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New Exact Solutions for the Generalized Combined Fractional mKdV, KdV Partial Differentional Equations with Variable Coefficients Using a new Fractional Sub-equation Method

Hussam E. Hashim[1] and Sherif El Bendary [2, 3]

[1]Faculty of Applied Medical Science, Medical Records, Taif University, Taif, El-Haweiah, P. O. Box 888, Zip Code 21974, Kingdom of Saudi Arabia (KSA)
E-mail: hosame590@gmail. com
[2]Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, El-Haweiah, P. O. Box 888, Zip Code 21974, Kingdom of Saudi Arabia (KSA)
[3]Department of Mathematics, Faculty of Science, Tant university, Egypt.

Abstract

The main objective of this paper is to apply the new Fractional Sub-equation Method of $\left(\frac{G^{'}}{G}\right)$-Expansion method to establish new exact solutions for the generalized combined fractional mKdV, KdV partial differentional equations with variable coefficients. As a result, new traveling wave solutions including hyperbolic trigonometric function obtained. Our solutions can be viewed as a generalization to the results which found in some recent published papers. Our solutions can be written in the form of infinite series, which make our solutions are advanced more than the other solutions which found in some recent published papers.

Keywords: Combined KdV and mKdV equation, Generalized $\left(\frac{G^{'}}{G}\right)$ -expansion method, Traveling wave solutions.

1. 1Introduction

Phenomena in physics and other fields are often described by nonlinear evolution equations(NLEEs). When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics [1, 2], plasma and elastic media [3, 4] and optical fibers [5, 6], etc. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been proposed, such as Hirota's bilinear method [7], Backlund transformation [8], Painlevé expansion [9], sine-cosine method [10], homogeneous balance method [11], homotopy perturbation method [12-14], variational iteration method [15-18], asymptotic methods [19], non-perturbative methods [20], Adomian decomposition method [21], tanh-function method [22-26], algebraic method [27-30], Jacobi elliptic function expansion method [31-33], F-expansion method [34-36] and auxiliary equation method [37-40]. Recently, Wang et al. [41] introduced a new direct method called the $\left(\frac{G'}{G}\right)$-expansion method to look for travelling wave solutions of NLEEs. Consider the fractional generalized mKdV and KdV partial differentional equation

$$D_t^{\alpha}u + a(t)\,D_x^{\alpha}u + b(t)\,u\,D_x^{\alpha}u + c(t)\,u^2\,D_x^{\alpha}u + e(t)\,D_x^{3\alpha}u = 0, \qquad (1.\ 1)$$

where $a(t), b(t), c(t)$ and $e(t)$ all are functions of t. Eq. (1. 1) ($a(t) = 0$ and $b(t), c(t), e(t)$ all constants) has been widely used in many physical fields such as plasma physics, fluid physics, solid-state physics and quantum field theory. When $a(t) = c(t) = 0$, and $b(t), e(t)$ are constants Eq. (1. 1) becomes fractional KdV equation. When $a(t) = b(t) = 0$, and $c(t), e(t)$ are constants Eq. (1. 1) is fractional mKdV equation.. The KdV equation and mKdV equation had been studied by many authors. Recently, Zhang [42] obtained some exact solutions of Eq. (1. 1) ($a(t) = 0$, and $b(t), c(t), e(t)$ are constants) by tanh function method and the direct method. Recently Eq. (1. 1) has solved by Zhenya [43] using a generalized approch based on Riccati equation when and $a(t), b(t), c(t), e(t)$ are all constants. In this paper we try to solve Eq. (1. 1) using generalized $\left(\frac{G'}{G}\right)$-expansion method when $a(t), b(t), c(t)$ and $e(t)$ all are functions of t. The $\left(\frac{G'}{G}\right)$-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (LODE): $G'' + \lambda G' + \mu G = 0,$ where $G' = \frac{dG(\xi)}{d\xi}, G'' = \frac{d^2G(\xi)}{d\xi^2}, \xi = x - Vt$, V is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the $\left(\frac{G'}{G}\right)$-expansion method, Wang et al. [41] successfully obtained more travelling wave solutions of four NLEEs. Very recently, Zhang et al. [44] proposed a generalized $\left(\frac{G'}{G}\right)$-expansion method to improve the work made in [41]. The main purpose of this paper is to we investigate the generalized $\left(\frac{G'}{G}\right)$-expansion method for construct an explicit Exact traveling wave solutions of fractional generalized mKdV and KdV partial differentional equation $D_t^{\alpha}u + a(t)D_x^{\alpha}u + b(t)uD_x^{\alpha}u + c(t)u^2D_x^{\alpha}u + e(t)D_x^{3\alpha}u = 0,$

where $a(t), b(t), c(t)$ and $e(t)$ all are functions of $t, 0<\alpha\le 1.$ where $G = G(\xi)$ satisfies a second order linear differential equation $G'' + \lambda G' + \mu G = 0$, $\xi(t) = p(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)} + q(t)$, where $p(t)$, $q(t)$ are functions of *t*. The performance of this method is reliable, simple and gives many new solutions, its also standard and computerizable method which enable us to solve complicated nonlinear evolution equations in mathematical physics. The paper is organized as follows. In Section 2, we describe briefly the generalized $\left(\frac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$, $\xi(t) = p(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)} + q(t)$. In section 3, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. In section 4, we give the constructions of the fractal index method. In Section 5, we apply this method to the generalized MkdV equation. In section 6, some conclusions are given.

2. 1Description The Generalized $\left(\frac{G'}{G}\right)$-Expansion Method

Suppose that we have the following nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{2. 1}$$

we suppose its solution can be expressed by a polynomial $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=1}^{n} \alpha_i(t)\left(\frac{G'}{G}\right)^i + \alpha_0(t), \qquad \alpha_j(t) \neq 0, \tag{2. 2}$$

where $\alpha_0(t)$ and $\alpha_j(t)$ are functions of t $(j = 1, 2, ..., n)$ and $\xi = \xi(x, t)$ is a function of x, t to be determine later, $G = G(\xi)$ satisfies the second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \tag{2. 3}$$

To determine u explicitly we take the following four steps.

Step 1. Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (2. 1).

Step 2. Substitute Eq. (2. 2) along with Eq. (2. 3) into Eq. (2. 1) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left handside of Eq. (2. 1) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Then set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for $\alpha_0(t)$, $\alpha_i(t)$ and ξ.

Step 3. Solve the system of all equations obtained in step 2 for $\alpha_0(t)$, $\alpha_i(t)$ and ξ by use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of Eq. (2. 3) depending on $\left(\frac{G'}{G}\right)$, since the solutions of this equation have been well known for us, then we can obtain exact solutions of Eq. (2. 1).

3. 1Preliminaries and Notation

In this section, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. For more details see [1]. For the finite derivative in $[a, b]$, we define the following fractional integral and derivatives.

Definition 3. 1 *A real function* $f(x), x>0$, *is said to be in the space* $C\mu, \mu \in R$, *if there exists a real number (* $p>\mu$ *) such that* $f(x)=x^p f_1(x)$, *where* $f_1(x)\in C(0,\infty)$, *and it is said to be in the space* $C_\mu^m\ \mu$ *if* $f^m \in C\mu$, $m\in N$.

Definition 3. 2*The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$ *of a function* $f\in C\mu, \mu\geq -1$, *is defined as*

$$J^\alpha\ x = \frac{1}{\Gamma\ \alpha}\int_0^x x-t^{\ \alpha-1} f\ t\ dt, \alpha>0, x>0, J^0\ x = f\ x .$$

Properties of the operator J^α can be found in [1]; we mention only the following:
For $f\in C\mu, \mu\geq -1, \alpha,\beta\geq 0$ and $\gamma>-1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by M. Caputo in his work on the theory of viscoelasticity [1].

Definition 3. 3*For* m *to be the smallest integer that exceeds* α, *the Caputo time fractional derivative operator of order* $\alpha>0$ *is defined as*

$$D_t^\alpha f\ \ x = \frac{\partial^\alpha u\ \ x,t}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma\ \ m-\alpha}\displaystyle\int_0^x x-t^{\ \ m-\alpha-1} f^{\ m}\ \ t\ \ dt, & \text{for } m-1<\alpha\leq m, m\in N \\ \dfrac{\partial^m u\ \ x,t}{\partial t^m}, & \text{for } \alpha=m. \end{cases}$$

4. 1Fractal Index Method

To understanding the fractional complex transform. Consider a plane with fractal structure (see Fig. 4. 1). The shortest path between two points is not a line and we have

$$ds_E = kds^\alpha, \tag{4. 1}$$

where is the actual distance between two points A and B(the green curve in Fig. 4. 1), isthe line distance between two points (the red line in Fig. 4. 1), is the fractal dimension and k is a constant.

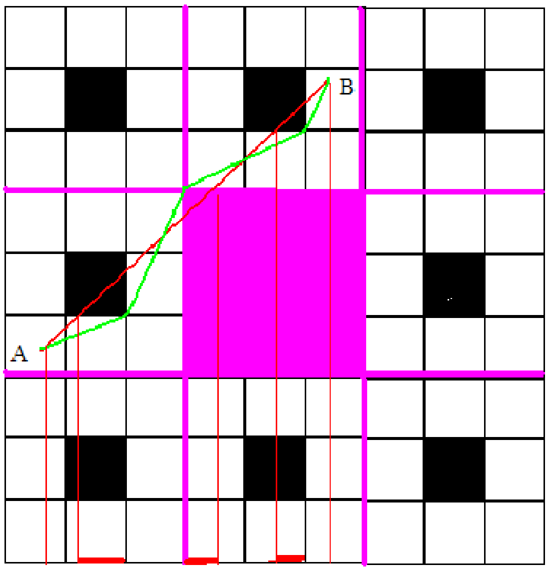

Fig. 4. 1 The distance between two points in a discontinuous space.

Projection the (the green curve) into horizontal direction yields Canton-like sets, and its length can be expressed as

$$\Delta_? AB = k_? ?^{\alpha}, \qquad (4.\ 2)$$

where are the fractal dimensions of the Canton-like sets in the horizontal direction, is a constant Eq. (4. 2) means the following transform

$$s_E = ks^{\alpha},$$

This idea leads to the fractional complex transform the fractal curve AB in Fig. 4. 1 is projected to Cantor-like sets in horizontal direction. From Fig. 4. 1, we have

$$\Delta_? AB = \cos\theta \, ds_E, \qquad (4.\ 3)$$

or

$$\Delta_? AB = \frac{dx}{ds} ds_E \qquad (4.\ 4)$$

where is the slope angle of straight line AB. From the relations Eqs. (4. 2) and (4. 4), we have

$$k_? d\,?^{\alpha} = k \frac{dx}{ds} ds^{\alpha}$$

or

$$?^{\alpha} = \frac{k}{k_?} \frac{d\,?}{ds} ds^{\alpha} = \sigma \frac{d\,?}{ds} ds$$

where and so called the fractal index, therefore, we have the following chain rule forfractional calculus

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \sigma \frac{\partial u}{\partial s} \frac{\partial^{\alpha} s}{\partial t^{\alpha}}$$

5. 1The generalized Combined Fractional mKdV, KdV Partial Differentional Equations

In this section, we apply the generalized $\left(\frac{G'}{G}\right)$-expansion method to solve the fractional generalized mKdV and KdV partial differentional equation, construct the traveling wave solutions for it as follows:

Let us first consider the following fractional generalized mKdV and KdV partial differentional equation

$$D_t^{\alpha}u + a(t)D_x^{\alpha}u + b(t)uD_x^{\alpha}u + c(t)u^2D_x^{\alpha}u + e(t)D_x^{3\alpha}u = 0, \tag{5. 1}$$

where $a(t), b(t), c(t)$ and $e(t)$ all are functions of t. There is no any method gave the exact solution of the above equation before. In order to look for the traveling wave solution of Eq. (5. 1) we suppose that

$$u(x,t) = u(\xi), \xi(t) = p(t)\frac{x^{\alpha}}{\Gamma(1+\alpha)} + q(t) \tag{5. 2}$$

By using the the chain rule $D_t^{\alpha}u = \sigma_t' \frac{du}{d\xi}D_t^{\alpha}\xi$ and $D_x^{\alpha}u = \sigma_x' \frac{du}{d\xi}D_t^{\alpha}\xi$, where σ_t' and σ_x' are called the fractal indexes (See section 3) for details see [16], without loss of generality we can take $\sigma_x' = \sigma_t' = l$, where l is a constant by using the definition of Capatu Derivative and the above modefied chain rule, equation (5. 1) convert to the ordinary differential equation

$$u_t + a(t)u_x + b(t)uu_x + c(t)u^2u_x + e(t)u_{xxx} = 0, \tag{5. 3}$$

Suppose that the solution of Eq. (5. 1) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows

$$u(\xi) = \sum_{i=1}^{n}\alpha_i(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^i + \alpha_0(t) \tag{5. 4}$$

considering the homogeneous balance between u_{xxx} and u^2u_x in Eq. (5. 1) we required that $n+3 = 2n+n+1$, then $n=1$. So we try to find a solution of the form

$$u(t,x) = \alpha_0(t) + \alpha_1(t)\frac{G'(\xi)}{G(\xi)}, \tag{5. 5}$$

where G satisfies

$$G'' + \lambda G' + \mu G = 0.$$

It is easy to see that $p(t)$ must be a constant function assuming that $\alpha_1(t)$ is not zero on any interval of positive length. Substituting Eq. (5. 4) into Eq. (5. 1) along with Eq.

(5. 3) and comparing ng coefficients of $\left(\frac{G'}{G}\right)^k, k = 0,1,2,3,4$ we obtain the following equations

$$\alpha_0' = \alpha_1\mu\, q' + ap + b\alpha_0 p + c\alpha_0^2 p + ep^3\ 2\mu + \lambda^2 \quad ; \tag{5. 6}$$

$$\alpha_1' = \alpha_1\ q'\lambda + ap\lambda + b\alpha_0 p\lambda + b\alpha_1 p\mu + c\alpha_0^2 p\lambda + 2c\alpha_0\alpha_1 p\mu + ep^3\lambda\ 8\mu + \lambda^2 \quad ; \tag{5. 7}$$

$$-\alpha_1 q' = \alpha_1 p\ a + b\alpha_0 + b\alpha_1\lambda + c\alpha_0^2 + 2c\alpha_0\alpha_1\lambda + c\alpha_1^2\mu + ep^2\ 8\mu + 7\lambda^2 \quad ; \tag{5. 8}$$

$$-\alpha_1 p\ b\alpha_1 + 2c\alpha_0\alpha_1 + c\alpha_1^2\lambda + 12ep^2\lambda\ = 0; \tag{5. 9}$$

$$-\alpha_1 p\ c\alpha_1^2 + 6ep^2\ = 0. \tag{5. 10}$$

We solve Eq. (5. 9) for α_1, Eq. (5. 8) for α_0 and Eq. (5. 7) for q'. We obtain (choosing one solution of Eq. (5. 9))

$$\alpha_1 = p\sqrt{\frac{-6e}{c}}; \tag{5. 11}$$

$$\alpha_0 = -\frac{b}{2c} + \frac{1}{2}p\lambda\sqrt{\frac{-6e}{c}}; \tag{5. 12}$$

$$q' = \frac{p}{4c}\ -4ac + b^2 + 2cep^2\lambda^2 - 8ep^2\mu c\ . \tag{5. 13}$$

Now we substitute Eq. (5. 9), Eq. (5. 10), Eq. (5. 11) into Eq. (5. 6) and obtain $\alpha_1' = 0$ implies that

$$e\ t\ = rc\ t\ , \quad \text{where } r \text{ is aconstant.} \tag{5. 14}$$

we substitute Eq. (5. 9), Eq. (5. 10), Eq. (5. 11) into Eq. (5. 5) and obtain $\alpha_0' = 0$ implies that

$$b\ t\ = sc\ t\ , \quad \text{where } s \text{ is aconstant.} \tag{5. 16}$$

Therefore, the solution of the Eq. (5. 5), Eq. (5. 6), Eq. (5. 7), Eq. (5. 8), Eq. (5. 9) is as follows. We must assume Eq. (5. 13) and Eq. (5. 14) otherwise there is no solution. Then $q\ t$ is obtained from Eq. (5. 12) by

$$q'\ t\ = \frac{p}{4}\ -4a\ t\ + s^2 c\ t\ + 2rp^2\lambda^2 c\ t\ - 8rp^2\mu c\ t\ . \tag{5. 17}$$

Moreover, α_0 and α_1 are constant functions

$$\alpha_0\ t\ = \frac{1}{2}\ -s + \sqrt{-6rp}\,\lambda \tag{5. 18}$$

and

$$\alpha_1\ t\ = \sqrt{-6rp}. \tag{5. 19}$$

As an example, take

$p = 1, r = -1, s = 1, \lambda = 0, \mu = -1$

and

$c(t) = t, \quad a(t) = t^2, \quad G(\xi) = \cosh\xi.$

Then

$\alpha_0 = -\frac{1}{2}, \quad \alpha_1 = \sqrt{6}, \quad q(t) = -\frac{1}{3}t^3 - \frac{7}{8}t^2.$

We obtain that

$$u(t,x) = -\frac{1}{2} + \sqrt{6}\tanh(\xi), \qquad (5.\ 20)$$

with

$$\xi(t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{1}{3}t^3 - \frac{7}{8}t^2$$

is a solution of equation Eq. (5. 1). One can check with the computer that u given by Eq. (5. 18) is really a solution of Eq. (5. 1).

6. 1 Conclusions

This study shows that the generalized $\left(\frac{G'}{G}\right)$-expansion method is quite efficient and practically will suited for use in finding exact solutions for the problem considered here. New and more general excat solutions for any arbitray functions $a(t), b(t), c(t)$ and $e(t)$ are obtained, there is no any method before, gave any exact solution for this equation. Also we construct an innovative explicit traveling wave solutions involving parameters of the generalized combined KdVand mKdV equation.

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