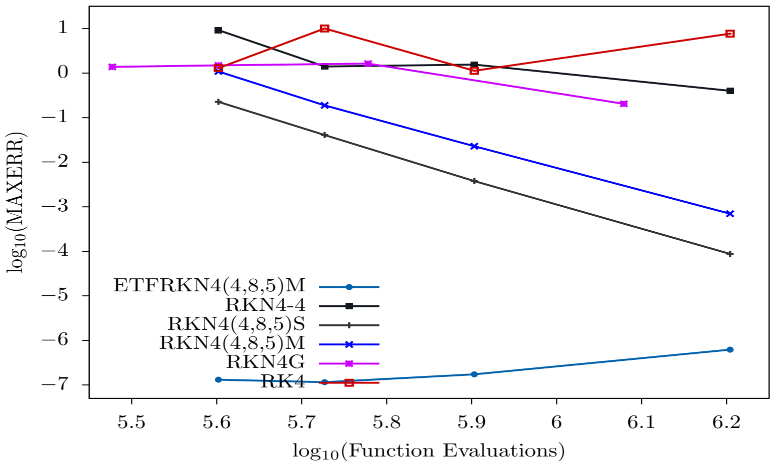

Figure 1: The efficiency curve for Problem 1 with $t_{end} = 10000$ and $h = i(0.025)$, $i = 1, 2, 3, 4$.

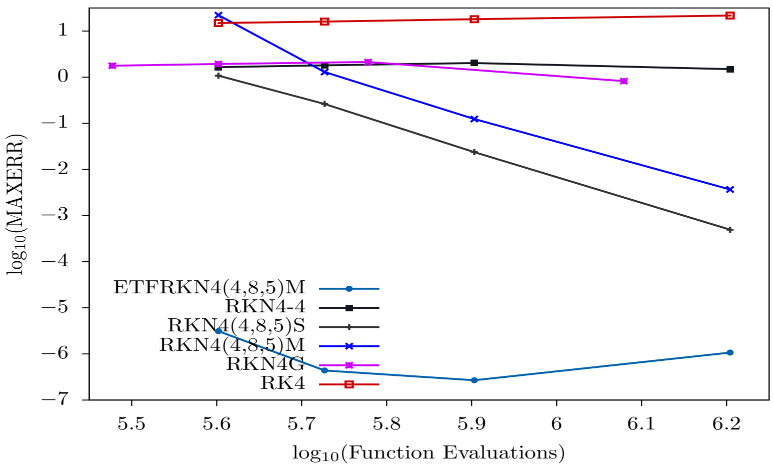

Figure 2: The efficiency curve for Problem 2 with $t_{end} = 10000$ and $h = i(0.025)$, $i = 1, 2, 3, 4$.

We also display the accuracy of these methods graphically in figures 1 to 4.

5. Conclusion

In this study, we have presented the fourth order four-stage explicit trigonometrically-fitted RKN method for the solutions of oscillatory problems. The numerical results obtained show clearly that the global error of the new method is smaller than that of the other existing methods; The new method is much more efficient than the other existing methods.

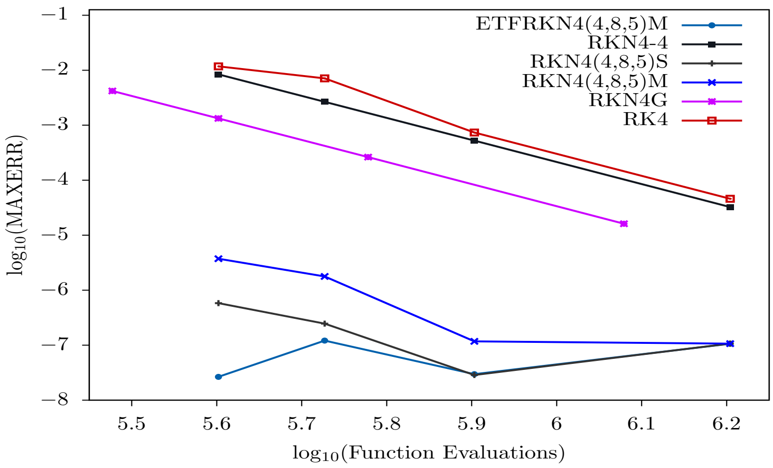

Figure 3: The efficiency curve for Problem 3 with $t_{end} = 10000$ and $h = i(0.025)$, $i = 1, 2, 3, 4$.

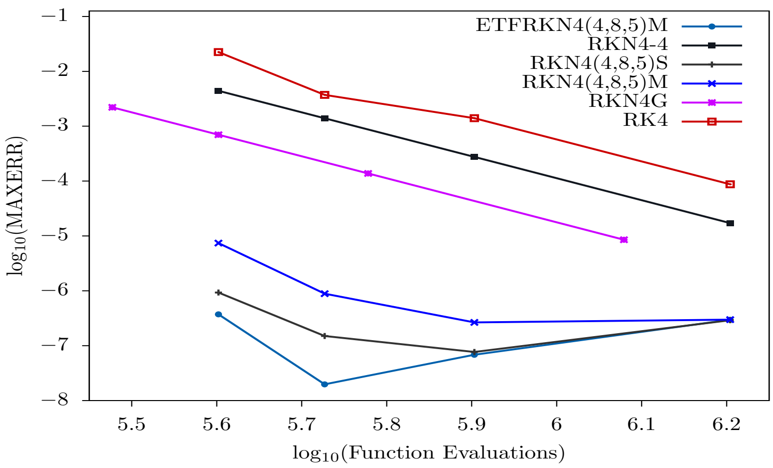

Figure 4: The efficiency curve for Problem 4 with $t_{end} = 10000$ and $h = i(0.025)$, $i = 1, 2, 3, 4$.

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