Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4949-4968
© Research India Publications
http://www.ripublication.com

Analysis of a Preemptive Priority Retrial Queue with two Types of Customers, Single Vacation and Service Interruption

S.Yuvarani and M.C. Saravanarajan*

School of Advanced Sciences, VIT University, Vellore, India.
Email: yuvasaran10@gmail.com and *mcsaravanarajan@vit.ac.in.*
*Corresponding Author**

Abstract

This paper concerned with performance analysis of single server preemptive priority retrial queue with single vacation where two types of customers are considered and they are called priority customers and ordinary customers**.** The priority customers do not form any queue and they have an exclusive preemptive priority to receive their services over ordinary customers. As soon as the system is empty, the server goes for vacation and the regular busy server can be subjected to breakdown. By using the supplementary variable technique, we obtain the steady state probability generating functions for the system/orbit size. Some important system performance measures and the stochastic decomposition are discussed. Finally, numerical examples are presented to visualize the effect of parameters on system performance measures.

Keywords: retrial queue; preemptive priority queue; single vacation; random breakdown; supplementary variable technique;

1. Introduction

In recent times retrial queues in queueing theory is recognized as an important research area due to many applications in several areas. Gomez-Corral [1], Artalejo and Gomez-Corral [2] and Artalejo [3] are given the general models of queueing in various aspects. Retrial queueing systems are consuming queues with repeated trials which are characterized by the fact that a new customer who finds the server busy is requested to leave the service area and join a trial group, called orbit. After some delay of time the customer in the orbit can repeat their request for service according to FCFS. An arbitrary customer in the orbit who repeats the request for service is independent of the rest of the customers in the orbit. Upon the arrival of a customer, if the server is busy or under repair or on vacation, the customer will join the orbit and try his luck again for some time later. This kind of (retrial) queue play a superior role in computer networks, telecommunication, telephone systems communication protocols, retail shopping queues, etc.

In the earlier years, retrial queues with two varieties of customers have been widely studied by many of the researchers, Artalejo et al [3], Liu et al [18], and Wang [22]. The high priority customers are formed in queue or not queue and served according to discipline of preemptive or non-preemptive. If blocked pool of customers, low priority customers (called as ordinary customers) leave the system and join the retrial group to retry its service after some time when the server is free. Moreover, in some of the systems, an arriving higher priority customer may push out the lower priority customers whose service is continuing to the queue or the orbit. In the above features, Choi and Park [7] first studied a non preemptive priority retrial queue, in which priority customers have non preemptive priority over ordinary customers and are queued in FCFS discipline. Krishna Kumar et al. [8] considered a single server retrial queueing system with two-phase service and preemptive resume. Liu and Wu [9] considered a Markovian arrival process of queues with preemptive resume, negative customers and multiple vacations which present the importance of preemptive resume in practical situations. In recent times, Wu and Lian [10] studied a single-server retrial *G*-queue with priority and unreliable server under Bernoulli vacation schedule. For a comprehensive analysis of priority queueing models, the reader may refer Liu et al. [11], Liu and Gao [12], Senthilkumar et al. [13] and Gao [14].

A vacation queueing model is considered as an extension of classical queueing system in which server may not be available for a period of time due to many reasons, like being checked for maintenance, any damage is occurred to the server or simply it is taking a break. The period of time taken to come out of the server absence is considered as a server vacation. The server vacation models are essential for the systems in which server wants to utilize the idle time for many purposes. Various authors have analyzed queueing models of server vacations with many combinations. A literature survey on queueing systems with server vacations is done by Doshi [18]. Most vacation models deals with the exhaustive policies (Doshi et al 1986) that is the system must be empty when the server starts vacation. Arivudainambi et al. [4], Krishnakumar et al. [20] have studied a single server retrial queue with general retrial times, single vacations. Recently Geo and Wang [10], Rajadurai et al. [17] have analyzed about the M/G/1 retrial queue with working vacations, balking and sever breakdown.

In most of the literature survey related to queueing theory, it is assumed that the server is available always in the system. Most of the cases, the server is assumed to be liable and always available for the customers to be served. Sometimes we come across the cases where the server may breakdown and resume its service after repair. In instance, in manufacturing systems the machine may breakdown due to Mechanical or job related problems. The computer systems may breakdown due to software related problems, like virus. Retrial queues that take into account server failures and repairs are introduced by Aissani [22], Kulkarni and choi [23], Choudhury and Deka [6] and

Rajadurai et al. [15] are discussed about the single server queue with two phases of service and the server is focus to the service interruption while providing the service to the customers. Thangaraj et al. [21] studied an M/G/1 queue with two stage heterogeneous service, compulsory vacations and random breakdowns. Retrial queues are used in many applications like real-time systems, operating systems, manufacturing system and simulations. The application of server vacation model can be found in manufacturing systems, designing of local area networks, data communication systems. Queueing systems with breakdown are very common in Manufacturing systems and computer Priority networks.
The rest of this paper is structured as follows. The detailed mathematical description of this model is given in section 2. The stability condition of the model is analysed in this section 3. In section 4, the steady state joint distribution of the server state and the number of customers in the orbit/system are obtained. Some system performance measures are discussed in section 5. The stochastic decomposition is shown good for our model in section 6. Important special cases are derived in section7. In section 8, the effects of various parameters on the system performance are analyzed numerically. Conclusion and summary of the paper are presented in section 9.

2. Description of the model

In this section, we consider a preemptive priority retrial queue with two types of customers, single vacation and service interruption. The detailed description of model is given as follows:

- ***The arrival process:*** There are two types of customers like priority customers and ordinary customers, arrives in to the system. Priority customers have preemptive priorities over ordinary customers in service time of busy server. We assume that both priority customers and ordinary customers arrive according to two independent Poisson processes with rates λ and δ, respectively.
- ***The retrial process:*** An arriving priority (or ordinary) customer finds the server is free, the customer begins its service immediately, otherwise in the arrival time of a priority customer, the server gives service for a priority customer, the newly arriving priority customer will depart the system directly without service. While the regular busy server is working with an ordinary customer, the arriving priority customer will interrupt the service of the ordinary customer and the server begins priority customer's service immediately. We assume that when an ordinary customer is preempted by a priority customer, the ordinary customer who was just being served before waits in the service area for the remaining service to complete.
- If an arriving ordinary customer finds the server is being busy or on vacation, the arrivals join pool of blocked customers called an orbit in accordance with FCFS discipline. That is, only one customer at the head of the orbit queue is allowed access to the server. Then measured from the instant the server becomes free, an external potential priority customer or ordinary customer and a retrial ordinary customer compete to enter the server. Inter-retrial times have

an arbitrary distribution $R(t)$ with corresponding Laplace Stieltjes-Transform (LST) $R^*(\vartheta)$. The retrial ordinary customer is required to give up the attempt for service if an external priority customer or ordinary customer arrives first. In that case, the retrial ordinary customer goes back to its position in the retrial queue.

- ***The vacation process:*** The server begins vacation each time when the orbit becomes empty. During the vacation period, the service time follows a general random variable V with distribution function $V(x)$. and LST $V^*(\vartheta)$ and finite k^{th} moment $v^{(k)}$ (k = 1,2).
- ***The breakdown process:*** While the server is working on any kind of service, it may breakdown at any time and the service will be stopped for a short interval of time. Let $\propto$ be the breakdown arising rate.
- ***Repair process:*** As soon as the breakdown occurs the server is sent for repair during that time it stops providing service to the customers till service channel is got repaired. The customer just being served before server breakdown will be waiting in the server, to complete the remaining service. The repair time (denoted by G) of the server is generally distributed with d.f $G(y)$, LST $G^*(\vartheta)$, and finite k^{th} moment $g^{(k)}$ (k = 1,2)
- ***The regular service process*:** In the normal busy period, there is a single server which provides regular service. The service time of priority customers follows a general distribution and denoted by the random variable S_p with distribution function $S_p(t)$, having LST $S_p^*(\vartheta)$ and the first and second moments are $\beta_p^{(1)}$ and $\beta_p^{(2)}$. The service time of ordinary customers follows a general distribution and denoted by the random variable S_b with distribution function $S_b(t)$, having LST $S_b^*(\vartheta)$ and the first and second moments are $\beta_b^{(1)}$ and $\beta_b^{(2)}$.
- Various stochastic processes involved in the system are assumed to be independent of each other.
- Throughout the rest of the paper, We denote by $\bar{F}(x) = 1 - F(x)$ the tail of distribution function $F(x)$. We also denote $F^*(s) = \int_0^\infty e^{-sx} dF(x),$ the Laplace-Stieltjes transform of $F(x)$ and $\tilde{F}(s) = \int_0^\infty e^{-sx} F(x)dx,$ to be the Laplace transform of $F(x)$ and we assume the notation $\bar{F}^*(s) = \frac{1-F^*(s)}{s}.$

3. Stability condition

In this section, we will carry out the discussion on the stability condition of the system by using embedded Markov chain technique. Let $\{t_n;\ n = 1, 2, \ldots\}$ be the sequence of epochs of the regular service completion times for priority customers, ordinary customers, a vacation period completion occurs or repair period ends. Then the state of the queueing system can be described by the bivariate Markov process $C(t), N(t);\ t \geq 0$, where $C(t)$ denotes the server state (0,1,2,3,4,5,6,7) depending on the server is free, busy on priority customers, busy on preemptive priority customers, busy on ordinary customers, on vacation, repair on priority customer, preemptive priority customer and ordinary customer. $N(t)$ denotes the number of ordinary customers in the orbit.

In addition, let $R^0(t),\ S_p^0(t),\ S_b^0(t),\ V^0(t)$ and $G^0(t)$ be the elapsed retrial time, elapsed service time of the priority customer, elapsed service time of the ordinary customer, elapsed vacation time of any customer and elapsed repair time of any customer respectively at time t. Further, we introduce the random variable,

$$C(t) = \begin{cases} 0, & \text{if the server is idle at time t} \\ 1, & \text{if the server is busy with a priority customer without preempting} \\ & \text{an ordinary customer and in regular service period at time } t, \\ 2, & \text{if the server is busy with a priority customer with preempting} \\ & \text{an ordinary customer and in regular service period at time } t, \\ 3, & \text{if the server is busy with an ordinary customer} \\ & \text{and in regular service period at time } t, \\ 4, & \text{if the server is on vacation at time } t, \\ 5, & \text{if the server is on repair at time } t, \text{when priority customer is on service} \\ 6, & \text{if the server is on repair at time } t, \text{when preemptive priority customer is on service} \\ 7, & \text{if the server is on repair at time } t, \text{when ordinary customer is on service} \end{cases}$$

If $C(t) = 0$ and $N(t) > 0$, then $R^0(t)$ represent the elapsed retrial time. If $C(t) = 1$ and $N(t) \geq 0$ then $S_p^0(t)$ corresponding to the elapsed service time of the priority customer being served in regular busy period. If $C(t) = 2$ and $N(t) \geq 0$ then $S_p^0(t)$ corresponding to the elapsed service time of the preemptive priority customer and $S_b^0(t)$ corresponding to the elapsed service time of the interrupted ordinary customer being served in regular busy period. If $C(t) = 3$ and $N(t) \geq 0$ then $S_b^0(t)$ corresponding to the elapsed service time of the ordinary customer being served in regular busy period. If $C(t) = 4$ and $N(t) \geq 0$ then $V^0(t)$ corresponding to the elapsed vacation time of any customer. If $C(t) = 5$ and $N(t) \geq 0,$ then $G^0(t)$ corresponding to the repair time of priority customer being served at that period of time. If C(t) 6 and $N(t) \geq 0,$ then $G^0(t)$ corresponding to the repair time of preemptive priority customer. If C(t) = 7 and $N(t) \geq 0,$ then $G^0(t)$ corresponding to the repair time of the ordinary customer. Then the sequence of random vectors $Z_n = C\ t_n + ,\ N\ t_n +$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 3.1: The embedded Markov chain $Z_n;\ n \in N$ is ergodic if and only if $\rho < R^*(\lambda+\delta)$, where $\rho = R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)\ (1+r)\lambda\beta_b^{(1)}\ 1+\delta\beta_p^{(1)}\ +\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}$

Proof To prove the sufficient condition of ergodicity, it is very convenient to use Foster's criterion (see Pakes [24]), which states that the chain $Z_n;\ n \in N$ is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function $f(j)$, $j \in N$ and $\varepsilon > 0$, such that mean drift $\psi_j = E\left[f(z_{n+1}) - f(z_n)/z_n = j\right]$ is finite for all $j \in N$ and $\psi_j \le -\varepsilon$ for all $j \in N$, except perhaps for a finite number j's. In our case, we consider the function $f(j) = j$. then we have

$$\psi_j = \begin{cases} \rho - 1, & \text{if } j = 0, \\ \rho - R^*(\lambda+\delta),, & \text{if } j = 1, 2 \ldots \end{cases}$$

Clearly the inequality $\rho < R^*(\lambda+\delta)$, is sufficient condition for Ergodicity.
To prove the necessary condition, As noted in Sennott et al. [25], if the Markov chain $Z_n;\ n \ge 1$ satisfies Kaplan's condition, namely, $\psi_j < \infty$ for all $j \ge 0$ and there exits $j_0 \in$ N such that $\psi_j \ge 0$ for $j \ge j_0$. Notice that, in our case, Kaplan's condition is satisfied because there is a k such that $m_{ij} = 0$ for $j < i-k$ and $i > 0$, where $M = (m_{ij})$ is the one step transition matrix of $Z_n;\ n \in N$. Then $\rho \ge R^*(\lambda+\delta)$, implies the non-Ergodicity of the Markov chain.

4. Steady state analysis of the system

In this section, we develop the steady state difference-differential equations for the retrial queueing system by treating the elapsed retrial times, the elapsed service times, the elapsed vacation time and the elapsed repair times as supplementary variables. Then we derive the probability generating function (PGF) for the server states, the PGF for number of customers in the system and orbit.
In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_p(0) = 0$, $S_p(\infty) = 1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $V(0)=0$, $V(\infty)=1$, $G(0)=0$, $G(\infty)=1$ are continuous at $x=0$. So that the functions $a(x)$, $\mu_p(x)$, $\mu_b(x)$, $\gamma(x)$ and $\xi(u)$ are the conditional completion rates (hazard rate) for retrial, service of a priority customer and ordinary customer, vacation completion rate, repair completion rate of a customer respectively.

$$i.e.,\ a(x)dx = \frac{dR(x)}{1-R(x)};\ \mu_p(x)dx = \frac{dS_p(x)}{1-S_p(x)};\ \mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)};\ \gamma(x)dx = \frac{dV(x)}{1-V(x)};\xi(u)dx = \frac{dG(x)}{1-G(x)}.$$

For the process, we define the limiting probabilities $Q_0(t) = P\ X(t) = 0,\ N(t) = 0$ and the probability densities

$P_n(x,t)dx \quad = P\{C(t)=0, N(t)=n,\ x<R^0(t)\le x+dx\}$, for $t\ge 0,\ x\ge 0$ and $n\ge 1$.

$\Pi_{1,n}(x,t)dx \quad = P\{C(t)=1, N(t)=n,\ x<S_p^0(t)\le x+dx\}$, for $t\ge 0,\ x\ge 0,\ n\ge 0$.

$\Pi_{2,n}(x,y,t)dx = P\{C(t)=2, N(t)=n,\ x<S_p^0(t)\le x+dx,\ y<S_b^0(t)\le y+dy\}$, for $t\ge 0,\ x\ge 0,\ y\ge 0,\ n\ge 0$.

$\Pi_{3,n}(x,t)dx \quad = P\{C(t)=3, N(t)=n,\ x<S_b^0(t)\le x+dx\}$, for $t\ge 0,\ x\ge 0,\ n\ge 0$.

$\Omega_n(x,t)dx \quad = P\{C(t)=4,\ N(t)=n,\ x<V^0(t)\le x+dx\}$, for $t\ge 0,\ x\ge 0$ and $n\ge 0$.

$R_{1,n}(u,x,t)dx \quad = P\{C(t)=5, N(t)=n,\ x<S_p^0(t)\le x+dx, u<G^0(t)\le u+du\}$, for $t\ge 0,\ x\ge 0, u\ge 0$ and $n\ge 0$.

$R_{2,n}(u,x,y,t)dx = P\{C(t)=6, N(t)=n,\ x<S_b^0(t)\le x+dx, y<S_p^0(t)\le y+dy, u<G^0(t)\le u+du\}$, for $t\ge 0,\ x\ge 0, y\ge 0, u\ge 0$.

$R_{3,n}(u,x,t)dx \quad = P\{C(t)=7, N(t)=n,\ x<S_b^0(t)\le x+dx, u<G^0(t)\le u+du\}$, for $t\ge 0,\ x\ge 0, u\ge 0$.

We assume that the stability condition is fulfilled in the sequel and so that we can set $P_0=\lim_{t\to\infty}P_0(t);$ and limiting densities for $t\ge 0$, $x\ge 0$ and $n\ge 1$.

$P_n(x)=\lim_{t\to\infty}P_n(x,t);\qquad \Pi_{1,n}(x)=\lim_{t\to\infty}\Pi_{1,n}(x,t);$

$\Pi_{2,n}(x,y)=\lim_{t\to\infty}\Pi_{2,n}(x,y,t);\quad \Pi_{3,n}(x)=\lim_{t\to\infty}\Pi_{3,n}(x,t)\ ;\ \Omega_n(x)=\lim_{t\to\infty}\Omega_n(x,t);$

$R_{1,n}(u,x)=\lim_{t\to\infty} R_{1,n}(u,x,t);\quad R_{2,n}(u,x,y)=\lim_{t\to\infty}R_{2,n}(u,x,y,t);\quad R_{3,n}(u,x)=\lim_{t\to\infty}R_{3,n}(u,x,t).$

4.1. The steady state equations :

By using the method of supplementary variable technique, we formulate the system of governing equations of this model as follows:

$$(\lambda+\delta)P_0=\int_0^\infty \Omega_0(x)\gamma(x)dx \tag{4.1}$$

$$\frac{dP_n(x)}{dx}+[\lambda+\delta+a(x)]P_n(x)=0,\ n\ge 1 \tag{4.2}$$

$$\frac{d\Pi_{1,0}(x)}{dx}+[\lambda+\delta+\mu_p(x)]\Pi_{1,0}(x)=0,\ n=0, \tag{4.3}$$

$$\frac{d\Pi_{1,n}(x)}{dx}+[\lambda+\delta+\mu_p(x)]\Pi_{1,n}(x)=\lambda\Pi_{1,n-1}(x)+\int_0^\infty \xi(u)R_{1,n}(u,x)\mathrm{du},\ n\ge 1 \tag{4.4}$$

$$\frac{\partial\Pi_{2,0}(x,y)}{\partial x}+[\lambda+\alpha+\mu_p(x)]\Pi_{2,0}(x,y)=0,\ n=0, \tag{4.5}$$

$$\frac{\partial\Pi_{2,n}(x,y)}{\partial x}+[\lambda+\alpha+\mu_p(x)]\Pi_{2,n}(x,y)=\lambda\Pi_{2,n-1}(x,y)+\int_0^\infty \xi(u)R_{2,n}(u,x)\mathrm{du},\ n\ge 1, \tag{4.6}$$

$$\frac{d\Pi_{3,0}(x)}{dx}+[\lambda+\delta+\mu_b(x)]\Pi_{3,0}(x)=\int_0^\infty \Pi_{2,0}(y,x)\mu_p(y)dy,\ n=0 \tag{4.7}$$

$$\frac{d\Pi_{3,n}(x)}{dx}+[\lambda+\delta+\mu_b(x)]\Pi_{3,n}(x)=\lambda\Pi_{3,n-1}(x)+\int_0^\infty \Pi_{2,n}(y,x)\mu_p(y)dy+\int_0^\infty R_{3,n}(u,x)\xi(u)du,\ n\ge 1 \tag{4.8}$$

$$\frac{d\Omega_0(x)}{dx}+[\lambda+\gamma(x)]\Omega_0(x)=0,\ n=0 \tag{4.9}$$

$$\frac{d\Omega_n(x)}{dx}+\left(\lambda+\gamma(x)\right)\Omega_n(x)=\lambda\Omega_{n-1}(x),\ n\geq 1 \tag{4.10}$$

$$\frac{\partial R_{1,n}(u,x)}{\partial x}+\left(\lambda+\xi(u)\right)R_{1,n}(u,x)=0,\ n=0 \tag{4.11}$$

$$\frac{\partial R_{1,n}(u,x)}{\partial x}+\left(\lambda+\xi(u)\right)R_{1,n}(u,x)=\lambda R_{1,n-1}(u,x),\ n\geq 1 \tag{4.12}$$

$$\frac{\partial R_{2,0}(u,x,y)}{\partial x}+(\lambda+\xi(u))R_{2,0}(u,x,y)=0, n=0 \tag{4.13}$$

$$\frac{\partial R_{2,n}(u,x,y)}{\partial x}+(\lambda+\xi(u))R_{2,n}(u,x,y)=\lambda R_{2,n-1}(u,x,y), n\geq 1 \tag{4.14}$$

$$\frac{\partial R_{3,0}(u,x)}{\partial x}+(\lambda+\xi(u))R_{3,0}(u,x)=0, n=0 \tag{4.15}$$

$$\frac{\partial R_{3,n}(u,x)}{\partial x}+(\lambda+\xi(u))R_{3,n}(u,x)=\lambda R_{3,n-1}(u,x), n\geq 1 \tag{4.16}$$

To solve the Eqns. (4.2) to (4.12), the steady state boundary conditions at x = 0 and y = 0 are followed,

$$P_n(0)=\int_0^\infty \Omega_n(x)\gamma(x)dx+\int_0^\infty \Pi_{1,n}(x)\mu_p(\mathrm{y})dy+\int_0^\infty \Pi_{3,n}(x)\mu_b(x)dx,\ n\geq 1 \tag{4.17}$$

$$\Pi_{1,n}(0)=\delta\int_0^\infty P_n(x)dx+\delta P_0,\ n\geq 0 \tag{4.18}$$

$$\Pi_{2,n}(0,x)=\delta\Pi_{3,n}(x),\ n\geq 0 \tag{4.19}$$

$$\Pi_{3,0}(0)=\int_0^\infty P_1(x)a(x)dx+\lambda P_0,\ n=0 \tag{4.20}$$

$$\Pi_{3,\mathrm{n}}(0)=\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\int_0^\infty P_n(x)dx+\lambda P_0,\ n\geq 1 \tag{4.21}$$

$$\Omega_n(0)=\int_0^\infty \Pi_{1,0}(y)\mu_p(y)dy+\int_0^\infty \Pi_{3,0}(x)\mu(x)dx,\ n\geq 0 \tag{4.22}$$

$$R_{1,n}(0,x)=\alpha\Pi_{1,n}(x), n\geq 0 \tag{4.23}$$

$$R_{2,n}(0,x,y)=\alpha\Pi_{2,n}(x,y),\quad n\geq 0 \tag{4.24}$$

$$R_{3,n}(0,x)=\alpha\Pi_{3,n}(x),\quad n\geq 0 \tag{4.25}$$

The normalizing condition is

$$P_0+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\left(\begin{array}{l}\int_0^\infty \Pi_{1,n}(x)dx+\int_0^\infty\int_0^\infty \Pi_{2,n}(x,y)dxdy+\int_0^\infty \Pi_{3,n}(x)dx+\int_0^\infty \Omega_n(x)dx+\\ \int_0^\infty\int_0^\infty R_{1,n}(u,x)dudx+\int_0^\infty\int_0^\infty R_{2,n}(u,x,y)\,\mathrm{du}\,dxdy+\int_0^\infty\int_0^\infty R_{3,n}(u,x)dxdy\end{array}\right)=1 \tag{4.26}$$

4.2. The steady state solution

The steady state solution of the retrial queueing model is obtained by using the probability generating function technique. To solve the above equations, the PGFs are defined for $|z| \leq 1$ as follows:

$$P(x,z)=\sum_{n=1}^{\infty}P_n(x)z^n;\ P(0,z)=\sum_{n=1}^{\infty}P_n(0)z^n;\ \Pi_1(x,z)=\sum_{n=0}^{\infty}\Pi_{1,n}(x)z^n;\ \ \Pi_1(0,z)=\sum_{n=0}^{\infty}\Pi_{1,n}(0)z^n;$$

$$\Pi_2(x,y,z)=\sum_{n=0}^{\infty}\Pi_{2,n}(x,y)z^n;\ \ \Pi_2(x,0,z)=\sum_{n=0}^{\infty}\Pi_{2,n}(x,0)z^n;\ \Pi_3(x,z)=\sum_{n=0}^{\infty}\Pi_{3,n}(x)z^n;\ \ \Pi_3(0,z)=\sum_{n=0}^{\infty}\Pi_{3,n}(0)z^n;$$

$$\Omega(x,z)=\sum_{n=0}^{\infty}\Omega_n(x)z^n;\ \ \Omega(0,z)=\sum_{n=0}^{\infty}\Omega_n(0)z^n;\ R_1(\mathrm{u},x,z)=\sum_{n=0}^{\infty}R_{1,n}(\mathrm{u},x)z^n;\ R_1(0,x,z)=\sum_{n=0}^{\infty}R_{1,n}(0,x)z^n;$$

$$R_2(u,x,\mathrm{y},z)=\sum_{n=0}^{\infty}R_{2,n}(u,x,y)z^n;R_2(0,x,y,z)=\sum_{n=0}^{\infty}R_2(0,x,y)z^n;R_3(u,x,z)=\sum_{n=0}^{\infty}R_{3,n}(u,x)z^n;R_3(0,x,z)=\sum_{n=0}^{\infty}R_3(0,x)z^n;$$

On multiplying the Eqns. (4.2)-(4.12) by z^n and summing over n, (n = 0, 1, 2...) and Solving the partial differential equations, we get

$$P(x,z)=P(0,z)[1-R(x)]e^{-(\lambda+\delta)x} \tag{4.27}$$

$$\Pi_1(x,z)=\Pi_1(0,z)[1-S_p(x)]e^{-A_p(z)x}, \tag{4.28}$$

$$\Pi_2(x,y,z)=\Pi_2(0,y,z)[1-S_p(x)]e^{-A_p(z)x}, \tag{4.29}$$

$$\Pi_3(x,z)=\Pi_3(0,z)[1-S_b(x)]e^{-A_b(z)x}, \tag{4.30}$$

$$\Omega(x,z)=\Omega(0,z)[1-V(x)]e^{-b(z)x}, \tag{4.31}$$

$$R_1(x,z)=R_1(0,z)[1-G(u)]e^{-b(z)u}, \tag{4.32}$$

$$R_2(u,x,y,z)=R_2(0,x,y,z)[1-G(u)]e^{-b(z)u} \tag{4.33}$$

$$R_3(u,x,z)=R_3(0,x,z)e^{-b(z)u}[1-G(u)], n\geq 0 \tag{4.34}$$

Where $A_p(z)=\lambda(1-z)+\alpha\,\mathrm{G}^*(\lambda(1-z))$, $A_b(z)=A_p(z)+\delta(1-S_p^*(A_p(z)))$ and $\mathrm{b}(z)=\lambda(1-z)$

From the Eqns. (4.17)-(4.25), we can obtain

$$P(0,z)=\int_0^{\infty}\Pi_1(x,z)\mu_p(x)dx+\int_0^{\infty}\Pi_3(x,z)\mu_b(x)dx+\int_0^{\infty}\Omega(x,z)\gamma(x)dx+\int_0^{\infty}\Omega(0,z)\gamma(x)dx-(\lambda+\delta)P_0 \tag{4.35}$$

$$\Pi_1(0,z)=\delta\int_0^{\infty}P(x,z)dx+\delta P_0, \tag{4.36}$$

$$\Pi_2(0,x,z)=\delta\Pi_3(x,z)+\delta P_0 \tag{4.37}$$

$$\Pi_3(0,z)=\left(\frac{1}{z}\int_0^{\infty}P(x,z)a(x)dx+\lambda\int_0^{\infty}P(x,z)dx+\lambda P_0\right), \tag{4.38}$$

$$\Omega(0,z)=\int_0^{\infty}\Pi_3(x,z)\mu_b(x)dx+\int_0^{\infty}\Pi_1(x,z)\mu_p(x)dx \tag{4.39}$$

$$R_1(0,x,z)=\alpha\Pi_1(x,z) \tag{4.40}$$

$$R_2(0,x,y,z)=\alpha\Pi_2(x,y,z) \tag{4.41}$$

$$R_3(0,x,z) = \alpha\Pi_3(x,z) \tag{4.42}$$

Inserting the Eqn. (4.26) in (4.35), we get

$$\Pi_1(0,z) = \delta P(0,z)\bar{R}^*(\lambda+\delta) + \delta P_0, \tag{4.43}$$

where

$$\bar{R}^*(\lambda+\delta) = \left(\frac{1-R^*(\lambda+\delta)}{\lambda+\delta}\right)$$

Inserting equation (4.27) in (4.35) and make some manipulation, finally we get,

$$\Pi_3(0,z) = \frac{P(0,z)}{z}\; R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta)\; + \lambda P_0 \tag{4.44}$$

Inserting the Eqn. (4.24) in (4.31), we get

$$\Omega(0,z) = \frac{(\lambda+\delta)}{V^*(\lambda)} P_0 \tag{4.45}$$

Using (4.28)-(4.31) in (4.35) and make some manipulation, we get

$$P(0,z) = \; \Pi_1(0,z)S_p^* \;\; A_p(z) \; + \Pi_3(0,z)S_b^* \;\; A_b(z) \; + \Omega(0,z)V^* \;\; \mathrm{b}(z) \; + \Omega(0,z) - \; \lambda+\delta \;\; Q_0 \tag{4.46}$$

Using the Eqns. (4.43)-(4.45) in (4.50), we get

$$P(0,z) = \frac{Nr(z)}{Dr(z)} \tag{4.47}$$

$$Nr(z) = \frac{zP_0}{V^*(\lambda)} \;\; (\lambda+\delta)[V^*(b(z)) - 1 - V^*(\lambda)] \; + \lambda V^*(\lambda)S_b^*(A_b(z)) + \delta V^*(\lambda)S_p^*(A_p(z))$$

$$Dr(z) = z - \; R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta) \;\; S_b^*(A_b(z)) - z\delta\bar{R}^*(\lambda+\delta) \;\; S_p^* \;\; A_p(z)$$

Using the equation (4.50) in (4.42), we get

$$\Pi_1(0,z) = \frac{\delta P_0}{V^*(\lambda)} \;\; z[V^*(b(z)) - 1] - zR^*(\lambda+\delta)[V^*(b(z)) - 1 - V^*(\lambda)] - R^*(\lambda+\delta)\mathrm{V}^*(\lambda)S_p^*(A_p(z)) \;\Big/ Dr(z) \tag{4.48}$$

Using the equation (4.51) in (4.45), we get

$$\Pi_3(0,z) = \frac{P_0}{V^*(\lambda)} \;\; R^*(\lambda+\delta) \;\; (\lambda - \lambda z + \delta)[V^*(b(z)) - 1 - V^*(\lambda)] \; + \lambda z[V^*(\mathrm{b}(z)) - 1] - \delta V^*(\lambda)S_p^*(A_p(z)) \;\Big/ Dr(z) \tag{4.49}$$

Using the equation (4.51), (4.44) in (4.46), we get

$$\Pi_2(0,x,z) = \frac{\delta P_0[1-S_b(x)]e^{-A_b(z)x}}{V^*(\lambda)} \;\; R^*(\lambda+\delta) \;\; (\lambda - \lambda z + \delta)[V^*(b(z)) - 1 - V^*(\lambda)] \; + \lambda z[V^*(b(z)) - 1] - \delta V^*(\lambda)S_p^*(A_p(z)) \;\Big/ Dr(z) \tag{4.50}$$

Using the equation (4.42) in (4.47) we get

$$R_1(0,x,z) = \frac{\alpha\delta P_0[1-S_p(x)]e^{-A_p(x)}}{V^*(\lambda)} \;\; z[V^*(b(z)) - 1] - zR^*(\lambda+\delta)[V^*(b(z)) - 1 - V^*(\lambda)] - R^*(\lambda+\delta)\mathrm{V}^*(\lambda)S_p^*(A_p(z)) \;\Big/ Dr(z) \tag{4.51}$$

Using the equation (4.43) in (4.55) we get

$$R_2(0,x,y,z) = \alpha\delta P_0[1 - S_p(x)]e^{A_p(z)y} \; [1 - S_b(x)]e^{-A_b(z)x} \left\{ \begin{matrix} R^*(\lambda+\delta) \; (\lambda - \lambda z + \delta)[V^*(b(z)) - 1 - V^*(\lambda)] \\ + \lambda z[V^*(b(z)) - 1] - \delta V^*(\lambda)S_p^*(A_p(z)) \end{matrix} \right\} \Big/ V^*(\lambda) \times Dr(z) \tag{4.52}$$

Using the equation (4.44) in (4.49) we get

$$R_3(0,x,z)=\frac{\alpha P_0[1-S_b(x)]e^{-A_b(z)x}}{V^*(\lambda)}\ R^*(\lambda+\delta)\ (\lambda-\lambda z+\delta)[V^*(b(z))-1-V^*(\lambda)]\ +\lambda z[V^*(b(z))-1]-\delta V^*(\lambda)S_p^*(A_p(z))\Big/Dr(z) \quad (4.53)$$

Using the equations (4.33) and (4.39)-(4.43) in (4.22)-(4.27), then we get the results for the following PGFs $P(x,z),\ \Pi_1(x,z),\ \Pi_2(x,y,z),\ \Pi_b(x,z),\ \Omega(x,z)$ and $Q_v(x,z).$ Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 4.1. *The marginal probability distributions of the number of customers in the orbit when server being idle, busy serving a priority customers with preempting an ordinary customer, busy serving a priority customers without preempting an ordinary customer, busy serving an ordinary customers, on vacation and repair is given by*

$$P(z)=\frac{Nr(z)}{Dr(z)} \quad (4.54)$$

$$Nr(z)=\frac{zP_0\bar{R}^*(\lambda+\delta)}{V^*(\lambda)}\quad (\lambda+\delta)[V^*(b(z))-1-V^*(\lambda)]\ +\lambda V^*(\lambda)S_b^*(A_b(z))+\delta V^*(\lambda)S_p^*(A_p(z))$$

$$Dr(z)=z-\ R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\ S_b^*(A_b(z))-z\delta\bar{R}^*(\lambda+\delta)\ S_p^*\ A_p(z)$$

$$\Pi_1(z)=\frac{\delta P_0[1-S_p^*(A_p(z))]}{V^*(\lambda)}\ z[V^*(b(z))-1]-zR^*(\lambda+\delta)[V^*(b(z))-1-V^*(\lambda)]-R^*(\lambda+\delta)V^*(\lambda)S_p^*(A_p(z))\ /\ A_p(z)\times Dr(z) \quad (4.55)$$

$$\Pi_2(z)=\frac{\delta P_0[1-S_b^*(A_b(z))][1-S_p^*(A_p(z)]}{V^*(\lambda)}\left\{\begin{matrix}R^*(\lambda+\delta)\ (\lambda-\lambda z+\delta)[V^*(b(z))-1-V^*(\lambda)]\\+\lambda z[V^*(b(z))-1]-\delta V^*(\lambda)S_p^*(A_p(z))\end{matrix}\right\}\Bigg/A_p(z)\times A_b(z)\times Dr(z) \quad (4.56)$$

$$\Pi_3(z)=\frac{P_0[1-S_b^*(A_b(z))]}{V^*(\lambda)}\ R^*(\lambda+\delta)\ (\lambda-\lambda z+\delta)[V^*(b(z))-1-V^*(\lambda)]\ +\lambda z[V^*(b(z))-1]-\delta V^*(\lambda)S_p^*(A_p(z))\Big/A_b(z)\times Dr(z) \quad (4.57)$$

$$\Omega(z)=\frac{(\lambda+\delta)P_0[1-V^*(b(z))]}{V^*(\lambda)b(z)} \quad (4.58)$$

$$R_1(z)=\alpha\delta P_0[1-S_p^*(A_p(z))][1-G^*(b(z))]\left\{\begin{matrix}z[V^*(b(z))-1]-zR^*(\lambda+\delta)[V^*(b(z))-1-V^*(\lambda)]\\-R^*(\lambda+\delta)V^*(\lambda)S_p^*(A_p(z))\end{matrix}\right\}\Bigg/V^*(\lambda)[A_p(z)xb(z)]Dr(z) \quad (4.59)$$

$$R_2(z)=\frac{\delta P_0[1-S_b^*(A_b(z))][1-S_p^*(A_p(z)][1-G^*(b(z))}{V^*(\lambda)}\left\{\begin{matrix}R^*(\lambda+\delta)\ (\lambda-\lambda z+\delta)[V^*(b(z))-1-V^*(\lambda)]\\+\lambda z[V^*(b(z))-1]-\delta V^*(\lambda)S_p^*(A_p(z))\end{matrix}\right\}\Bigg/A_p(z)\times A_b(z)\times b(z)\times Dr(z) \quad (4.60)$$

$$R_3(z)=\frac{\delta P_0[1-S_p^*(A_p(z)][1-G^*(z)]}{V^*(\lambda)}\left\{\begin{matrix}R^*(\lambda+\delta)\ (\lambda-\lambda z+\delta)[V^*(b(z))-1-V^*(\lambda)]\\+\lambda z[V^*(b(z))-1]-\delta V^*(\lambda)S_p^*(A_p(z))\end{matrix}\right\}\Bigg/A_p(z)\times b(z)\times Dr(z) \quad (4.61)$$

$$P_0 = \frac{R^*(\lambda+\delta)-\rho}{\left\{\begin{array}{l} V^*(\lambda)R^*(\lambda+\delta)\left[1+(1+\alpha g^{(1)})\ \lambda\alpha g^{(1)}\ \beta_b^{(1)}(1+\delta\beta_p^{(1)})+\delta\beta_p^{(1)}\ \right] \\ +v^{(1)}(1-R^*(\lambda+\delta))\ 1-\delta\beta_p^{(1)}\ \ \lambda(1+\alpha g^{(1)})+1\ \ -\lambda v^{(1)}\beta_b^{(1)}(1+\delta\beta_p^{(1)}) \\ +v^{(1)}R^*(\lambda+\delta)\ \ (\lambda+\delta)-\delta\ \ \beta_b^{(1)}(1+\delta\beta_p^{(1)})-\lambda(\lambda+\delta)(1+\alpha g^{(1)}) \end{array}\right\}} \tag{4.62}$$

where

$$\rho=\lambda\ 1+\alpha g^{(1)}\ \ [R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)]\beta_b^{(1)}(1+\delta\beta_p^{(1)})+\delta\bar{R}^*(\lambda+\delta)\beta_p^{(1)}\ ;\ b(z)=\lambda(1-z);$$

$$A_p(z)=\ \lambda(1-z)+\alpha(1-G^*(b(z))\ \ and\ A_b(z)=\ \lambda(1-z)+\alpha(1-G^*(b(z))+\delta\ 1-S_p^*\ \ A_p(z)$$

Proof. Integrating the Eqns. (4.44)-(4.49) with respect to *x*, we define the PGFs as, $P(z)=\int_0^\infty P(x,z)dx,\ \Pi_1(z)=\int_0^\infty \Pi_1(x,z)dx,\ \Pi_2(z)=\int_0^\infty \Pi_2(x,z)dx,\ \Pi_b(z)=\int_0^\infty \Pi_b(x,z)dx,\ \Omega(z)=\int_0^\infty \Omega(x,z)dx,$ $Q_v(z)=\int_0^\infty Q_v(x,z)dx.$

By using the normalized condition, we can be determined the probability that the server is idle (P_0). Thus, by setting $z = 1$ in (4.44)-(4.50) and applying L-Hospital's rule whenever necessary and we get $P_0+P(z)+\Pi_1(z)+\Pi_2(z)+\Pi_3(z)+R_1(z)+R_2(z)+R_3(z)+\Omega(z)=1.$

Corollary 4.1.: *The probability generating function of number of customers in the system and orbit size distribution at stationary point of time is*

$$K_s(z)=\frac{Nr_s(z)}{Dr_s(z)}=P_0+P(z)+z\ \ \Pi_1(z)+\Pi_2(z)+\Pi_3(z)+R_1(z)+R_2(z)+R_3(z)\ \ +\Omega(z) \tag{4.63}$$

$$Nr_s(z)=\frac{P_0}{V^*(\lambda)b(z)}\left\{\begin{array}{l} \lambda\ 1-z\ \ [R^*(\lambda+\delta)V^*(\lambda)(z-1)+z\delta\ 1-R^*(\lambda+\delta)\ [V^*(b(z))-1] \\ +(\lambda+\delta)(1-V^*(b(z))\times Dr_s(z)+\delta z(1-S_p^*(A_p(z))\left[\begin{array}{l} z[(V^*(b(z))-1)-R^*(\lambda+\delta)(V^*(b(z))-1-V^*(\lambda))] \\ -V^*(\lambda)R^*(\lambda+\delta)S_b^*(z)) \end{array}\right] \\ +z(1-S_b^*(A_b(z)))\left[R^*(\lambda+\delta)\ \left\{\delta+(\lambda-\lambda z)(V^*(b(z))-1-V^*(\lambda))-\delta V^*(\lambda)S_p^*(A_p(z))\right\}-\lambda z(V^*(b(z))-1)\right] \end{array}\right\}$$

$$Dr_s(z)=\left\{z-\left(R^*(\lambda+\delta)+\lambda z\bar{R}^*(\lambda+\delta)\right)S_b^*\left(A_b(z)\right)-z\delta\bar{R}^*(\lambda+\delta)\left(S_p^*\left(A_p(z)\right)\right)\right\}$$

$$K_o(z)=\frac{Nr_o(z)}{Dr_s(z)}=P_0+P(z)+\Pi_1(z)+\Pi_2(z)+\Pi_3(z)+\Omega(z)+R_1(z)+R_2(z)+R_3(z) \tag{4.64}$$

$$Nr_o(z)=\frac{P_0}{V^*(\lambda)b(z)}\left\{\begin{array}{l} \lambda\ 1-z\ \ [R^*(\lambda+\delta)V^*(\lambda)(z-1)+z\delta\ 1-R^*(\lambda+\delta)\ [V^*(b(z))-1] \\ +(\lambda+\delta)(1-V^*(b(z))\times Dr_s(z)+\delta(1-S_p^*(A_p(z))\left[\begin{array}{l} z[(V^*(b(z))-1)-R^*(\lambda+\delta)(V^*(b(z))-1-V^*(\lambda))] \\ -V^*(\lambda)R^*(\lambda+\delta)S_b^*(z)) \end{array}\right] \\ +(1-S_b^*(A_b(z)))\left[R^*(\lambda+\delta)\ \left\{\delta+(\lambda-\lambda z)(V^*(b(z))-1-V^*(\lambda))-\delta V^*(\lambda)S_p^*(A_p(z))\right\}-\lambda z(V^*(b(z))-1)\right] \end{array}\right\}$$

where P_0 is given in Eq. (4.62).

5. System Performance Measures

In this section, we derive some system probabilities, mean number of customers in the orbit/system, mean busy period and busy cycle of the model.

5.1. System state probabilities

From Eqns. (4.55)-(4.64), by setting $z \to 1$ and applying L-Hospital's rule whenever necessary, then we get the following results,

(i) The probability that the server is idle during the retrial, is given by,

$$P = P(1) = \frac{P_0 \bar{R}^*(\lambda+\delta)\ \ (\lambda+\delta)\lambda v^{(1)} + V^*(\lambda)\lambda(1+\alpha g^{(1)})(1+\delta\beta_p^{(1)})(\delta\beta_p^{(1)} + \lambda\beta_b^{(1)})}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(ii) The probability that the server is busy serving a priority customers without preempting an ordinary customer, is given by,

$$\Pi_1 = \Pi_1(1) = \frac{\delta P_0 \beta_p^{(1)}\ \ \lambda v^{(1)}(1-R^*(\lambda+\delta) + V^*(\lambda)R^*(\lambda+\delta)[1-\lambda\beta_b^{(1)}(1+\alpha g^{(1)})(1+\delta\beta_p^{(1)})]}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(iii) The probability that the server is busy serving a priority customers with preempting an ordinary customer, is given by,

$$\Pi_2 = \Pi_2(1) = \frac{\lambda\delta P_0 \beta_p^{(1)}\beta_b^{(1)}\ \ R^*(\lambda+\delta)\left[(\lambda+\delta)v^{(1)} + V^*(\lambda)(1+\delta\beta_p^{(1)})(1+\alpha g^{(1)})\right]}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(iv) The probability that the server is busy serving an ordinary customers, is given by,

$$\Pi_3 = \Pi_3(1) = \frac{P_0\lambda\beta_b^{(1)}\left[R^*(\lambda+\delta)(\delta v^{(1)} + V^*(\lambda)(1+\delta\beta_p^{(1)})(1+\alpha g^{(1)})) + \lambda v^{(1)}\right]}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(v) The probability that the server is on vacation, is given by $\Omega = \Omega(1) = \dfrac{(\lambda+\delta)P_0 v^{(1)}}{V^*(\lambda)}$

(vi) The probability that the server is on repair, when priority customer on service is given by,

$$R_1 = R_1(1) = \frac{\lambda\delta\alpha P_0 \beta_p^{(1)} g^{(1)}\ \ \lambda v^{(1)}(1-R^*(\lambda+\delta)) + V^*(\lambda)R^*(\lambda+\delta)(1-\lambda\beta_b^{(1)})(1+\alpha g^{(1)})(1+\delta\beta_p^{(1)})}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(vii) The probability that the server is on repair, when pre-emptive priority customer is on service is given by

$$R_2 = R_2(1) = \frac{\lambda\delta\alpha P_0 \beta_p^{(1)} g^{(1)}\ \ R^*(\lambda+\delta)\left[(\lambda+\delta)v^{(1)} + V^*(\lambda)(1+\delta\beta_p^{(1)})(1+\alpha g^{(1)})\right]}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

(viii) The probability that the server is on repair, when ordinary customer is on service is given by

$$R_3 = R_3(1) = \frac{\lambda \alpha P_0 \beta_b^{(1)} g^{(1)} \ \ R^*(\lambda+\delta)\left[(\lambda+\delta)v^{(1)} + V^*(\lambda)(1+\delta\beta_p^{(1)})(1+\alpha g^{(1)})\right]}{V^*(\lambda)[R^*(\lambda+\delta)-\rho]}$$

5.2. Mean system size and orbit size

If the system is in steady state condition,

(i) The expected number of customers in the orbit (L_q) is obtained by differentiating (4.64) with respect to z and evaluating at $z = 1$

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z) = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\ \ Dr_q''(1)^{\ 2}}\right]$$

$$Nr_q''(1) = -2\left\{\begin{array}{l} \lambda\left[R^*(\lambda+\delta)V^*(\lambda) - \lambda(1-R^*(\lambda+\delta)v^{(1)}\right] \\ +\lambda v^{(1)}(\lambda+\delta)\left\{\begin{array}{l}\left[1- \ R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta) \ \ \beta_b^{(1)}A'(1) - \lambda\bar{R}^*(\lambda+\delta)\right. \\ -\delta R^*(\lambda+\delta)+\lambda\delta\bar{R}^*(\lambda+\delta)\beta_p^{(1)}(1+\alpha g^{(1)})]\end{array}\right\} \\ +\lambda\delta\beta_p^{(1)}(1+\alpha g^{(1)}) \ \left\{R^*(\lambda+\delta)V^*(\lambda)+\lambda v^{(1)} \ \ 1-R^*(\lambda+\delta) \ \ +V^*(\lambda)R^*(\lambda+\delta)A'(1)\beta_p^{(1)}\right\} \\ +A'(1)\beta_b^{(1)} \ \left\{R^*(\lambda+\delta)[\lambda\delta v^{(1)}+\lambda V^*(\lambda)-\delta\lambda V^*(\lambda)(1+\alpha g^{(1)})]+\lambda^2 v^{(1)}\right\} \end{array}\right\}$$

$$Nr_q'''(1) = 3\left\{\begin{array}{l} 2\lambda v^{(1)}+\lambda^2 v^{(2)} \ \left[\lambda\beta_b^{(1)}A'(1)-\lambda \ \ 1-R^*(\lambda+\delta) \ \ [\beta_p^{(1)}(1+\alpha\,\mathrm{g}^{(1)})-1]\right] \\ + \ R^*(\lambda+\delta) - \ R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta) \ \ \beta_b^{(1)}A'(1)+\lambda\delta\bar{R}^*(\lambda+\delta)(1+\alpha\,\mathrm{g}^{(1)}) \\ +\left[\beta_b^{(2)}A'(1)^2+\beta_b^{(1)}A''(1)\right] \left\{(\lambda+\delta)[R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta)]v^{(1)} - R^*(\lambda+\delta)\left[\lambda\delta v^{(1)}+\lambda V^*(\lambda)\right]\right\} \\ -\lambda\delta\left(\beta_p^{(2)}(1+\alpha\,\mathrm{g}^{(1)})+\beta_p^{(1)}\alpha\lambda g^{(2)}\right)\left\{\bar{R}^*(\lambda+\delta)v^{(1)}+\lambda(1-R^*(\lambda+\delta))v^{(1)}+R^*(\lambda+\delta)V^*(\lambda)\right\} \\ +\lambda^2 R^*(\lambda+\delta)\beta_b^{(1)}A'(1)\left(v^{(1)}-\delta v^{(2)}\right) \end{array}\right\}$$

$$Dr_q''(1) = -2\lambda V^*(\lambda) \ \ R^*(\lambda+\delta)-\rho$$

$$Dr_q'''(1) = 3\lambda V^*(\lambda)\left(\begin{array}{l} R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta) \ \left[\lambda^2(1+\delta g^{(1)})(1+\delta\beta_p^{(1)})S_b^{(2)}+\delta\lambda g^{(2)}(1+\delta\beta_p^{(1)})(1+\delta\alpha\,\mathrm{g}^{(1)})\right] \\ +\lambda^2\bar{R}^*(\lambda+\delta)\beta_b^{(1)}(1+\alpha g^{(1)})(1+\delta\beta_p^{(1)}) - \delta\lambda^2\bar{R}^*(\lambda+\delta) \ \ 1+\delta\beta_p^{(1)} \ \ \beta_p^{(2)}+\alpha\beta_p^{(1)}g^{(1)} \end{array}\right)$$

Where,

$$A'(1) = \lambda\left[(1+\alpha g^{(1)})(1+\delta\beta_p^{(1)})\right]; \ A''(1) = \lambda^2 \ \ \alpha g^{(2)}(1+\delta\beta_p^{(1)})+\delta(1+\alpha g^{(1)})\beta_p^{(2)}$$

$$\rho = \ R^*(\lambda+\delta)+\lambda\bar{R}^*(\lambda+\delta) \ \ \lambda\beta_b^{(1)}(1+\alpha g^{(1)}) \ \ 1+\delta\beta_p^{(1)} \ \ +\delta\bar{R}^*(\lambda+\delta)\lambda\beta_p^{(1)}(1+\alpha g^{(1)});$$

(ii) The excepted number of customers in the system (L_s) is obtained by differentiating (4.50) with respect to z and evaluating at $z = 1$

$$L_s \ = \ K_s'(1) \ = \ \lim_{z\to 1}\frac{d}{dz}K_s(z) \ = \ P_0\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\ \ Dr_q''(1)^{\ 2}}\right]$$

where

$$Nr_s'''(1) = \ Nr_q'''(1) - 6\lambda^2 V^*(\lambda)\beta_b^{(1)}A'(1)\left[R^*(\lambda+\delta) \ \ \delta v^{(1)} + V^*(\lambda) \ \ \delta\beta_p^{(1)} \ \ 1+\alpha g^{(1)} \ \ -1 \ \ +\lambda v^{(1)}\right]$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) can be found by using the Little's formula

$$W_s = \frac{L_s}{\lambda} \text{ and } W_q = \frac{L_q}{\lambda}$$

6. Stochastic Decomposition

Stochastic decomposition has been widely observed among M/G/1 type queueing models with server vacations by Fuhrman and Cooper[8]. A key result in these examines the number of customers in the system in the steady state at a random point in time is distributed as the sum of two independent random variables, one of which is the number of customers in the corresponding standard queueing system at random point in time, the other-random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled. Stochastic decomposition has also been observed to hold on for some M/G/1 retrial queueing models `.

Let $K(z)$ be the stationary size distribution of M/G/1 retrial queueing system with two type of customers, single vacation and service interruptions is convolution of two independent random variables $\chi(z)$ and $\varphi(z)$.

The mathematical version of the stochastic decomposition law is $K(z) = \chi(z).\varphi(z)$ and is expressed in the form

(i) The system size distribution of M/G/1 queueing system with two type of customers, single vacation and service interruption.(Represented in the first term of $K(z)$)

(ii) The conditional distribution of the number of customers in the vacation system at random point in time given the server is idle. (Represented in second term of $K(z)$).

The number of arrivals in the vacation system at a random point in time given that the server is on vacation or idle. In fact, the second term can also obtained through the vacation definition of our system, i.e.,

$$\varphi(z) = \frac{K2(z)}{D2(z)} = (P_0 + P(z) + \Omega(z))/(P_0 + P(1) + \Omega(1))$$

$$\varphi(z) = \begin{Bmatrix} V^*(\lambda)b(z)Dr(z) + z\bar{R}^*(\lambda+\delta)\left[(\lambda+\delta)[V^*(b(z)) - 1 - V^*(\lambda)] + V^*(\lambda)[\lambda S_b^*(A(z)) + \delta S_p^*(A_p(z))\right] \\ +(\lambda+\delta)[1 - V^*(b(z))] \end{Bmatrix}$$

$$\times \; V^*(\lambda)Dr(z) + \bar{R}^*(\lambda+\delta)\left[(\lambda+\delta)\lambda v^{(1)} + V^*(\lambda) \;\; \lambda(1+\alpha g^{(1)}) \;\; \delta\beta_p^{(1)} + \lambda\beta_b^{(1)}[1+\delta\beta_p^{(1)}] \;\; \right]$$

The first term can be obtained through the without vacation definition of our system

$$\chi(z) = \frac{R^*(\lambda+\delta)S_b^* \;\; A_b(z) \;\; (z-1) \;\; 1 + \delta S_p^* \;\; A_p(z)}{z - \;\; R^*(\lambda+\delta) + \lambda z\bar{R}^*(\lambda+\delta) \;\; S_b^* \;\; A_b(z) \;\; - z\delta\bar{R}^*(\lambda+\delta) \;\; S_p^* \;\; A_p(z)}$$

From above stochastic decomposition law, we observe that $K(z) = \chi(z).\varphi(z)$ which confirm that the decomposition result of Fuhrman and Cooper [8] also valid for this special vacation system.

7. Special cases

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

***Case* (i):** *No vacation and No breakdowns*

In this case, we put $Pr\,[V = 0]=1$; $\alpha = 0$, our model can be reduced to a single server retrial queueing system with working vacations and $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{P_0 R^*(\lambda+\delta) S_b^* \;\; A_b(z) \;\; (z-1) \;\; 1+\delta S_p^* \;\; A_p(z)}{z - \;\; R^*(\lambda+\delta) + \lambda z \bar{R}^*(\lambda+\delta) \;\; S_b^* \;\; A_b(z) \;\; - z\delta \bar{R}^*(\lambda+\delta) \;\; S_p^* \;\; A_p(z)}$$

This coincides with the result of Gao [14].

***Case* (ii):** *No priority arrival and No breakdowns*

In this case, we put $\delta = \alpha = 0$, our model can be reduced to a single server retrial queueing system with single vacation and $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{\left[R^*(\lambda) - \lambda \beta_b^{(1)} \right] S_b^* \;\; A_b(z) \;\; (z-1)}{V^*(\lambda) \;\; z - \;\; R^*(\lambda) + z(1-R^*(\lambda)) \;\; S_b^* \;\; A_b(z)}$$

***Case* (iii):** *No priority arrival and No vacation*

In this case, we put $Pr\,[V = 0] =1$; $\delta = 0$, our model can be reduced to a single server retrial queueing system with breakdowns and repairs and $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{\left[R^*(\lambda) - \lambda \beta_b^{(1)} \;\; 1+\alpha g^{(1)} \;\; \right] S_b^* \;\; A_b(z) \;\; (z-1)}{z - \;\; R^*(\lambda) + z(1-R^*(\lambda)) \;\; S_b^* \;\; A_b(z)}$$

***Case* (iv):** *No priority arrival, No vacation and No breakdowns*

In this case, we put $Pr\,[V = 0] =1$; $\delta = \alpha = 0$, our model can be reduced to an M/G/1 retrial queue and $K_s(z)$ can be obtained as follows,

$$K_s(z) = \frac{\left[R^*(\lambda) - \lambda \beta_b^{(1)} \right] S_b^* \;\; \lambda - \lambda z \;\; (z-1)}{z - \;\; R^*(\lambda) + z(1-R^*(\lambda)) \;\; S_b^* \;\; \lambda - \lambda z}$$

***Case* (v):** *No priority arrival, No retrial; No vacation and No breakdowns*

In this case, we put $R^*(\lambda) \to 1$; $Pr\,[V = 0] =1$; $\delta = \alpha = 0$, our model can be reduced to an M/G/1 queue and $K_s(z)$ can be obtained as follows,

$$K_s(z)=\frac{\left[1-\lambda\beta_b^{(1)}\right]S_b^{*}\ \ \lambda-\lambda z\ \ (z-1)}{z-S_b^{*}\ \ \lambda-\lambda z}$$

8. Numerical examples

In this section, we present some numerical examples to study the effect of various parameters in the system performance measures of our system where all retrial times, service times, vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied. MATLAB software has been used to illustrate the results numerically. Where the exponential distribution is $f(x)=\upsilon e^{-\upsilon x}, x>0$, Erlang-2 stage distribution is $f(x)=\upsilon^2 x e^{-\upsilon x}, x>0$ and hyper-exponential distribution is $f(x)=c\upsilon e^{-\upsilon x}+(1-c)\upsilon^2 e^{-\upsilon^2 x}, x>0.$

For the effect of the parameters λ, a, δ, ξ and γ on the system performance measures, two and three dimensional graphs are illustrated in Figure 1-Figure 6. Figure 1 and Figure 3 show that the idle probability P_0 increases for the increasing values of the retrial rate (a) and repair rate (ξ). Figure 2 shows that the probability that the server is idle during retrial time (P) is increasing for the increasing values of the arrival rate (λ). Figure 4 shows that the probability that the mean vacation time (Ω) is decreasing for the increasing values of the vacation rate (γ). In Figure 5, the surface displays an upward trend as expected for increasing value of arrival rate (λ) and priority arrival rate (δ) against the mean orbit size (L_q). Figure 6 shows that the probability that server is idle (P_0) increases for increasing the value of repair rate (ξ) and vacation rate (γ).

From the numerical examples, we can find the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

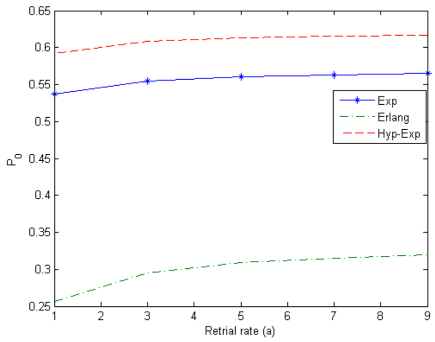

Figure 1. P_0 versus a

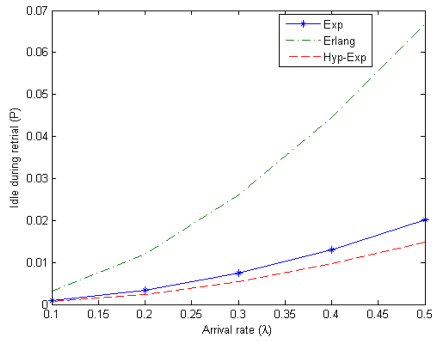

Figure 2. P versus λ

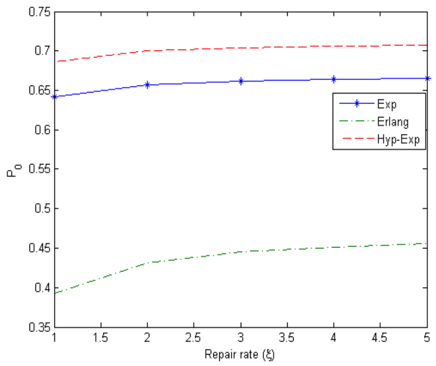

Figure 3. P_0 **versus** ξ

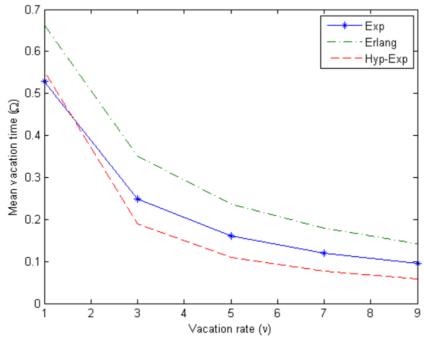

Figure 4. Ω **versus** γ

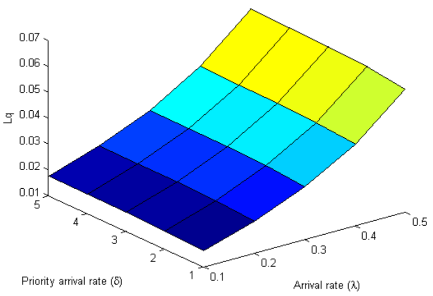

Figure 5. L_q **versus** λ **and** δ

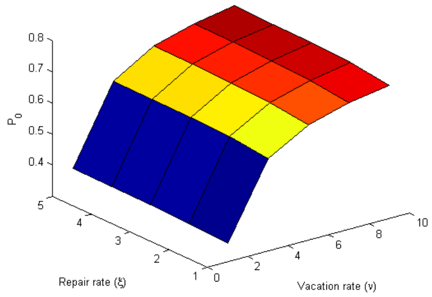

Figure 6. P_0 **versus** ξ **and** γ

9. Conclusion

In this paper, we have analyzed a single server retrial queueing system with single vacation and service interruption, where the server is subject to breakdown and repair. Using the method of supplementary variable technique, the probability generating functions for the numbers of customers in the system when it is free, busy, on vacations and under repairs is found. Some important system performance measures and stochastic decomposition law are discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, the analytical results are validated with the help of numerical illustrations.

References

[1] Artalejo J.R. and Gomez-Corral A. 2008, “Retrial queueing systems: a computational approach”. Springer, Berlin.

[2] Artalejo, J.R. 2010, “Accessible bibliography on retrial queues: progress in 2000-2009”. *Math Computer Model*, 51, pp. 1071-1081

[3] Artalejo, J.R, Dudin, A.N. and Klimenok, V. I. 2001, "Stationary analysis of a retrial queue with preemptive repeated attempts", *Operations Research Letters*, 28, pp. 173-180.

[4] Arivudainambi, D, Godhandaraman, P. and Rajadurai, P. 2014, "Performance analysis of a single server retrial queue with working vacation", *OPSEARCH*, 51, pp. 434-462.

[5] Choi, B.D. and Chang, Y, 1999, "Single Server Retrial Queues with Priority Calls", *Mathematical and Computer Modelling*, 30, pp. 7-32

[6] Choudhury, G. and Deka, K., 2010, "A Batch arrival retrial queueing system with two phases of service subject to the server breakdown and repair". *Performance Evaluation*, 65, pp. 714-724.

[7] Doshi, B.T. 1986, "Queueing systems with vacations-a survey". *Queuing Syst.* 1, pp. 29-66.

[8] Fuhrmann, S.W., and Cooper, R.B., 1985, "Stochastic decompositions in the M/G/1 queue with general retrial times", *Operations Research*, 33, pp. 1117-1129.

[9] Gomez-Corral A. 2006. "A bibliographical guide to analysis of retrial queues through matrix analytic techniques". *Ann Oper Res*, 141, pp. 163-191

[10] Gao S, Wang J, and Li W. 2014, "An M/G/1 retrial queue with general retrial times, working vacations and vacation interruption". *Asia-Pacific J Oper Res*. 31, pp. 6-31.

[11] Gao S. 2015. "A preemptive priority retrial queue with two classes of customers and general retrial times", *Oper Res Int J*. DOI 10.1007/s12351-015-0175-z

[12] Gomez-correl, A. 1999, "Stochastic analysis of a single server retrial queue with general retrial times", *Naval Res.Logist.* 46(5), pp. 561-581

[13] Krishnakumar, B., and Arivudainambi, D. 2002. "The M/G/1 retrial queue with Bernoulli Schedule and general retrial times". *Computers and Mathematics with Applications*, 43, 15-30.

[14] Krishnakumar, B, Vijaykumar, A and Arivudainambi, D. 2002, "An *M/G/*1 retrial queueing system with two phase service and preemptive resume", *Annals of Operations Research*. 113, pp. 61-79.

[15] Rajadurai, P., Chandrasekaran, V.M. and Saravanarajan, M.C. 2015, "Steady state analysis of batch arrival feedback retrial queue with two phases of service, negative customers, Bernoulli vacation and server breakdown". *Int. J. Math. Oper. Res.* 7(5), pp. 519-546.

[16] Rajadurai, P., Varalakshmi, M., Saravanarajan, M.C. and Chandrasekaran, V.M. 2015. "Analysis of M[X]/G/1 retrial queue with two phase service under Bernoulli vacation schedule and random breakdown", *Int. J. Math. Oper. Res*, 7(1), pp. 19-41.

[17] Rajadurai P., Saravanarajan, M.C. and Chandrasekaran, V.M. 2015. "Analysis of M/G/1 retrial queue with balking, negative customers, working vacation, and server breakdown", *International Journal of Applied Engineering Research*, 10(55), pp. 4130-4136.

[18] Liu, Z and Wu, J. 2009, "An MAP/G/1 G-queue with preemptive resume and multiple vacations", *Applied Mathematical Modeling*. 33, pp. 1738-1748

[19] Liu, Z and Gao, S. 2011. "Discrete-time Geo/Geo/1 retrial queue with two classes of customers and feedback". *Math Comput Model*, 53, pp. 1208-1220

[20] Senthilkumar, M, Chakravarthy, S.R. and Arumuganathan R. 2013. "Preemptive resume priority retrial queue with two classes of MAP arrivals", *Applied Mathematical Sciences,* 7, pp. 2569-2589

[21] Thangaraj, V. and Vanitha, S. 2010. "M/G/1 queue with two stage heterogeneous service, compulsory vacation and random breakdowns", *Int, J. Contemp. Math. Sciences*, 5(7), pp. 307-322.

[22] Wang J. 2008. "On the single server retrial queue with priority subscribers and server breakdowns", *Jrl Syst Sci & Complexity*, 21, pp. 304-315

[23] Wu, J. and Lian, W. 2013. "A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule", *Computers & Industrial Engineering*, 64, pp. 84-93

[24] Zhang, M. and Hou, Z. 2012. "M/G/1 queue with single working vacation", *Journal of Appl Math Comput*. 39, pp. 221-234.