Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4975-4996

http://www.ripublication.com

Numerical Modelling For time fractional nonlinear partial differential equation by Homotopy Analysis Fractional Sumudu Transform Method

Rishi Kumar Pandey and Hradyesh Kumar Mishra*

Department of Mathematics Jaypee University of Engineering and Technology Guna-473226 (M. P.), INDIA
E-mail: rishipandey.9@rediffmail.com, hk.mishra@juet.ac.in,
Corresponding Author: Hradyesh Kumar Mishra,(+91-9407570623),*
Department of Mathematics,E-mail hk.mishra@juet.ac.in

Abstract

In this article, we implement new analytical technique, the homotopy analysis fractional sumudu transform method (HAFSTM), for solving nonlinear partial differential equations of fractional order. The fractional derivatives are taken in caputo sense. The method in applied mathematics can be used as alternative methods for obtaining analytic and approximate solutions for various types of differential equations. The purpose of this study is to avoid the restrictive assumptions and rounding off errors in numerical computation of problems. The numerical solutions obtained by the HAFSTM method indicate that the approach is easy to implement and computationally very attractive and accurate.

Keywords: Homotopy Analysis Method, Homotopy Analysis Fractional Sumudu Transform Method, Linear and Nonlinear partial differential equation, Fractional partial differential equation.

AMS Subject Classification: 26A33; 34A08; 60G22; 65Gxx.

1. Introduction

In past few decades considerable interest showed by many researcher in the field of fractional calculus specially application of ordinary and partial differential equations of fractional order in modelling and simulation of problems due to their valuable applications in field of modelling of science and engineering. These applications in interdisciplinary sciences show the importance and necessity of fractional calculus. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier [1], Miller and Ross [2], Podlubny [3], Kilbas, Srivastava and Trujillo[4] and others V. Parthiban and K. Balachandran [5], Samko et al. [6], Caponetto et al. [7], Diethelm [8]. All mentioned authors provide systematic understanding of the fractional calculus such as the existence and the uniqueness of solutions, some analytical methods for solving fractional differential equations like Green's function method, the Mellin transform method, the power series method etc. Yet presently no method available that yields an exact solution for nonlinear fractional partial differential equations. Only approximate solutions can be derived using linearization or perturbation methods. Many mathematical methods such as Adomian decomposition method (ADM) [9-13], homotopy perturbation method (HPM) [14-19], variational iteration method (VIM) [20-25], homotopy analysis method (HAM)[26-30], Laplace decomposition method (LDM) [31-33], homotopy perturbation transform method (HPTM) [34], homotopy perturbation sumudu transform method (HPSTM) [35] and homotopy analysis transform method (HATM) [36-38] have been proposed to obtain exact and approximate analytical solutions of nonlinear equations. Inspired by all above discussion we have applied HAFSTM [39] for the solution of fractional partial differential equation.

The main objective of this paper to extend the application of homotopy analysis fractional sumudu transform method to provide approximate solution of initial value problems of nonlinear partial differential equation of fractional order.

2. Basic Definition of Fractional Calculus and Sumudu transform

Definition 2.1 A real function $f(t), t>0,$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in N$

Definition 2.2 The Riemann Liouville Fractional integral operator of order $\alpha \ge 0,$ of a function $f(t) \in C_\mu,$ and $\mu \ge -1$ is defined as [40,41]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha>0,\ x>0 \text{ and } J^0 f(t) = f(t).$$

For the Riemann-Liouville fractional integral, we have

$$J^\alpha t^y = \frac{\Gamma(y+1)}{\Gamma(y+\alpha+1)} t^{\alpha+y}$$

Definition 2.3 The fractional derivative of $f(t)$ in the Caputo sense is defined as [42]

$$D_t^{\alpha} f(t) = \begin{cases} J^{m-\alpha} D^n f(t), \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-T)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \end{cases}$$

where $m-1<\alpha\le m,\ m\in N,\ t>0.$

Definition 2.4 In early 90's, Watugala [43] introduced an incipient integral transforms. The sumudu transform is defined over the set of functions

$$A=\left\{ f(t) \,\middle|\, \exists M,\tau_1,\tau_2>0, |f(t)|<Me^{\frac{|t|}{\tau_j}}, if\ t\in(-1)^j\times[0,\infty) \right\},$$

by the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^{\infty} f(ut)\, e^{-t}\, dt, u\in(-\tau_1,\tau_2).$$

Definition 2.5 The sumudu transform of $f(t)=t^{\alpha}$ is defined as [44]

$$S\left[t^{\alpha}\right] = \int_0^{\infty} e^{-t} t^{\alpha}\, dt = \Gamma(\alpha+1)\, u^{\alpha}, R(\alpha)>0.$$

Definition 2.6 The Sumudu transform $S[f(t)]$ of the Riemann-Liouville fractional integral is defined as [44]

$$S\left[I^{\alpha} f(t)\right] = u^{-\alpha} F(u).$$

Definition 2.7 The Sumudu transform $S[f(t)]$ of the Caputo fractional derivative is defined as [44]

$$S\left[D_t^{\alpha} f(t)\right] = u^{-\alpha} S\left[f(t)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^{+}), \text{ where } m-1<\alpha\le m.$$

3. Solution by Homotopy Analysis Fractional Sumudu Transform Method

To illustrate the rudimental conception of the HAFSTM for the fractional partial differential equation, we consider the following fractional partial differential equation as

$$D_t^{n\alpha} U(x,t) + R(x)\, U(x,t) + N(x)\, U(x,t) = G(x,t);\ t>0, x\in R, n-1<\alpha\le n, \quad (3.1)$$

where $D_t^{n\alpha} = \dfrac{\partial^{n\alpha}}{\partial x^{n\alpha}}$, $R(x)$ is the linear operation in $x, N(x)$ is the general nonlinear operation in x and $G(x,t)$ is a continuous function.

For simplicity, we ignore all initial and boundary conditions, which can be treated in a homogeneous way. Now the methodology consists of applying the Sumudu transform first on both sides of the equation (3.1), we get

$$S\left[D_t^{n\alpha}U(x,t)\right]+S\left[R(x)U(x,t)\right]+S\left[N(x)U(x,t)\right]=S\left[G(x,t)\right];$$

$$t>0, x\in\mathbb{R}, n-1<\alpha\le n, \tag{3.2}$$

Using the differentiation property of the Sumudu transform

$$\frac{S\left[U(x,t)\right]}{u^{\alpha}}-\sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{\alpha-k}}+S\left[R(x)U(x,t)\right]+S\left[N(x)U(x,t)\right]-S\left[G(x,t)\right]=0,$$

$$S\left[U(x,t)\right]-u^{\alpha}\sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{\alpha-k}}+u^{\alpha}S\left[R(x)U(x,t)+N(x)U(x,t)-G(x,t)\right]=0, \tag{3.3}$$

we define nonlinear operator as

$$N\left[\phi(x,t;q)\right]=S\left[\phi(x,t;q)\right]-u^{\alpha}\sum_{k=0}^{n-1}\frac{U^{(k)}(0)}{u^{\alpha-k}}+u^{\alpha}S\left[R(x)\phi(x,t;q)+N(x)\phi(x,t;q)-G(x,t;q)\right] \tag{3.4}$$

where $q\in[0,1]$ be an embedding parameter and $\phi(x,t;q)$ is a real function of x,t and q.

we construct a homotopy as follow:

$$(1-q)S\left[\phi(x,t;q)-U_0(x,t)\right]=\hbar qH(x,t)N\left[\phi(x,t;q)\right] \tag{3.5}$$

where $\hbar$ is a nonzero auxiliary parameter and $H(x,t)\neq 0$. An auxiliary function $U_0(x,t)$ is an initial guess of $U(x,t)$ and $\phi(x,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HAFSTM. Obviously, when $q=0$ and $q=1$ it holds

$$\phi(x,t;0)=U_0(x,t), \qquad \phi(x,t;1)=U(x,t) \tag{3.6}$$

Thus, as q incre ases from 0 to 1, the solution varies from initial guess $U_0(x,t)$ to the solution $U(x,t)$. Now, expanding $\phi(x,t;q)$ on Taylor's series with respect to q, we get

$$\phi(x,t;q)=U_0(x,t)+\sum_{m=1}^{\infty}q^mU_m(x,t), \tag{3.7}$$

where

$$U_m(x,t)=\frac{1}{\lfloor m}\frac{\partial^m\phi(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{3.8}$$

The convergence of the series solution (3.7) is controlled by $\hbar$. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are properly chosen, the series (3.7) converges at $q=1$. Hence we obtain

$$U(x,t)=U_0(x,t)+\sum_{m=1}^{\infty}U_m(x,t), \tag{3.9}$$

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess $U_0(x,t)$ and the

exact solution $U(x,t)$ by means of the terms $U_m(x,t)\ (m=1,2,3,...)$, which are still to be determined.
Define the vectors

$$\vec{U}=\{U_0(x,t),U_1(x,t),U_2(x,t),...,U_m(x,t)\}. \qquad (3.10)$$

Differentiating the zero order deformation eq. (3.5) m times with respect to embedding parameter q and then setting $q=0$, and finally dividing them by $m!$, we obtain the m^{th} order deformation equation as follows:

$$S\left[U_m(x,t)-\chi_m U_{m-1}(x,t)\right]=\hbar H(x,t)R_m(\vec{U}_{m-1},x,t). \qquad (3.11)$$

Operating the inverse Sumudu transform of both sides, we get

$$U_m(x,t)=\chi_m U_{m-1}(x,t)+\hbar S^{-1}\left[H(x,t)R_m(\vec{U}_{m-1},x,t)\right], \qquad (3.12)$$

where

$$R_m(\vec{U}_{m-1},x,t)=\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x,t;q)}{\partial q^{m-1}}\bigg|_{q=0} \qquad (3.13)$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1 & m>1.\end{cases}$$

In this way, it is easy to obtain $U_m(x,t)$ for $m\ge 1$, at M^{th} order, we have

$$U(x,t)=\sum_{m=0}^{M}U_m(x,t), \qquad (3.14)$$

where $M\to\infty$, we obtain an accurate approximation of the original equation (3.1).

4. Illustrative Examples

In this section we shall illustrate the technique by three examples. These examples are somewhat artificial in the sense that the exact answer, for the special cases, is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the time-fractional derivative on the behaviour of the solution.

Example4.1. We borrow the nonlinear time-fractional advection partial differential equation [45]

$$D_t^{\alpha}U(x,t)+U(x,t)U_x(x,t)=x+xt^2,\quad t>0, x\in R,\ 0\le\alpha\le 1, \qquad (4.1)$$

subject to the initial condition

$$U(x,0)=0. \qquad (4.2)$$

Operating the Sumudu transform on both sides in eq. (4.1) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$S\left[U(x,t)\right]+u^{\alpha}S\left[U(x,t)\,U_x(x,t)\right]=x(1+2u^2), \tag{4.3}$$

The nonlinear operator is

$$N\left[\phi(x,t;q)\right]=S\left[\phi(x,t;q)\right]+u^{\alpha}S\left[\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x}\right]-x(1+2u^2), \tag{4.4}$$

and thus

$$R_m(\vec{U}_{m-1})=S\left[U_{m-1}(x,t)\right]-x(1+2t^2)(1-\chi_m)+u^{\alpha}S\left[\sum_{j=0}^{m-1}U_j(x,t)\,(U_j(x,t))_x\right]. \tag{4.5}$$

The $m^{th}-$ order deformation equation is given by

$$S\left[U_m(x,t)-\chi_m U_{m-1}(x,t)\right]=\hbar H(x,t)\,R_m(\vec{U}_{m-1}(x,t)).$$

Applying the inverse Sumudu transform, we have

$$U_m(x,t)=\chi_m U_{m-1}(x,t)+S^{-1}\left[\hbar H(x,t)\,R_m(\vec{U}_{m-1}(x,t))\right]. \tag{4.6}$$

On solving above equation from $m=1,2,...,$ we get

$$U_1(x,t)=-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+3)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$

$$U_2(x,t)=-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^2 x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$

$$\begin{aligned}U_3(x,t)&=-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^2 x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\\&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^2 x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\\&+\hbar^3 x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\&\left.+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^2}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right],\end{aligned}$$

$$
\begin{aligned}
U_4(x,t) ={}& -\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\\
&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\\
&+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right.\\
&\left.+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\right]\\
&+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+2x\left(\hbar^{3}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\right.\\
&\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+\hbar^{4}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]\right),
\end{aligned}
$$

$$
\begin{aligned}
U_5(x,t) ={}& -\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\\
&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\\
&+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]\\
&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right.\right.\\
&\left.\left.-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\right]+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+2x\left(\hbar^{3}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\right.\\
&\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+\hbar^{4}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]\right)+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right.\\
&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\\
&+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]\\
&+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)+\hbar\left[-\hbar x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right.\right.\\
&\left.\left.-\hbar^{2}x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right)\right]\right]+\hbar^{3}x\left[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.+\frac{4\Gamma(2\alpha+5)}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+2x\left(\hbar^{3}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\right.\\
&\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]+\hbar^{4}\left[\frac{1}{\Gamma(\alpha+1)^{2}}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(\alpha+3)}\frac{t^{3\alpha+2}}{\Gamma(3\alpha+3)}\right.\\
&\left.\left.\left.+\frac{4}{\Gamma(\alpha+3)^{2}}\frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)}\right]\right)\right],
\end{aligned}
\tag{4.7}
$$

etc. proceed by same manner the rest of components of the iteration can be obtained. Setting the $\hbar = -1,$ in eq. (4.7) the above expressions are exactly the same as given by ADM [41].

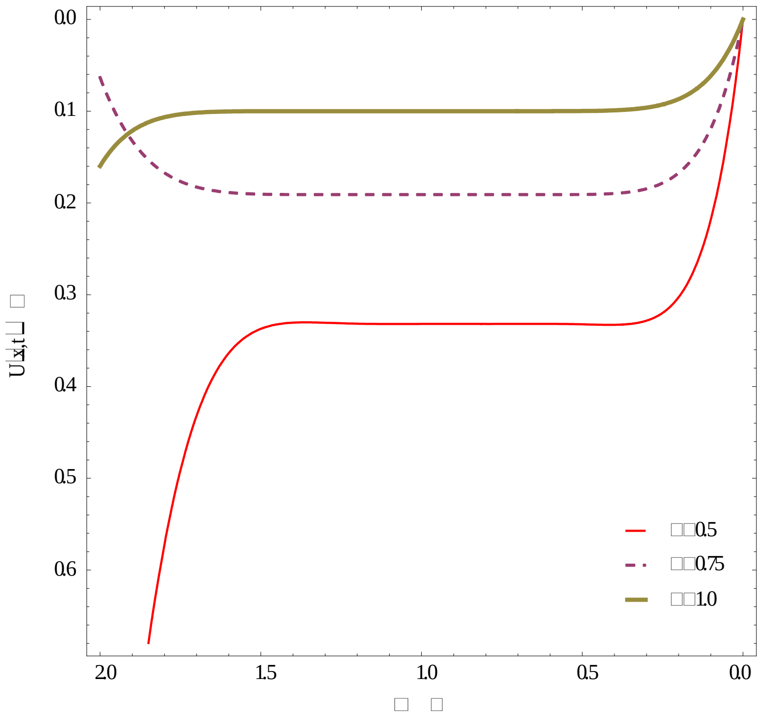

Figure 1 $\hbar$ curve for different values of α .

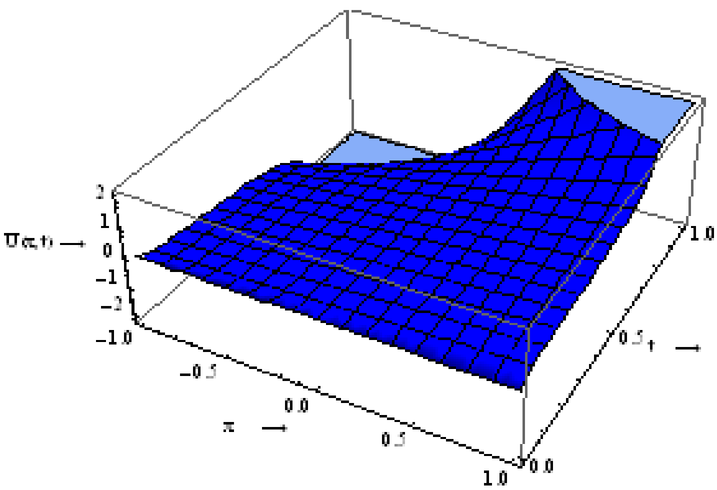

Figure 2 Plot of approximate solution for value $\alpha = 0.5$.

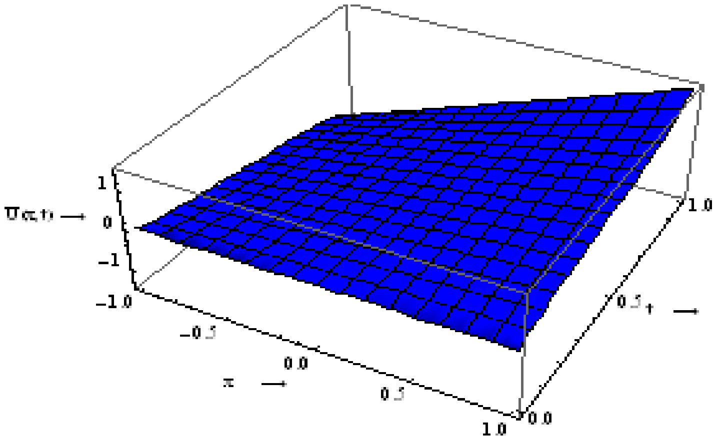

Figure 3 Plot of approximate solution for value $\alpha = 0.75$.

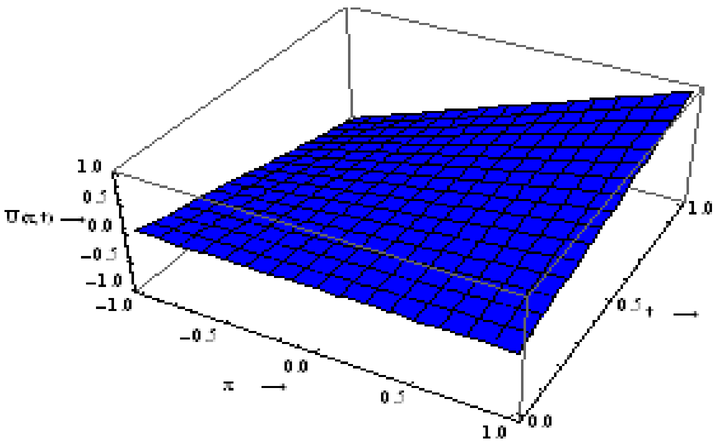

Figure 4 Plot of approximate solution for value $\alpha = 1$.

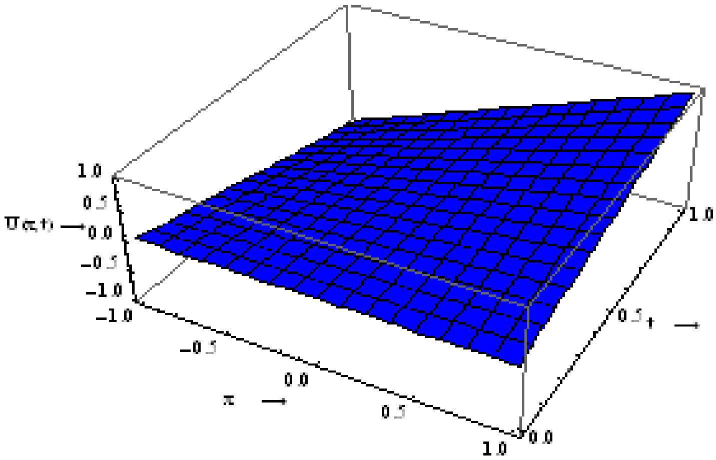

Figure 5 Plot of exact solution for value $\alpha = 1$.

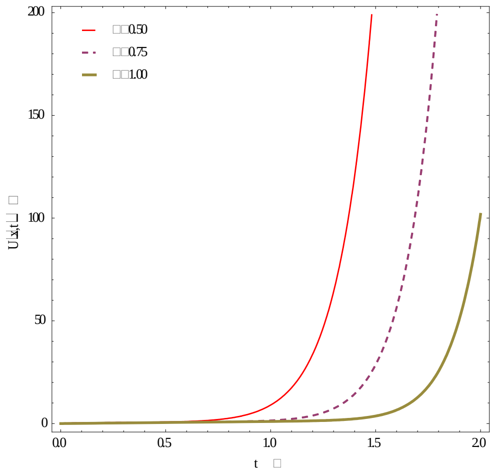

Figure 6 The behaviour of solution for different values α at $x = 1, \hbar = -1.$

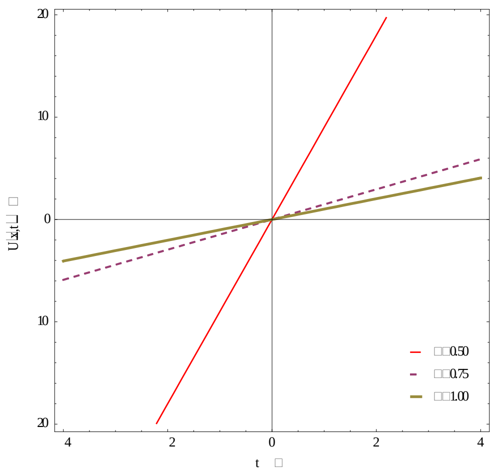

Figure 7 The behaviour of solution for different values α at $t = 1, \hbar = -1.$

Fig. 1 shows that the $\hbar$ values admissible between $-1.6 \le \hbar \le -0.4$ obtained from the fifth order solution $U \;\; x,t$ for different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1.$ Figs. 2-5 shows the behaviour of approximate solution of $U \;\; x,t$ for different fractional Brownian motions $\alpha = 0.5, 0.75$ and at standard motions, also exact solution $\alpha = 1.$

Figs. 6-7 showing the behaviour of approximate solutions at $t = 1$ and $x = 1$ respectively. It is seen that $U \;\; x,t$ increases very rapidly after point $t \ge 1$ and constant nature $t < 1.3.$ Also linear behaviour is observe in different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1.$

Table 1 Numerical values when $\alpha = 0.5, 0.75$ and 1.0 and comparison with [45]

t	x	$\alpha = 0.5$			$\alpha = 0.75$			$\alpha = 1.0$				
		u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	Exact	$\lvert u_9 - u \rvert$
0.2	0.25	0.112844	0.103750	0.111585	0.078787	0.077933	0.078766	0.050000	0.050309	0.050000	0.050000	4.63123×10^{-11}
	0.50	0.225688	0.207499	0.223170	0.157574	0.155865	0.157532	0.100000	0.100619	0.100000	0.100000	9.26246×10^{-11}
	0.75	0.311249	0.311249	0.334755	0.236361	0.233798	0.236299	0.150001	0.150928	0.150000	0.150000	1.39937×10^{-10}
	1.0	0.451375	0.414999	0.446341	0.315148	0.311730	0.315065	0.200001	0.201237	0.200000	0.200000	1.85249×10^{-10}
0.4	0.25	0.164004	0.172012	0.155966	0.128941	0.134855	0.128203	0.1000230	0.101894	0.100000	0.100000	1.00770×10^{-7}
	0.50	0.328008	0.344025	0.311932	0.257881	0.269710	0.256406	0.200046	0.203787	0.200000	0.200000	2.01540×10^{-7}
	0.75	0.492011	0.516037	0.467898	0.386821	0.404565	0.384608	0.300069	0.305681	0.300000	0.300000	3.02310×10^{-7}
	1.0	0.656015	0.688050	0.623864	0.515762	0.539420	0.512811	0.400092	0.407575	0.400000	0.400000	4.03080×10^{-10}
0.6	0.25	0.243862	0.215641	0.250596	0.177238	0.179990	0.171831	0.150411	0.153094	0.150010	0.150000	9.26744×10^{-6}
	0.50	0.487721	0.431283	0.501189	0.354477	0.359979	0.343663	0.300823	0.306188	0.300019	0.300000	1.92549×10^{-5}
	0.75	0.731581	0.646924	0.751784	0.531715	0.539969	0.515494	0.451234	0.459282	0.450029	0.450000	2.88822×10^{-5}
	1.0	0.975441	0.862566	1.00238	0.7089541	0.7089541	0.687326	0.601646	0.612376	0.600039	0.600000	3.85098×10^{-5}

Example 4.2. We borrow the nonlinear time-fractional hyperbolic equation[45]

$$D_t^{\alpha} U(x,t) = \frac{\partial}{\partial x}\left(U(x,t) \frac{\partial U(x,t)}{\partial x} \right), \qquad t > 0,\ x \in R,\ 1 < \alpha \le 2, \tag{4.8}$$

ubject to the initial condition

$$U(x,0) = x^2, \qquad U_t(x,0) = -2x^2. \tag{4.9}$$

Operating the Sumudu transform on both sides in equation (4.8) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$S\left[U(x,t)\right] = u^{\alpha} S\left[\frac{\partial}{\partial x}\left(U(x,t) \frac{\partial U(x,t)}{\partial x} \right) \right],$$

The nonlinear operator is

$$N\left[\phi(x,t;q)\right] = S\left[\phi(x,t;q)\right] - u^{\alpha} S\left[\frac{\partial}{\partial x}\left(\phi(x,t;q) \frac{\partial \phi(x,t;q)}{\partial x} \right) \right], \tag{4.10}$$

and thus

$$R_m(\vec{U}_{m-1}) = S\left[U_{m-1}(x,t)\right] - x^2(1-2t)(1-\chi_m) + u^{\alpha} S\left[\sum_{j=0}^{m-1} U_j(x,t)\left(U_{m-1-j}(x,t)\right)_{xx} + \left(U_j(x,t)\right)_x \left(U_{m-1-j}(x,t)\right) \right]. \tag{4.11}$$

The $m^{th}-$ order deformation equation is given by

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar H(x,t) R_m(\vec{U}_{m-1}(x,t)).$$

Applying the inverse Sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + S^{-1}\left[\hbar H(x,t) R_m(\vec{U}_{m-1}(x,t))\right]. \tag{4.12}$$

On solving above equation from $m = 1, 2, \ldots,$ we get

$$U_0(x,t) = x^2(1-2t),$$

$$U_1(x,t) = -6\hbar x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{4t^{\alpha}}{\Gamma(\alpha+2)}-\frac{8t^{\alpha}}{\Gamma(\alpha+3)}\right),$$

$$\begin{aligned} U_2(x,t) &= -6\hbar(1+\hbar)x^2\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{4t^{\alpha}}{\Gamma(\alpha+2)}+\frac{8t^{\alpha}}{\Gamma(\alpha+3)}\right) \\ &+\frac{72t^{2\alpha}x^2\hbar^2}{\Gamma(2\alpha+1)}-\frac{288t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{144t^{2\alpha+1}x^2\hbar^2\Gamma(\alpha+2)}{\Gamma(2\alpha+2)\Gamma(\alpha+1)} \\ &+\frac{576t^{2\alpha+2}x^2\hbar^2}{\Gamma(2\alpha+3)}+\frac{576t^{2\alpha+2}x^2\hbar^2\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}-\frac{1152t^{2\alpha+3}x^2\hbar^2\Gamma(\alpha+4)}{\Gamma(2\alpha+4)\Gamma(\alpha+3)}, \end{aligned} \quad (4.13)$$

etc. proceed by same manner the rest of components of the iteration can be obtained. Setting the $\hbar = -1$, in eq. (4.13) the above expressions are exactly the same as given by ADM [45].

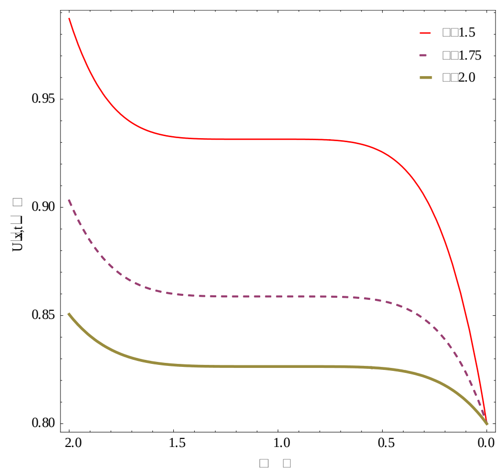

Figure 8 $\hbar$ curve for different values of α.

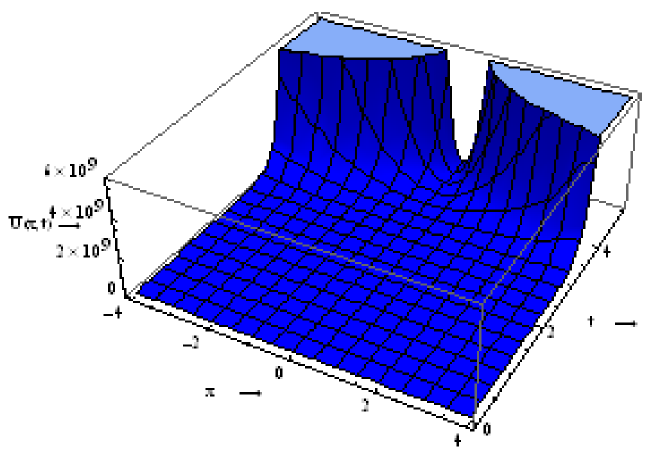

Figure 9 Plot of approximate solution for value $\alpha = 0.5$.

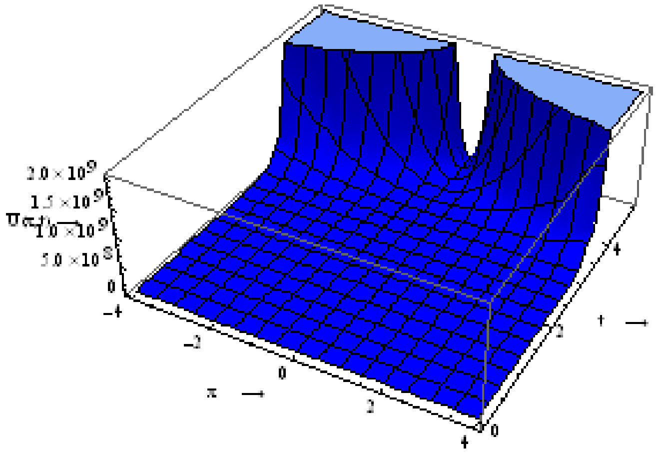

Figure 10 Plot of approximate solution for value $\alpha = 0.75$.

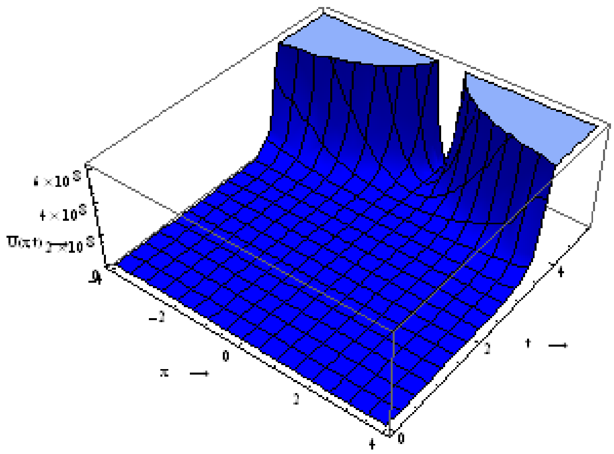

Figure 11 Plot of approximate solution for value $\alpha = 1$.

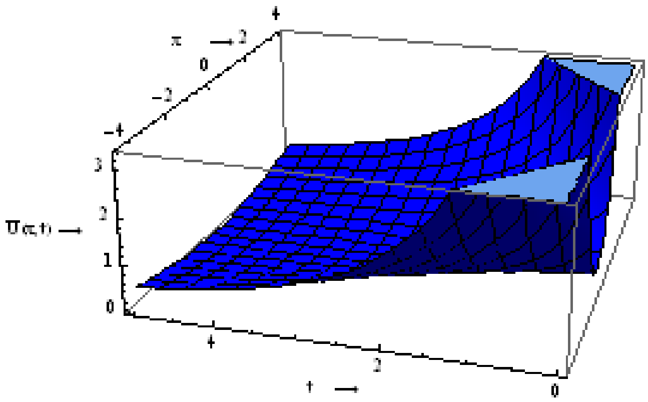

Figure 12 Plot of exact solution for value $\alpha = 1$.

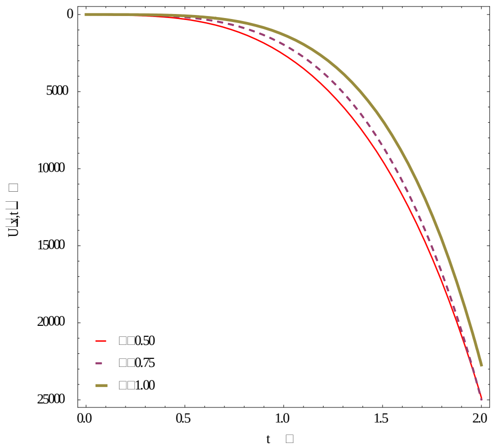

Figure 13 The behaviour of solution for different values α at $x = 1, \hbar = -1$.

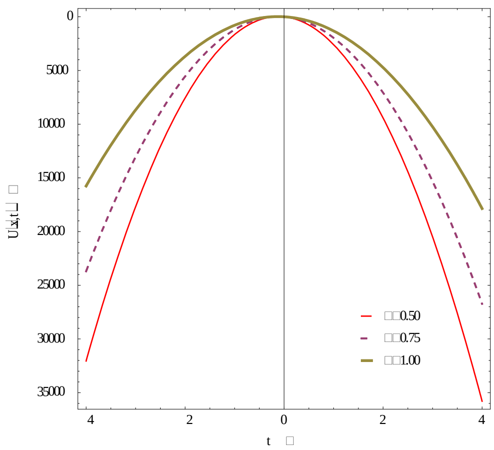

Figure 14 The behaviour of solution for different values α at $t = 1, \hbar = -1$.

Fig. 8 shows that the $\hbar$ values admissible between $-1.5 \le \hbar \le -0.5$ obtained from the fifth order solution $U(x,t)$ for different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1$. Shows approximately nearer solution to exact solution at $\alpha = 1$. Figs. 8-10 shows the behaviour of approximate solution of $U(x,t)$ for different fractional Brownian motions $\alpha = 0.5, 0.75$ and at standard motions $\alpha = 1$.

Figs. 13-14 showing the behaviour of approximate solutions at $t = 1$ and $x = 1$ respectively. It is seen that $U(x,t)$ increases very rapidly after point $t \ge 0$ and constant nature $t < 0$ Also quadratic behaviour is observe in different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1$. around origin.

Table 2 Numerical values when $\alpha = 1.5, 1.75$ and 2.0 and comparison with [45]

t	x	$\alpha = 1.5$			$\alpha = 1.75$			$\alpha = 2.0$				
		u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	Exact	$\lvert u_5 - u \rvert$
0.2	0.25	0.0592832	0.047502	0.060225	0.0497012	0.043403	0.048787	0.0433951	0.043400	0.043403	0.043403	2.10900×10^{-10}
	0.50	0.237133	0.190007	0.240900	0.194805	0.184170	0.195146	0.173580	0.173600	0.173611	0.173611	8.43600×10^{-10}
	0.75	0.533549	0.427517	0.542025	0.438311	0.414383	0.439078	0.390556	0.390600	0.390625	0.390625	1.898100×10^{-9}
	1.0	0.948532	0.760029	0.963600	0.779220	0.736680	0.780584	0.694321	0.694400	0.694444	0.694444	3.33740×10^{-9}
0.4	0.25	0.0654119	0.041853	0.081026	0.037742	0.037742	0.045918	0.031567	0.031779	0.031887	0.031888	4.08226×10^{-7}
	0.50	0.261647	0.167412	0.324099	0.174992	0.150968	0.183674	0.126268	0.127118	0.127549	0.127551	1.63291×10^{-6}
	0.75	0.588707	0.376676	0.729222	0.393732	0.339679	0.413266	0.284103	0.286015	0.286986	0.286990	3.67404×10^{-6}
	1.0	1.04659	0.669647	1.29639	0.699969	0.603873	0.734695	0.505072	0.508471	0.508471	0.508471	6.53162×10^{-6}
0.6	0.25	0.063177	0.037722	0.128961	0.381836	0.031457	0.050262	0.022005	0.023665	0.024490	0.024414	2.41791×10^{-5}
	0.50	0.252710	0.150888	0.515844	0.152735	0.125829	0.201050	0.088018	0.094660	0.097560	0.097656	9.67162×10^{-5}
	0.75	0.568598	0.339499	1.16065	0.343653	0.283114	0.452362	0.198040	0.212984	0.219509	0.219727	2.17611×10^{-4}
	1.0	1.01084	0.603553	2.06337	0.610938	0.503314	0.804199	0.352071	0.378638	0.390238	0.390625	3.86865×10^{-4}

Example 4.3. We borrow the nonlinear time-fractional Fisher's equation [45]

$$D_t^{\alpha} U(x,t) = U_{xx}(x,t) + 6U(x,t)\left(1 - U(x,t)\right), \qquad t > 0,\ x \in R,\ 0 < \alpha \le 1, \tag{4.14}$$

subject to the initial condition

$$U(x,0) = \frac{1}{\left(1+e^{x}\right)^{2}}, \tag{4.15}$$

Operating the Sumudu transform of both sides in Eq. (4.14) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$S\left[U(x,t)\right] - u^{\alpha} S\left[U_{xx}(x,t) + 6U(x,t) - 6\left(U(x,t)\right)^{2}\right] = 0,$$

The nonlinear operator is

$$N\left[\phi(x,t;q)\right] = S\left[\phi(x,t;q)\right] - u^{\alpha} S\left[\phi_{xx}(x,t;q) + 6\phi(x,t;q) - 6\left(\phi(x,t;q)\right)^{2}\right], \tag{4.16}$$

and thus

$$R_m\left(\vec{U}_{m-1}\right) = S\left[U_{m-1}(x,t)\right] - u^{\alpha} S\left[\left(U_{m-1}(x,t)\right)_{xx} + 6U_{m-1}(x,t) - 6\sum_{j=0}^{m-1} U_j(x,t)\, U_{m-1-j}(x,t)\right], \tag{4.17}$$

The $m^{th}-$ order deformation equation is given by

$$S\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] = \hbar H(x,t)\, R_m\left(\vec{U}_{m-1}(x,t)\right).$$

Applying the inverse Sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + S^{-1}\left[\hbar H(x,t)\, R_m\left(\vec{U}_{m-1}(x,t)\right)\right]. \tag{4.18}$$

On solving above equation from $m = 1, 2, \ldots,$ we get

$$U_0(x,t) = \frac{1}{(1+e^x)^2},$$

$$U_1(x,t) = \frac{1}{(1+e^x)^2} + \hbar\left(\frac{1}{(1+e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{10}{(1+e^x)^3}\right),$$

$$\begin{aligned} U_2(x,t) &= \frac{1}{(1+e^x)^2} + \hbar\left(\frac{1}{(1+e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{10}{(1+e^x)^3}\right) \\ &+ \hbar\left(\frac{1}{(1+e^x)^2} + \hbar\left(\frac{1}{(1+e^x)^2} - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{10}{(1+e^x)^3}\right)\right) \\ &- \frac{6\hbar e^{2x}t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} + \frac{2\hbar e^x t^\alpha}{(1+e^x)^3\Gamma(\alpha+1)} - \frac{6\hbar^2 e^{2x}t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} \\ &+ \frac{120\hbar^2 e^{2x}t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} - \frac{30\hbar^2 e^x t^{2\alpha}}{(1+e^x)^4\Gamma(2\alpha+1)} - \frac{6\hbar t^\alpha}{(1+e^x)^2\Gamma(\alpha+1)} \\ &- \frac{6\hbar^2 t^\alpha}{(1+e^x)^2\Gamma(\alpha+1)} - \frac{6\hbar t^{2\alpha}}{(1+e^x)^2\Gamma(\alpha+1)} - \frac{12\hbar t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} \\ &- \frac{12\hbar t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} - \frac{12\hbar^2 t^\alpha}{(1+e^x)^4\Gamma(\alpha+1)} + \frac{120\hbar^2 t^{2\alpha}}{(1+e^x)^5\Gamma(2\alpha+1)}, \end{aligned} \tag{4.19}$$

etc. proceed by same manner the rest of components of the iteration can be obtained. Setting the $\hbar = -1$, in eq. (4.19) the above expressions are exactly the same as given by ADM [45].

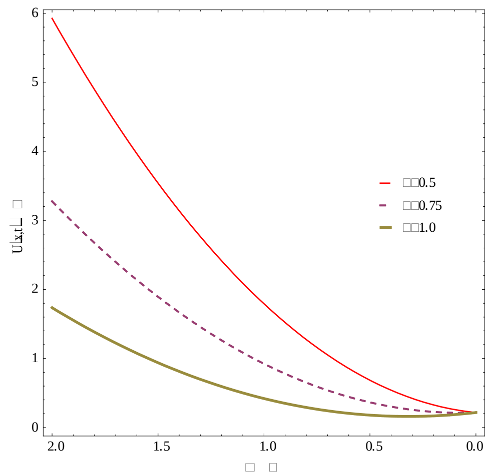

Figure 15 $\hbar$ curve for different values of α.

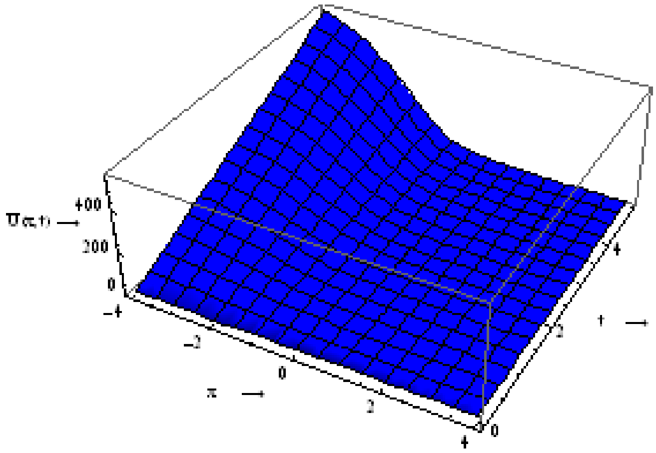

Figure 16 Plot of approximate solution for value $\alpha = 0.5$.

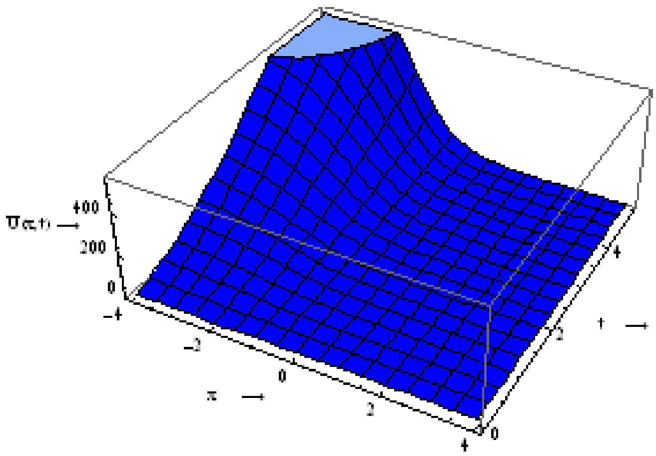

Figure 17 Plot of approximate solution for value $\alpha = 0.75$.

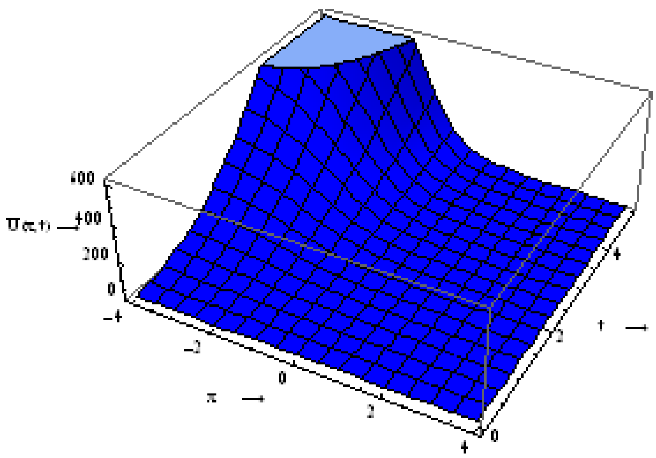

Figure 18 Plot of approximate solution for value $\alpha = 1$.

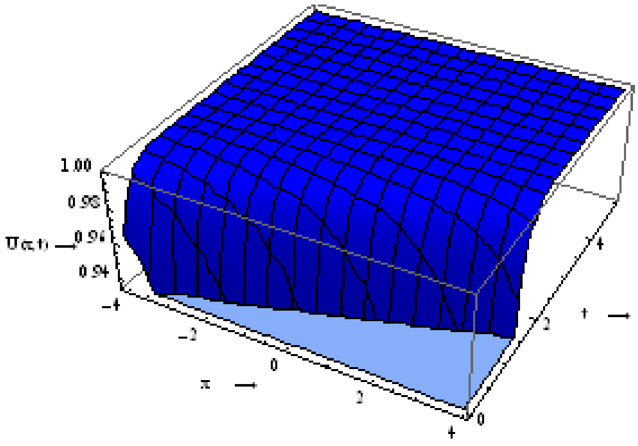

Figure 19 Plot of exact solution for value $\alpha = 1$.

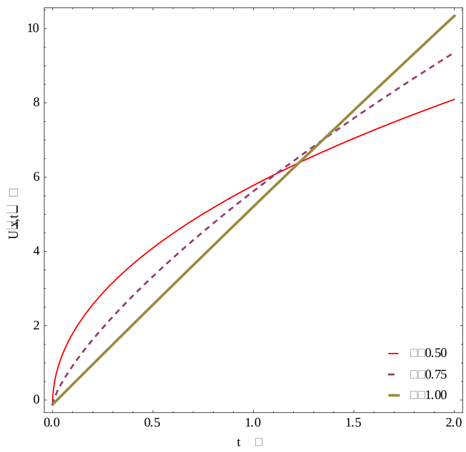

Figure 20 The behaviour of solution for different values α at $x = 1, \hbar = -1$.

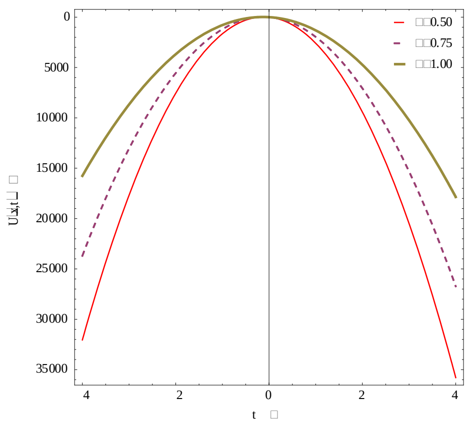

Figure 21 The behaviour of solution for different values α at $t = 1, \hbar = -1$.

Fig. 15 shows that the $\hbar$ values admissible between $-1 \le \hbar \le 0$ obtained from the fifth order solution $U \;\; x,t$ for different fractional Brownian motion $\alpha = 0.5, 0.75$ and for standard motion, that is, at $\alpha = 1.$ Figs. 16-19 shows the behaviour of approximate solution of $U \;\; x,t$ for different fractional Brownian motions $\alpha = 0.5, 0.75$ and at standard motions $\alpha = 1.$ Shows approximately nearer solution to exact solution at $\alpha = 1.$ Figs. 20-21 showing the behaviour of approximate solutions at $t = 1$ and $x = 1$ respectively.

Table 2 Numerical values when $\alpha = 0.5, 0.75$ and 1.0 and comparison with [45]

t	x	$\alpha = 0.5$			$\alpha = 0.75$			$\alpha = 1.0$				
		u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	u_{ADM} [45]	u_{VIM} [45]	u_{HASTM}	Exact	$\lvert u_9 - u \rvert$
0.1	0.25	0.946129	0.482361	0.483450	0.488195	0.412450	0.458618	0.317948	0.315940	0.316018	0.316042	2.40905×10^{-5}
	0.50	0.843908	0.394446	0.356433	0.405740	0.334514	0.390582	0.250500	0.249926	0.249982	0.250000	1.77145×10^{-5}
	0.75	0.715013	0.311106	0.367574	0.324457	0.262103	0.325749	0.190964	0.191606	0.191683	0.191689	6.11468×10^{-6}
	1.0	0.576466	0.236710	0.490698	0.249683	0.198407	0.265455	0.140979	0.142411	0.142541	0.142537	3.83664×10^{-6}
0.2	0.25	1.47532	0.746994	-0.326863	0.791250	0.617790	0.581424	0.481199	0.459320	0.459795	0.461284	1.48902×10^{-3}
	0.50	1.35983	0.653476	-1.13129	0.690142	0.536231	0.519219	0.396941	0.386450	0.386202	0.387456	1.25324×10^{-3}
	0.75	1.18098	0.548977	-0.309751	0.574404	0.448264	0.483538	0.315266	0.315478	0.315433	0.316042	6.09277×10^{-4}
	1.0	0.970076	0.441936	-0.309751	0.456647	0.359905	0.461939	0.241175	0.249092	0.250066	0.250000	6.55487×10^{-5}
0.3	0.25	1.96745	0.935741	-2.04701	1.12423	0.774999	0.445118	0.681440	0.591179	0.588679	0.60415	1.55156×10^{-2}
	0.50	1.845231	0.878473	-4.60302	1.00948	0.720112	0.322053	0.581861	0.527635	0.519763	0.534447	1.46838×10^{-2}
	0.75	1.622910	0.788974	-4.87857	0.859509	0.643697	0.355934	0.475833	0.459719	0.452525	0.461284	8.75903×10^{-3}
	1.0	1.345510	0.673844	-2.90245	0.695479	0.372917	0.495115	0.372917	0.387025	0.386067	0.387456	1.38825×10^{-3}

5. Conclusion

The new modification of homotopy analysis method is powerful tool to search the solution of various linear and nonlinear problems arising in science and engineering. The main aim of this article is to provide the approximate analytic solution of the time-fractional partial differential equation by using the HASTM. The proposed method is very efficient and easily computable. Three examples were investigated to demonstrate the ease and versatility of our new approach. The illustrative examples show that the method is easy to use and is an effective tool to solve fractional partial differential equations numerically.

References:-

[1] K. B. Oldham and J. Spanier (1974). *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order,* Academic Press, New York.

[2] Miller, K. S. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations,*Wiley, New York.

[3] Podlubny, 1. (1999). *Fractional Differential Equations*, Academic Press, San Diego.
[4] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of the Fractional Differential Equations*. North-Holland Mathematical Studies, Vol. 204, Elsevier (2006).
[5] V. Parthiban and K. Balachandran, Solutions of System of Fractional Partial Differential Equations. Applications & Applied Mathematics, vol. 8(1) 289-304(2013).
[6] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
[7] R. Caponetto, G. Dongola,, L. Fortuna, Fractional Order Systems: Modeling and Control Applications, World Scientific Publishing, Singapore vol. 72, (2010).
[8] Diethelm, K.: The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type, vol. 2004, Springer, Heidelberg (2010).
[9] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl., vol. 135, 501-544, 1988.
[10] N.T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Appl. Math. Comput., vol. 131, 517-529, 2002.
[11] S.S. Ray, R.K. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, Appl. Math. Comput., vol. 167, 561-571, 2005.
[12] H. Jafari, V. Daftardar-Gejji, Revised Adomian decomposition method for solving systems of ordinary and fractional differential equations, Appl. Math. Comput., vol. 181(1), 598-608, 2006.
[13] H. Jafari, V. Daftardar-Gejji, Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method, Appl. Math. Comput., vol. 180 (2), 700-706, 2006.
[14] A. Golbabai, M. Javidi, Application of homotopy perturbation method for solving eighth-order boundary value problems, Appl. Math. Comput., vol. 191(1), 334-346, 2007.
[15] J. H. He, A coupling method of a homotopy technique and a perturbation technique for nonlinear problems, Int. J. Non-Linear Mech., vol. 35(1), 37-43, 2000.
[16] J. H. He, The homotopy perturbation method for non-linear oscillators with discontinuities, Appl. Math. Comput., vol. 151(1), 287-292, 2004.
[17] J. H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals, vol. 26(3), 695-700, 2005.
[18] J. H. He, Asymptotology by homotopy perturbation method, Appl. Math. Comput., vol. 156 (3), 591-596, 2004.
[19] J. H. He, Homotopy perturbation method for solving boundary problems, Phys. Lett. A, vol. 350(12), 87-88, 2006.

[20] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Acad. Publ., Boston, 1994.
[21] J.H. He, Variational iteration method-a kind of nonlinear analytical technique: some examples, International Journal of Nonlinear Mechanics, vol. 34, 699-708, 1999.
[22] J.H. He, X.H. Wu, Variational iteration method: new development and applications,Computers & Mathematics with Applications, vol. 54, 881-894, 2007.
[23] J.H. He, G.C.Wu, F. Austin, The variational iteration method which should be followed, Nonlinear Science Letters A, vol. 1, 1-30, 2009.
[24] E. Hesameddini, H. Latifizadeh, An optimal choice of initial solutions in the homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 10, 1389-1398, 2009.
[25] E. Hesameddini, H. Latifizadeh, Reconstruction of variational iteration algorithms using the Laplace transform, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 10, 1377-1382, 2009.
[26] Sh. S. Behzadi, Iterative methods for solving nonlinear Fokker-Plank equation, Int. J. Industrial Mathematics, vol. 3, 143-156, 2011.
[27] Sh. S. Behzadi, Numerical solution of Sawada-Kotera equation by using iterative methods, Int. J. Industrial Mathematics, vol. 4, 269-288, 2012.
[28] M. Ghanbari, Approximate analytical solutions of fuzzy linear Fredholm integral equations by HAM, Int. J. Industrial Mathematics, vol. 4, 53-67, 2012.
[29] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[30] M. Zurigat, S. Momani, A. Alawneh, Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method, Comput. Math. Appl., vol. 59 (3), 1227-1235, 2010.
[31] Yasir Khan, An effective modification of the Laplace decomposition method for nonlinear equations, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 10,1373-1376, 2009.
[32] S.A. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, Journal of Applied Mathematics, vol. 1, 141-155, 2001.
[33] AR. Vahidi, Gh. A. Cordshooli, On the Laplace transform decomposition algorithm for solving nonlinear differential equations, Int. J. Industrial Mathematics, vol. 3, 17-23, 2011.
[34] Y. Khan, Qingbiao. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials, Computers & Mathematics with Applications, vol.61(8), 1963-1967, 2011.
[35] J. Singh, D. Kumar, Sushila, Homotopy perturbation sumudu transform method for nonlinear equations, Adv. Theor. Appl. Mech., vol. 4, 165-175, 2011.
[36] S. Kumar, A new analytical modelling for fractional telegraph equation via Laplace transform, Applied Mathematical Modelling vol. 38(13), 3154-3163, 2014.

[37] M. S. Mohamed, F. Al-malki, M. Al-humyani, Homotopy Analysis Transform Method for Time-Space Fractional Gas Dynamics Equation, Gen. Math. Notes, vol. 24(1), 1-16, 2014.

[38] S. Kumar, M. M. Rashidi, New analytical method for gas dynamics equation arising in shock fronts, Computer Physics Communications, vol. 185 (7), 1947-1954, 2014.

[39] R. K. Pandey,H. K. Mishra, Numerical simulation of time-fractional fourth order differential equations via Homotopy Analysis Fractional Sumudu Transform Method American Journal of Numerical Analysis, Vol. 3, No. 3, 52-64, 2015.

[40] Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, Acta Math. Vietnamica, 24 (1999) 207-233.

[41] O. L. Moustafa, On the Cauchy problem for some fractional order partial differential equations, Chaos Solitons& Fractals, 18 (2003) 135-140.

[42] I. Podlubny, Fractional Differential Equations, Academic, New York, 1999.

[43] G.K. Watugala, "Sumudu transform a new integral transform to solve differential equations and control engineering problems," Mathematical Engineering in Industry, vol. 6(4), 319-329, 1998.

[44] F. B. M. BELGACEM and A. A. KARABALLI, Sumudu transform fundamental properties investigations and applications, Journal of Applied Mathematics and Stochastic Analysis, Hindawi Publishing Corporation (2006), 2006, 1-23.

[45] Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Applied Mathematical Modelling Vol. 32,2008,28-39.