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WZI rings and strong regularity

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Abstract

The concept of right SSF-ring is introduced in this note as a generalization of right SF-rings, and we focus on the structure and properties in relation with related topics. It is shown that R is a strongly regular ring if and only if R is a WZI right SF-ring if and only if R is a WZI right SSF-ring.

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1. Introduction

Every ring in this article is an associative ring with identity unless otherwise stated. Let $J(R)$ denote the Jacobson radical, $Z_r(R)$ denote the right singular ideal and $E(R)$ denote the set of all idempotent elements of R, respectively. For any nonempty subset S of R, $r(S) = r_R(S)$ and $l(S) = l_R(S)$ denote the set of right annihilators og S in R, respectively. Especially, $S = \{a\}$, then we write $r_R(a)$ in place of $r_R\{(a)\}$.

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Following Cohn [5], a ring R is called *reversible* (or ZC) if $ab = 0$ implies $ba = 0$ for $a, b \in R$; and R is said to be *semicommutative* (or ZI) [10] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$.

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It is easily checked that reduced rings are ZC and ZC rings are ZI. It is well-known that each converse is not true in general. Note that R is a semicommutative ring if and only if $l(a)$ is an ideal of R for any $a \in R$.

Recall that a ring R is *semiprime* if $aRa = 0$ implies $a = 0$ for $a \in R$. Following Birkenmeier et al. [3], a ring R is called *right (left) principally quasi-Baer* if the right (resp., left) annihilator of a principal right (resp., left) ideal of R is generated by an idempotent. A ring R is usually called right (resp., left) *principally projective* if the right (res p., left) annihilator of an element of R is generated by an idempotent. Following Goodearl [8], a ring R is called *(von Neumann) regular* if for every $a \in R$, there exists $b \in R$ such that $a = aba$. Following Goodearl [8], a ring R is called *strongly regular* if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$. It is shown in [8, Theorem 3.2] that a ring R is a strongly regular ring if and only if R is a reduced regular ring.

According to Ramamurthi [12], a ring R is called a *left (resp., right) SF-ring* if each simple left (resp., right) R-module is flat. It is well-known that regular rings are left and right SF-rings. And R is strongly regular if and only if R is a reduced right SF-ring. Ramamurthi in [12] initiated the study of left (right) SF-rings and of the question whether a left (right) SF-ring is necessarily regular.

Following Mahmood [11], a ring R is called a *left (resp., right) SSF-ring* if every simple singular left (resp., right) R-module is flat. In [7], Goodearl showed that regular rings are always right (left) SF-rings. While, Rege [13] proved that reduced right (left) SF rings are strongly regular. Rings, whose simple singular left (right) R-modules are flat, were studied by authors in [1], [9], and [11].

Indeed, Abdul-Jabbar and Mahmood [1] proved that a ring R is strongly regular if and only if R is reduced and right (left) SSF-ring if and only if R is ZC and right SSF-ring. Also Guangshi [9] proved that R is strongly regular if and only if R is ZI and right (left) SSF-ring.

Following Zhou [15], a left ideal L of a ring R is called a *weakly ideal* (simply, *W-ideal*) if for any $0 \neq a \in L$, there exists a positive integer n such that $a^n \neq 0$ and $a^n R \subseteq L$. A right ideal K of a ring R is defined similarly to be a weakly ideal.

A ring R is called a left (resp., right) WZI if for any $a \in R$, $l(a)$ (resp., r(a)) is a W-ideal of R. Obviously, ZI rings are WZI rings.

The purpose of this paper is to study the regularity of left SSF-ring in terms of WZI rings. The following is the main result of this note.

For a ring R, the following conditions are equivalent:

(1) R is a strongly regular ring.

(2) R is a WZI and right (left) SF-ring.

(3) R is a WZI and right (left) SSF-ring.

2. Some property of WZI rings

In this section we study the structure of WZI rings, focusing on the relation to strong regularity. We write $N_2(R) = \{a \in R | a^2 = 0\}$ as in the literature.

Lemma 2.1. Let R be a ring. For any $a \in N_2(R)$, we have the following.

(1) If R is a left WZI ring, then $r(a)$ is an ideal.

(2) If R is a right WZI ring, then $l(a)$ is an ideal.

Proof.

(1) If $a = 0$, then we are done. Let $x \in r(a)$ and $a \neq 0$. Then $a \in l(x)$. Since $l(x)$ is a W-ideal and $a^2 = 0$, $aR \subseteq l(x)$. Therefore $Rx \subseteq r(a)$ and so $r(a)$ is an ideal.

(2) Similar to the proof of (1). ■

A ring is usually said to be *abelian* if every idempotent is central. ZI rings are clearly abelian. It is shown in [8, Theorem 3.2] that a ring R is a strongly regular ring if and only if R is an abelian regular ring. The following lemma is well-known.

Lemma 2.2. If R is a left WZI ring, then R is an abelian ring.

In fact, letting $e = e^2 \in E(R)$ and $r \in R$, we have that $1 - e \in l(e)$. If R is WZI ring, then $(1 - e)^n r \in l(e)$, entailing $ere = re$. Similarly $ere = er$, and thus R is an abelian ring. But the converse need not be true.

Example 2.3. There exists an abelian ring which is not WZI. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Due to Antoine [2, Example 4.8], let I be the ideal of A generated by b^2 and set $R = A/I$. Identify a and b with their images in R for simplicity.

Then R is an abelian ring by the arguments in [2]. But R is not left WZI. Indeed, consider $b \in R$. Then $b \in l(b)$ and $b^2 = 0$. So there cannot exist $n \geq 1$ such that $b^n \neq 0$ and $b^n R \subseteq l(b)$, noting $ba \notin l(b)$. In fact, n must be 1, and $bab \neq 0$, noting $ba \in bR$.

Following the literature, a ring R is called directly finite if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. It is easily checked that abelian rings are directly finite. Hence left (right) WZI rings are directly finite by Lemma 2.2.

Recall that reduced rings are left WZI. The aim of our next theorem is to find the conditions under which left WZI rings are reduced.

Theorem 2.4. A left WZI ring R is reduced if R satisfies any of the following conditions.

(1) R is a semiprime ring.

(2) R is a right (left) principally projective ring.

(3) R is a right (left) principally quasi-Baer ring.

Proof.

(1) Let $a^2 = 0$ for $a \in R$. If $a = 0$, then we are done. If $a \neq 0$, then $a \in l(a)$ and $a^2 = 0$. So R is left WZI ring, $aR \subseteq l(a)$, and so $aRa = 0$. Since R is a semiprime ring, $a = 0$. Hence R is reduced ring.

(2) Let $a^2 = 0$ for all $a \in R$. If $a = 0$, then we are done. If $a \neq 0$, then $a \in r(a)$. Since R is a right principally projective ring, then there exists an idempotent $e \in R$ such that $r(a) = eR$. Thus $a = ea$. Since R is an abelian ring by Lemma 2.2, $a = ea = ae = 0$, and so R is a reduced ring. A similar proof may be given for left principally projective rings.

(3) Similar to the proof of (2). ■

The following results can be easily obtained by Theorem 2.4.

Corollary 2.5. Let R be a semiprime ring. The following conditions are equivalent [6]:

(1) R is a reduced ring.

(2) R is a ZI ring.

(3) R is a left WZI ring.

Following Goodearl [8], a ring R is called (von Neumann) regular if for each $a \in R$, there exists $b \in R$ with $aba = a$, while R is strongly regular if for each there exists $b \in R$ with $a^2b = a$. Regular rings are semiprimitive (hence semiprime) by [8]. So we get the following from Corollary 2.5.

Corollary 2.6. Let R be a regular ring. The following conditions are equivalent:

(1) R is reduced ring.

(2) R is ZI ring.

(3) R is left WZI ring.

3. Strong regularity of SSF-rings

In this section, we study the strong regularity of right SSF-rings via W-ideals, and improve the results of Guangshi [9] and Mahmood [11]. The notions of W-ideals are very useful for the study of SF-rings.

In [7], Goodearl showed that regular rings are always right (left) SF-rings. According to Rege [13], reduced right (left) SF-rings are strongly regular. It is shown in [8] that R is strongly regular if and only if R is a reduced regular ring if and only if R is an abelian regular ring. Hence, we have the following Lemmas.

Lemma 3.1. [9] Let R be a right (left) SSF-ring. If I is an ideal of R, then R/I is a right (left) SSF-ring.

Lemma 3.2. [9] If R is a reduced and right (left) SSF-ring, then R is a strongly regular ring.

Lemma 3.3. [13] Let R be a ring, and I be a right ideal of R. Then R/I is a flat right R-module if and only if for each $a \in I$ there exists $b \in I$ such that $a = ba$.

The following lemma shows some properties of right (left) SSF-ring which have roles in this note.

Lemma 3.4. Let R be a right (left) SSF-ring. Then we have the following:

(1) The center $C(R)$ of R is strongly regular.

(2) $J(R)$ is a reduced ring without identity if for every $a \in N_2(R)$, $l(a) = r(a)$.

(3) R is a semiprimitive ring if for every $a \in J(R)$, $l(a) = r(a)$.

Proof. We apply the method in the proof of [1, Theorem 2.10].

(1) We claim that $aR + r(a) = R$ for any $a \in C(R)$. If not, there exists a maximal right ideal L such that $aR + r(a) \subseteq L$. First observe that L is an essential right ideal of R. If not, then L is a direct summand of R. So we can write $L = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in aR \subseteq L = r(e)$, $ea = 0$ and $a \in C(R)$. Then $ae = 0$, and $e \in r(a) \subseteq L = r(e)$; whence $e = 0$. It is a contradiction. Therefore L is an essential right ideal of R. Therefore L is an essential right ideal of R. Thus R/L is a simple singular right R-module and flat by hypothesis. Now since R is a right SSF-ring, there exist $c \in L$ such that $a = ca$ by Lemma 3.3. Thus $(1-c)a = 0$ and $a \in C(R)$, $a(1-c) = 0$ and so $1-c \in r(a) \subseteq L$; whence $1 \in L$. It is a contradiction. Therefore $aR + l(a) = R$ for any $a \in C(R)$ and so R is strongly regular.

(2) Let $a^2 = 0$ for $0 \neq a \in J(R)$. Then $l(a) \neq R$. There exists a maximal right ideal K such that $l(a) = r(a) \subseteq K$ by hypothesis. First observe that K is an essential right ideal of R. If not, then K is a direct summand of R. So we can write $K = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in M$, $ea = 0$, and $e \in l(a) \subseteq K = r(e)$; whence $e = 0$. It is a contradiction. Therefore K is an essential right ideal of R. Therefore K is an essential right ideal of R. Thus R/K is a simple singular right R-module and flat by hypothesis. Now since R is a right SSF-ring, there exist $c \in K$ such that $a = ca$ by Lemma 3.3. Thus $(1-c)a = 0$, and $1-c \in l(a) \subseteq K$; whence $1 \in K$. It is a contradiction. Therefore J(R) is reduced.

(3) We claim that $aR+r(a) = R$ for any $a \in J(R)$. If not, then there exist $a \in R$ such that $aR+r(a) \neq R$. There exists a maximal right ideal L such that $aR+r(a) \subseteq L$.

First observe that L is an essential right ideal of R. If not, then L is a direct summand of R. So we can write $L = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in L$, $ea = 0$, and $e \in l(a) = r(a) \subseteq L = r(e)$; whence $e = 0$. It is a contradiction. Therefore L is an essential right ideal of R. Therefore L is an essential right ideal of R. Thus R/L is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in L$ such that $a = ca$ by Lemma 3.3. Thus $(1-c)a = 0$, and $1 - c \in l(a) = r(a) \subseteq L$; whence $1 \in L$. It is a contradiction. Hence $aR + r(a) = R$ for any $a \in J(R)$. Thus there exists $x \in r(a)$ and $t \in R$ such that $at + x = 1$. So $a^2t + ax = a$, and $a(at - 1) = 0$. Since $a \in J(R)$ and $at - 1$ is invertible, hence $a = 0$. Therefore J(R)=0 and so R is a semiprimitive ring. ■

The following theorem extends the results of [1], [9], and [11].

Theorem 3.5. Let R be a right SSF-ring.

(1) If R is a right WZI ring, then R is a reduced ring.

(2) If R is a left WZI ring, then R is a reduced ring.

Proof.

(1) Let $a^2 = 0$. If $a = 0$, then we are done. If $a \neq 0$, then $l(a) \neq R$. Here if R is a right WZI, then $l(a)$ is an ideal by Lemma 2.1, and so there exists a maximal right ideal M such that $l(a) \subseteq M$. First observe that M is an essential right ideal of R. If not, then M is a direct summand of R. So we can write $M = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in M$, $ea = 0$, and $e \in l(a) \subseteq M = r(e)$; whence $e = 0$. It is a contradiction. Therefore M is an essential right ideal of R. Thus R/M is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in M$ such that $a = ca$ by lemma 3.4. Thus $(1-c)a = 0$. If $1 - c = 0$, then $1 = c \in M$; It is a contradiction. If $1 - c \neq 0$, then $1 - c \in l(a) \subseteq M$; whence $1 \in M$. It is a contradiction. Therefore $a = 0$ and so R is reduced ring.

(2) It suffices to prove that R is semiprime by help of Coroally 2.5. Let $aRa = 0$. If $a = 0$, then we are done. If $a \neq 0$, then $l(Ra) \neq R$. There exists a maximal right ideal K of R such that $a \in l(Ra) \subseteq K$. First observe that K is an essential right ideal of R. If not, then K is a direct summand of R. So we can write $K = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in K$, $ea = 0$ and $e \in l(a)$. Since R is a left WZI, there exists $n > 0$ such that $e^nR \subseteq l(a)$, and $e \subseteq l(Ra) \subseteq K = r(e)$; whence $e = 0$. It is a contradiction. Therefore K is an essential right ideal of R. Thus R/K is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in K$ such that $a = ca$ by lemma 3.4. Thus $(1-c)a = 0$. If $1 - c = 0$, then $1 = c \in M$; It is a contradiction. If $1 - c \neq 0$, then $1 - c \in l(a)$. Since R is a left WZI, there exists $n > 0$ such

that $(1-c)^n R \subseteq l(a)$, and $(1-c)^n \subseteq l(Ra) \subseteq K$; whence $1 \in K$. It is a contradiction. Hence $a = 0$ and so R is semiprime ring. Therefore R is reduced by Coroally 2.5. ■

Theorem 3.6. The following conditions are equivalent:

(1) R is a strongly regular ring.

(2) R is a ZI and right (left) SSF-ring.

(3) R is a WZI and right (left) SSF-ring.

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (1): Let R be a WZI and right SSF-ring. By Theorem 3.5, R is reduced. If R is not regular, then there exist $a \in R$ such that $aR + l(a) \neq R$. Since R is reduced, $l(a) = r(a)$. There exists a maximal right ideal L such that $aR + l(a) \subseteq L$. First observe that L is an essential right ideal of R. If not, then L is a direct summand of R. So we can write $L = r(e)$ for some $0 \neq e = e^2 \in R$. Since $a \in L$, $ea = 0$, and $e \in l(a) \subseteq L = r(e)$; whence $e = 0$. It is a contradiction. Therefore L is an essential right ideal of R. Therefore L is an essential right ideal of R. Thus R/L is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in L$ such that $a = ca$ by Lemma 3.4 Thus $(1-c)a = 0$, and $1 - c \in l(a) \subseteq L$; whence $1 \in L$. It is a contradiction. Hence $aR + l(a) = R$ for any $a \in R$ and so R is regular. Therefore R is strongly regular. ■

From Theorem 3.6, we can obtain the following.

Corollary 3.7. ([8], [9]) The following conditions are equivalent:

(1) R is a strongly regular ring.

(2) R is a ZC and right(left) SSF-ring.

(3) R is a ZI and right(left) SSF-ring.

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