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Correction and generalization: "Fixed points of α_admissible Meir-Keeler contraction mappings on quasi-metric spaces"

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Abstract

In a recent paper titled "Fixed points of $\boldsymbol{\alpha}$_ admissible Meir-Keeler contraction mappings on quasi-metric spaces", proofs of main results are incorrect and the uniqueness is not discussed. In this paper we suggest a new fixed point theorem. Our main result is a generalization of the main results in the original paper. Moreover, we introduce the concept of "separated mappings" to obtain the uniqueness of the fixed point.

Index Terms— quasi-metric; Meir-Keeler; contraction; admissible; separated mappings.

I. INTRODUCTION AND PRELIMINARIES

Throughout this paper, the terminology and the notations are the same as in [1]. Exceptionally, we recall the following definitions:

Definition 1.1. Let (X, d) be a quasi-metric space. Let $T : X \to X$ be a triangular α_admissible mapping.

Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$\varepsilon \leq N_1(x, y) < \varepsilon + \delta \; implies \; \alpha(x, y)d(Tx, Ty) < \varepsilon,$

where

$N_1(x, y) = max\{d(x, y), d(Tx, x), d(Ty, y)\},$

for all $x, y \in X$. Then T is called a generalized α_Meir-Keeler contraction of type (I).

Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$\varepsilon \leq N_2(x, y) < \varepsilon + \delta \; implies \; \alpha(x, y)d(Tx, Ty) < \varepsilon,$

where

$$N_2(x,y) = max\left\{d(x,y), \frac{1}{2}[d(Tx,x), d(Ty,y)]\right\},$$

for all $x, y \in X$. Then T is called a generalized α_Meir-Keeler contraction of type (II).

We consider the following definition:

Definition 1.2. Let (X,d) be a quasi-metric space. Let $T : X \to X$ be a triangular α_admissible mapping. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \le M(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y)d(Tx,Ty) < \varepsilon, \quad 1.1$$

where

$$M(x,y) = \max \left\{d(x,y), \alpha d(Tx,x), \beta d(Ty,y), \frac{\gamma}{2}[d(Tx,x), d(Ty,y)]\right\}, \quad 1.2$$

for all $x, y \in X$ and α, β, γ in $[0,1]$. Then T is called a generalized α_Meir-Keeler contraction of type (III).

Remark 1.3. M represents a class of functions containing both function N_1 and N_2. In fact, if $(\alpha,\beta,\gamma) = (1,1,0)$ we get $M = N_1$ and if $(\alpha,\beta,\gamma) = (0,0,1)$ we get $M = N_2$.

Now, we introduce the concept of separated mapping as:

Definition 1.4. A function $\alpha : X \times X \to [0,+\infty[$ is called separated if it verifies for all $a, b \in X$:

$\alpha(a,b) = 0 \Rightarrow a = b$.

In [1] authors stated the following theorems (as main results):

Theorem 1.5. Let (X,d) be a complete quasi-metric space and $T : X \to X$ be a continuous generalized α_Meir-Keeler contraction of type (I). If $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(Tx_0, x_0) \ge 1$ for some $x_0 \in X$, then T has a fixed point in X.

Theorem 1.6. Let (X,d) be a complete quasi-metric space and $T : X \to X$ be a continuous generalized α_Meir-Keeler contraction of type (II). If $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(Tx_0, x_0) \ge 1$ for some $x_0 \in X$, then T has a fixed point in X.

Theorem 1.7. Let (X,d) be a complete quasi-metric space and $T : X \to X$ be a generalized α_Meir-Keeler contraction of type (II) and let (X,d) satisfies the following conditions: "If $\{x_n\}_n$ is a sequence in X which converges to x and satisfies $\alpha(x_{n+1}, x_n) \ge 1$ and $\alpha(x_n, x_{n+1}) \ge 1$ for all n then there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $\alpha(x_{n_k}, x) \ge 1$ and $\alpha(x, x_{n_k}) \ge 1$ for all k.

If $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(Tx_0, x_0) \ge 1$ for some $x_0 \in X$, then T has a fixed point in X.

The proofs of the three theorems stated above in the original paper [1] are based on the following -clearly incorrect- reasoning: Putting $r_n = max\{x_n, y_n\}$ for all $n \in \mathbb{N}$, if x_n and y_n verify some property (P) for all $n \in \mathbb{N}$ then r_n verifies the property (P) for all $n \in \mathbb{N}$.

The following example is for clarification: We consider the following sequences:

$$r_n = max\{x_n, y_n\},\ x_n = \begin{cases} n; & if\ n\ is\ even \\ 0; & if\ n\ is\ odd \end{cases}\ and\ y_n = x_{n+1},$$

and the property (P) by: If there exists a subsequence $\{r_{n_k}\}_k$ of the sequence $\{r_n\}_n$ which converges to "0" , then the sequence $\{x_n\}_n$ converges to "0".

This error makes almost all the proofs incorrect. The remaining results in [1] (corollaries 18, 19 and theorems 23, 28, 29) are direct consequences from 1.5, 1.6 and 1.7, then their proofs still correct.

In this paper, we consider the following fixed point theorem. This theorem is a generalization of the results obtained in [1]. Moreover, by adding a simple and natural condition, we got the uniqueness of the fixed point.

II. MAIN RESULT

Theorem 2.1. Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a generalized α_Meir-Keeler contraction of type (III), we assume that one of the two following conditions holds:

(A) T is continuous.

(B) If $\{x_n\}_n$ is a sequence in X which converges to x and satisfies: $min\{\alpha(x_{n+1}, x_n), \alpha(x_n, x_{n+1})\} \geq 1$ for all n, then there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $min\{\alpha(x, x_{n_k}), \alpha(x_{n_k}, x)\} \geq 1$ for all k.

If $min\{\alpha(x_0, Tx_0), \alpha(Tx_0, x_0)\} \geq 1$ for some $x_0 \in X$, then T has a fixed point in X. Furthermore, if α is separated, this fixed point is unique.

Remark 2.2. Theorems 1.5, 1.6 and 1.7 are a special case of theorem 2.1.

Proof. Let $x_0 \in X$ such that $min\{\alpha(x_0, Tx_0), \alpha(Tx_0, x_0)\} \geq 1$ and define the sequence $\{x_n\}_n$ in X as $x_{n+1} = Tx_n$. Throughout this paper we assume -without loss of any generality- that

$$x_n \neq x_{n+1}. \quad 2.1$$

For all n, and since T is α_admissible, it follows that

$$min\{\alpha(x_n, x_{n+1}), \alpha(x_{n+1}, x_n)\} \geq 1, \quad 2.2$$

for all $n \in \mathbb{N}$. Using 1.1 we can easily show that

$$\alpha(x_{n+1}, x_n)d(x_{n+2}, x_{n+1}) < M(x_{n+1}, x_n), \quad 2.3$$

and that

$$\alpha(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) < M(x_n, x_{n+1}), \quad 2.4$$

for all $n \in \mathbb{N}$. Having in mind inequality 2.3 and relation 1.2, we estimate

$d(x_{n+2},x_{n+1}) < M(x_{n+1},x_n)$
$= max\{d(x_{n+1},x_n), \alpha d(x_{n+2},x_{n+1}), \beta d(x_{n+1},x_n), \frac{\gamma}{2}[d(x_{n+2},x_{n+1}), d(x_{n+1},x_n)]\}$
$= d(x_{n+1},x_n),$ 2.5

for all $n \in \mathbb{N}$. Consequently, $\{d(x_{n+1},x_n)\}_n$ is a decreasing positive sequence, then it converges to some $r_1 \geq 0$. Remember that $\{M(x_{n+1},x_n)\}_n = \{d(x_{n+1},x_n)\}_n$ and supposing that $r_1 > 0$, we deduce that for all $\delta > 0$ there exists $n \in \mathbb{N}$ such that

$r_1 \leq M(x_{n+1},x_n) < r_1 + \delta \Rightarrow \alpha(x_{n+1},x_n)d(x_{n+2},x_{n+1}) < r_1$
$\Rightarrow d(x_{n+2},x_{n+1}) < r_1,$

which is a contradiction with the fact that the sequence $\{d(x_{n+1},x_n)\}_n$ is decreasing to r_1, then

$\lim_{n\to\infty} d(x_{n+1},x_n) = \lim_{n\to\infty} M(x_{n+1},x_n) = 0.$ 2.6

Similarly, taking into account inequality 2.4 and relation 1.2, we get

$d(x_{n+1},x_{n+2}) < M(x_n,x_{n+1})$
$= max\{d(x_n,x_{n+1}), \alpha d(x_{n+1},x_n), \beta d(x_{n+2},x_{n+1}), \frac{\gamma}{2}[d(x_{n+1},x_n), d(x_{n+2},x_{n+1})]\}$
$\leq \max \{d(x_n,x_{n+1}), d(x_{n+1},x_n)\}.$ 2.7

This estimate together with 2.5 gives

$\max\{d(x_{n+1},x_{n+2}), d(x_{n+2},x_{n+1})\} < \max\{d(x_n,x_{n+1}), d(x_{n+1},x_n)\},$

accordingly, the sequence $(\max \{d(x_n,x_{n+1}), d(x_{n+1},x_n)\})_n$ is a decreasing positive sequence and consequently it converges to some $r_2 \geq 0$. Assume that $r_2 > 0$, thanks to limits 2.6 we conclude that there exists $n_0 \in \mathbb{N}$ for which the sequence $\{d(x_n,x_{n+1})\}_{n>n_0}$ is decreasing to r_2. On the other hand, remember that

$d(x_{n+1},x_{n+2}) < M(x_n,x_{n+1})$
$\leq max \{d(x_n,x_{n+1}), d(x_{n+1},x_n)\},$

we deduce that $\{M(x_n,x_{n+1})\}_n$ converges also to r_2. This result, together with 1.1, allows us to write for all $\delta > 0$ and for all $n \in \mathbb{N}$ sufficiently large that

$r_2 \leq M(x_n,x_{n+1}) < r_2 + \delta \Rightarrow \alpha(x_n,x_{n+1})d(x_{n+1},x_{n+2}) < r_2$
$\Rightarrow d(x_{n+1},x_{n+2}) < r_2,$

which is a contradiction with the fact that the sequence $\{d(x_n,x_{n+1})\}_{n>n_0}$ is decreasing to r_2. Then

$\lim_{n\to\infty} d(x_n,x_{n+1}) = \lim_{n\to\infty} M(x_n,x_{n+1}) = 0.$ 2.8

Now, let's show that $\{x_n\}_n$ is a Cauchy sequence.

Lemma 2.3. Using the previous notations we obtain:

I)- For every positive real number $\varepsilon > 0$ there is a positive integer N such that for all positive integers $n > N$ and $l \in \mathbb{N}$, if the distance $d(x_{n+l},x_n) < \varepsilon$ then the distance $d(x_{n+l+1},x_{n+1}) < \varepsilon$, i.e.

$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall N \in \mathbb{N}; d(x_{n+l}, x_n) < \varepsilon \Rightarrow d(x_{n+l+1}, x_{n+1}) < \varepsilon.$ 2.9

II) For every positive real number $\varepsilon > 0$ there is a positive integer N such that for all positive integers $n > N$ and $l \in \mathbb{N}$, if the distance $d(x_n, x_{n+l}) < \varepsilon$ then the distance $d(x_{n+1}, x_{n+l+1}) < \varepsilon$, i.e.

$\forall\varepsilon > 0, \exists N > 0, \forall n \geq N, d(x_n, x_{n+l}) < \varepsilon \Rightarrow d(x_{n+1}, x_{n+l+1}) < \varepsilon.$ 2.10

Proof. Let $\varepsilon > 0$ and $n, l \in \mathbb{N}$. Throughout this proof, n is considered sufficiently large.

I) Supposing that

$d(x_{n+l}, x_n) < \varepsilon \text{ and } d(x_{n+l+1}, x_{n+1}) \geq \varepsilon,$ 2.11

and invoking relations 1.2, 2.6 it follows

$M(x_{n+l}, x_n) = \max\ \{d(x_{n+l}, x_n), \alpha d(x_{n+l+1}, x_{n+l}), \beta d(x_{n+1}, x_n),$

$\frac{\gamma}{2}[d(x_{n+l+1}, x_{n+l}) + d(x_{n+1}, x_n)]\Big\} < \varepsilon.$

Now, returning to 1.2, we can write thanks to 2.11 that

$M(x_{n+l}, x_n) \geq d(x_{n+l}, x_n)$

$\geq d(x_{n+l+1}, x_{n+1}) - d(x_{n+l+1}, x_{n+l}) - d(x_n, x_{n+1})$

$\geq \varepsilon - d(x_{n+l+1}, x_{n+l}) - d(x_n, x_{n+1}).$

The last two estimates mean that

$\lambda \leq M(x_{n+l}, x_n) < \lambda + d(x_{n+l+1}, x_{n+l}) + d(x_n, x_{n+1}),$ 2.12

where

$\lambda = \varepsilon - d(x_{n+l+1}, x_{n+l}) - d(x_n, x_{n+1}).$

Recalling relations 1.1 and 2.2, the double inequality 2.12 implies that

$d(x_{n+l+1}, x_{n+l}) \leq \varepsilon - d(x_{n+l}, x_{n+l+1}) - d(x_n, x_{n+1}) < \varepsilon.$

Which is in contradiction with 2.11. This finishes the proof of the first part of the lemma.

II)- Similarly as for the first part, assume the contrary, i.e.

$d(x_n, x_{n+l}) < \varepsilon \text{ and } d(x_{n+1}, x_{n+l+1}) \geq \varepsilon.$ 2.13

Accordingly, it comes due to relations 1.2, 2.6 that

$M(x_n, x_{n+l}) = \max\{d(x_n, x_{n+l}), \alpha d(x_{n+1}, x_n), \beta d(x_{n+l+1}, x_{n+l}),$

$\frac{\gamma}{2}\big(d(x_{n+1}, x_n) + d(x_{n+l+1}, x_{n+l})\big)\big\} < \varepsilon.$

On the other hand, owing to relations 1.2 and 2.13 we obtain

$M(x_n, x_{n+l}) \geq d(x_n, x_{n+l})$

$\geq d(x_{n+1}, x_{n+l+1}) - d(x_{n+l}, x_{n+l+1}) - d(x_{n+1}, x_n)$

$\geq \varepsilon - d(x_{n+l}, x_{n+l+1}) - d(x_{n+1}, x_n).$

By the two previous inequalities we deduce that

$\mu \leq M(x_n, x_{n+l}) < \mu + d(x_{n+l}, x_{n+l+1}) + d(x_{n+1}, x_n),$ 2.14

where

$\mu := \varepsilon - d(x_{n+l}, x_{n+l+1}) - d(x_{n+1}, x_n).$

Invoking relation 1.1 from definition 1.2, the double inequality 2.14 implies that

$d(x_{n+1}, x_{n+l+1}) \leq \varepsilon - d(x_{n+l}, x_{n+l+1}) - d(x_{n+1}, x_n) < \varepsilon,$

which is a contradiction with 2.13. This finishes the proof of the lemma.

Now, suppose that the sequence $\{x_n\}_n$ has not the left-Cauchy property, i.e.

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \exists k \in \mathbb{N}; d(x_{n+k}, x_n) > \varepsilon.$

From where, we can assert that there exists a subsequence $\{x_{n_k}\}_k$ with the following property:

$\exists \varepsilon > 0, \forall k \in N, \exists n < n_k \mid d(x_{n_k}, x_n) > \varepsilon \text{ and } d(x_{n_k-1}, x_n) \leq \varepsilon.$ 2.17

By virtue of the first part in lemma 2.3 and property 2.15, we get

$d(x_{n_k-1}, x_{n-1}) > \varepsilon,$ 2.16

for sufficiently large values of n. Now, thanks to relations 1.2, 2.6, 2.15 and 2.16, we conclude that

$\varepsilon < M(x_{n_k-1}, x_{n-1})$

$= \max \{d(x_{n_k-1}, x_{n-1}), \alpha d(x_{n_k}, x_{n_k-1}), \beta d(x_n, x_{n-1}),$

$\frac{\gamma}{2}\left(d(x_{n_k}, x_{n_k-1}) + d(x_n, x_{n-1})\right)\}$

$= d(x_{n_k-1}, x_{n-1})$

$\leq d(x_{n_k-1}, x_n) + d(x_n, x_{n-1})$

$\leq \varepsilon + \mathrm{d}(x_n, x_{n-1}),$

for sufficiently large values of n. From which it comes due to limit 2.6 and relation 1.1 from definition 1.2 that

$d(x_{n_k}, x_n) \leq \varepsilon.$

This contradiction implies that $\{x_n\}_n$ is a left-Cauchy sequence. Now we have to establish that $\{x_n\}_n$ is a right-Cauchy sequence, to do so, we following the same lines as above. So, assume that the sequence $\{x_n\}_n$ is not a right-Cauchy sequence, and then we can assert that there exists a subsequence $\{x_{n_k}\}_\mathrm{k}$ with the following property:

$\exists \varepsilon > 0, \forall k \in \mathbb{N}, \exists n < n_k \mid d(x_n, x_{n_k}) > \varepsilon \text{ and } d(x_n, x_{n_k-1}) \leq \varepsilon.$

Returning to lemma 2.3 and having in mind relations 1.2 and 2.17, we obtain

$\varepsilon < M(x_{n-1}, x_{n_k-1})$

$= \mathrm{d}(x_{n-1}, x_{n_k-1})$

$\leq \varepsilon + d(x_{n-1}, x_n),$

for sufficiently large values of n. This result with relations 1.1, 2.8 and 2.17 drives to an obvious contradiction. In light of what precedes, we deduce the existence of $a \in X$, such that

$x_n \to a$. 2.18

It's clear that if T is a continuous mapping then a is a fixed point of T. So, to achieve the poof we need prove that a is a fixed point of T under the condition (B). Owing to relations

1.2, 2.6, and 2.18 we can write

$M(x_n, a) = \max\ \{d(x_n, \mathrm{a}), \alpha d(x_{n+1}, x_n), \beta d(Ta, a),$

$\frac{\gamma}{2}\big(d(x_{n+1}, x_n) + d(Ta, a)\big)\Big\}$

$\leq \max\left\{\beta, \frac{\gamma}{2}\right\} \mathrm{d}(Ta, a),$ 2.19

which in turn, according to relation 1.1 implies that

$\alpha(x_n, a) d(x_{n+1}, Ta) \leq max\left\{\beta, \frac{\gamma}{2}\right\} d(Ta, a).$

Having in mind relations 2.2 and 2.18, the condition above assert the existence of a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $\alpha(x_{n_k}, a) \geq 1$ for all k. Hence, 2.19 becomes

$d\big(x_{n_k+1}, Ta\big) < max\left\{\beta, \frac{\gamma}{2}\right\} d(Ta, a).$ 2.20

Following the same line of thought, we can show that

$d(Ta, x_{n_k+1}) < max\left\{\alpha, \frac{\gamma}{2}\right\} d(a, Ta),$ 2.21

passing to the limit in 2.20 and 2.21, we get a clear contradiction with the fact that $max\left\{\beta, \frac{\gamma}{2}\right\} \in [0,1]$. Consequently a is a fixed point of T.

Finally, regarding the uniqueness, let us suppose for all x, y in X:

$\alpha(x, y) = 0 \Rightarrow x = y.$ 2.22

Consider $b\ (\neq a) \in X$ another fixed point of T, i.e.

$Ta = a, Tb = b \text{ and } d(a, b) > 0.$ 2.23

Combining relations 1.2 and 2.23 yields

$M(a, b) = d(a, b).$

This means that

$$\varepsilon \leq M(a, b) < \varepsilon + \frac{1}{n+1},$$

for all $n \in \mathbb{N}$, where

$\varepsilon = d(a, b) - d(a, b)\frac{n}{n+1}.$

Then, we can assert, due to relation 1.1, that

$\alpha(a, b) < 1 - \frac{n}{n+1},$ 2.24

for all $n \in \mathbb{N}$. Passing to the limit in 2.24, we get $\alpha(a, b) = 0$. From which, with relation 2.22, we obtain

$a = b$.
This achieves the proof.

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