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Some Aspects of $ij - \delta$ Semi open sets in Bitopological Spaces

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Abstract

In this paper, the authors study and introduce the concepts of $ij - \delta$ semi open sets, $ij - \delta s$ continuous functions, $ij - \delta s$ irresolute functions in bitopological Spaces and investigate some of their properties. Also we study and investigate the following separation axioms in bitopological spaces: $ij - \delta s - T_i$ and $ij - \delta s - R_i$ for $i = 0, 1, 2.$

Key words: $ij - \delta$ semi open sets, $ij - \delta s$ continuous function, $ij - \delta s$ irresolute function, $ij - \delta s - T_0$, $ij - \delta s - T_1$, $ij - \delta s - T_2$, $ij - \delta s - R_0$, $P - \delta s - R_1$ spaces.

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1. Introduction

In recent year, there are large numbers of research papers have been done to generalize the topological concepts to bitopological setting. The first step of semi open sets in bitopological spaces was done by Maheshwari and Prasad [18] in 1977. Palaniappan and Pious Missier [21] introduced and studied the concept of $\delta -$ semi open sets in bitopological spaces. Separation axioms Tk and Rk in bitopological spaces was introduced and studied in some manners in [1]. Khedr [12] introduced and study the separation axioms pairwise $\delta - R_0$, pairwise $\delta - R_1$ and $ij - \delta - T_k$ spaces, k = 0, 1, 2. The aim of this paper is to continue the study of the above way by introducing a kind of $ij - \delta$ semi open sets in bitopological spaces and we study the basic properties of this concept. Also we study the separation axioms based on such sets in bitopological spaces and investigate some of their properties.

2. Preliminaries

Throughout the present paper, (X, τ_1, τ_2) (or briefly X) always mean a bitopological space on which no separation axioms are assumed unless explicitly stated. Also $i, j = 1,2$ and $i \neq j$. Let A be a subset of (X, τ_1, τ_2). By $i - Int(A)$ and $i - Cl(A)$, we mean respectively the interior and the closure of A in the topological space (X, τ_i) for $i = 1,2$. A subset A of X is called $ij -$ semi open[11,12] (resp. $ij -$ regular open) if $A \subseteq j - cl[i - int(A)]$ (resp. $A = i - int[j - cl(A)]$). A point x of X is called an $ij - \delta -$ cluster point of A if $i - Int(j - Cl(U)) \cap A \neq \emptyset$ for every $\tau_i -$ open set U containing x. The set of all $ij - \delta -$ cluster points of A is called the $ij - \delta -$ closure of A and is denoted by $ij - \delta Cl(A)$. A point x of X is called an $ij - \theta -$ cluster point of A [2] if $A \cap j - Cl(U) \neq \emptyset$ for every $\tau_i -$ open set U containing x. The set of all $ij - \theta -$ cluster points of A is called the $ij - \theta -$ closure of A and is denoted by $ij - \theta Cl(A)$.

Definition 2.1 [12] A subset A is said to be $ij - \delta$ closed if $ij - \delta Cl(A) = A$. The complement of an $ij - \delta$ closed set is said to be $ij - \delta$ open. The set of all $ij - \delta$ open (resp. $ij - \delta$ closed) sets of X will be denoted by $ij - \delta$O(X)(resp. $ij - \delta$C(X)).

Definition 2.2 [12] A subset A is said to be $ij - \theta$ closed if $ij - \theta Cl(A) = A$. The complement of an $ij - \theta$ closed set is said to be $ij - \theta$ open. The set of all $ij - \theta$ open (resp. $ij - \theta$ closed) sets of X will be denoted by $ij - \theta$O(X)(resp. $ij - \theta$C(X)).

Definition 2.3[15] A function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $ij - \delta$ continuous, if $f^{-1}(V)$ is $ij - \delta$ open set in X for every $ij -$ regular open set V of Y.

Definition 2.4[16] A function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $ij -$ super continuous, if $f^{-1}(V)$ is $ij - \delta$ open set in X for every $i -$ open set V of Y.

3. Properties of $ij - \delta$ semi open sets

Definition 3.1 A subset A of a bitopological space (X, τ_1, τ_2) is called $ij - \delta$ semi open if there exists an $ij - \delta$ open set U such that $U \subseteq A \subseteq j - Cl(U)$. The complement of $ij-\delta$ semi open is called $ij-\delta$ semi closed. The family of $ij-\delta$ semi open (resp. $ij-\delta$ semi closed) set of X is denoted by $ij-\delta SO(X)$ (resp. $ij-\delta SC(X)$).

Example 3.2 Let $X=\{a,b,c,d\}$,$\tau_1=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$, and $\tau_2=\{\emptyset,X,\{c\},\{d\},\{c,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$, Then $12-\delta$ semi open sets are $\{a,b\},\{a,b,c\},\{a,b,d\}$ and $21-\delta$ semi open sets are $\{c\},\{d\},\{c,d\}$.

Definition 3.3

(a) The union of all $ij-\delta$ semi open sets contained in a subset A of a bitopological space (X,τ_1,τ_2) is called $ij-\delta$ semi interior of A. It is denoted by $ij-\delta sint(A)$.

(b) The intersection of all $ij-\delta$ semi closed sets containing a subset A of a bitopological space (X,τ_1,τ_2) is called $ij-\delta$ semi closure of A. It is denoted by $ij-\delta scl(A)$.

Theorem 3.4 For every subset A of a bitopological space (X,τ_1,τ_2), we have the following

(a) $X\backslash ij-\delta sint(A)=ij-\delta scl(X\backslash A)$

(b) $X\backslash ij-\delta scl(A)=ij-\delta sint(X\backslash A)$

(c) $ij-\delta sint(A)\cap ij-\delta sint(B)=ij-\delta sint(A\cap B)$

Proof.

(a) Let $x\notin ij-\delta scl(X\backslash A)$, then there exists $U\in ij-\delta SO(X)$ containing x such that $U\cap(X\backslash A)=\emptyset$. Thus $x\in U\subseteq A$ and $x\in ij-\delta sint(A)$. Hence $x\notin X\backslash ij-\delta sint(A)$. Now, let $x\notin X\backslash ij-\delta sint(A)$. Thus $x\in ij-\delta sint(A)$ and there exists $U\in ij-\delta SO(X)$ such that $x\in U\subseteq A$. Hence $U\cap(X\backslash A)=\emptyset$ and $x\notin ij-\delta scl(X\backslash A)$.

(b) The proof is similar to that of (a).

(c) Since $A\subset B$, then $ij-\delta sint(A)\subset ij-\delta sint(B)$. So $A\cap B\subset A$ implies that $ij-\delta sint(A\cap B)\subset ij-\delta sint(A)$ and $A\cap B\subset B$ implies that $ij-\delta sint(A\cap B)\subset ij-\delta sint(B)$. Now, let $x\in ij-\delta sint(A)\cap ij-\delta sint(B)$. Thus $x\in ij-\delta sint(A)$ and $x\in ij-\delta sint(B)$. Then $x\in A$ and $x\in B$. Hence $x\in ij-\delta sint(A\cap B)$ and $ij-\delta sint(A)\cap ij-\delta sint(B)=ij-\delta sint(A\cap B)$.

Theorem 3.5 Every $ij-\delta$ open set is $ij-\delta$ semi open set.

Proof. Obvious.

Remark 3.6 The converse of the theorem 3.5 needs not be true in general as is seen from the following example.

Example 3.7 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, c, d\}\}$, and $\tau_2 = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$, Then by $12 - \delta$ semi open sets are $\{a, b\}, \{a, b, c\}$ and $21 - \delta$ semi open sets are $\{d\}, \{c, d\}$. Here $\{a, b, c\}$ is $12 - \delta$ semi open set but not $12 - \delta$ open set and $\{c, d\}$ is $21 - \delta$ semi open set but not $21 - \delta$ open set.

Theorem 3.8 A subset A of a bitopological space (X, τ_1, τ_2) is called $ij - \delta$ semi open if and only if $A \subseteq j - cl[ij - \delta int(A)]$.

Proof. Let $A \subseteq j - cl[ij - \delta int(A)]$. Then for $U = ij - \delta int(A)$, we have $U \subseteq A \subseteq j - Cl(U)$. Therefore A is $ij - \delta$ semi open set.
Conversely, Let A be a $ij - \delta$ semi open set. Then $U \subseteq A \subseteq j - Cl(U)$, for some $ij - \delta$ open set U. But $U \subseteq ij - \delta int(A)$ and thus $j - Cl(U) \subseteq j - cl[ij - \delta int(A)$. Hence $A \subseteq j - Cl(U) \subseteq j - cl[ij - \delta int(A)$.

Theorem 3.9 A subset $\{A\alpha\}\alpha \in I$ be a collection of $ij - \delta$ semi open in a bitopological space (X, τ_1, τ_2). Then $\bigcup_{\alpha \in I} A_\alpha$ is $ij - \delta$ semi open.

Proof. For each $\alpha \in I$, we have an Uα, such that . Then $\bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} j - cl(U_\alpha) \subseteq j - cl(\bigcup_{\alpha \in I} U_\alpha)$. Hence $U = \bigcup_{\alpha \in I} U_\alpha$ is $ij - \delta$ semi open.

Theorem 3.10 If A is a $ji - \delta$ closed subset of a bitopological space (X, τ_1, τ_2) , then A is $ij - \delta$ semi open.

Proof. Obvious.

Theorem 3.11 Let A be a $ij - \delta$ semi open set of a bitopological space (X, τ_1, τ_2) and suppose $A \subseteq B \subseteq j - cl(A)$. Then B is $ij - \delta$ semi open.

Proof. There exists an $ij - \delta$ open set U such that $U \subseteq A \subseteq j - Cl(U)$. Then $U \subseteq B$. But $j - Cl(A) \subseteq j - Cl(U)$ and thus $B \subseteq j - Cl(U)$. Hence $U \subseteq B \subseteq j - Cl(U)$. Therefore B is $ij - \delta$ semi open.

Theorem 3.12 If A and B are $ij - \delta$ semi open sets in a bitopological space (X, τ_1, τ_2), then A $\cap$ B is also $ij - \delta$ semi open set.

Proof. Suppose that A and B are $ij-\delta$ semi open sets. Let U be $ij-\delta$ open and U $\subseteq$ A $\cap$ B. Since U $\subseteq$ A $\cap$ B, we have U $\subseteq$ A and U $\subseteq$ B. Then U $\subseteq$ $j-$cl(A) and U $\subseteq$ $j-$cl(B), since A and B are $ij-\delta$ semi open sets.

$\Rightarrow$ U $\cap$ U $\subseteq$ ($j-$cl(A)) $\cap$ ($j-$cl(B))

$\Rightarrow$ U $\subseteq$ $j-$cl(A$\cap$B)

Hence A$\cap$B is $ij-\delta$ semi open set.

Theorem 3.13 For any subset A of a bitopological space (X,τ_1,τ_2), A $\cap$ $j-cl[ij-\delta int(A)]$ is $ij-\delta$ semi open and A $\cup$ $j-int[ij-\delta cl(A)]$ is $ij-\delta$ semi closed set.

Proof. $j-cl[ij-\delta int(A\cap j-cl[ij-\delta int(A)])]$

$= j-cl[ij-\delta int(A)\cap ij-\delta int(j-cl[ij-\delta int(A)])]$

$= j-cl[ij-\delta int(A)]$

$A\cap j-cl[ij-\delta int(A)]$

$=$ A $\cap$ $j-cl[ij-\delta int(A\cap j-cl[ij-\delta int(A)])]\subseteq j-cl[ij-\delta int(A\cap j-cl[ij-$
δint

$\Rightarrow A\cap j-cl[ij-\delta int(A)]$ is $ij-\delta$ semi open.

This implies that,

$X\setminus[A\cup j-int[ij-\delta cl(A)]] = (X\setminus A)\cap j-cl[ij-\delta int(X\setminus A)]$

is $ij-\delta$ semi open.

This proves that $A\cup j-int[ij-\delta cl(A)]$ is $ij-\delta$ semi closed set.

Theorem 3.14 The intersection of a $ij-\delta$ semi open and a $ij-\delta$ open is always $ij-\delta$ semi open.

Proof. Suppose that A is $ij-\delta$ semi open and B is $ij-\delta$ open set. Since every $ij-\delta$ open set is $ij-\delta$ semi open. Also since intersection of two $ij-\delta$ semi open is $ij-\delta$ semi open. Therefore we have, A $\cap$ B is $ij-\delta$ semi open.

Theorem 3.15 The union of a $ij-\delta$ semi open and a $ij-\delta$ open is always $ij-\delta$ semi open.

Proof. Suppose that A is $ij-\delta$ semi open and B is $ij-\delta$ open set. Since every $ij-\delta$ open set is $ij-\delta$ semi open, B is $ij-\delta$ semi open. Also since union of two $ij-\delta$ semi open is $ij-\delta$ semi open. Therefore we have, A $\cup$ B is $ij-\delta$ semi open.

Theorem 3.16 Let A be a subset of Y and Y be a subspace of X. If Y $\in ij-\delta O(X)$ and A $\in ij-\delta SO(Y)$, then A $\in ij-\delta SO(X)$.

Proof. Suppose that A $\subset$ Y $\subset$ X, where A is a $ij-\delta$ semi open in Y and Y is a $ij-\delta$ open set in X. Let U $\subseteq$ A and U is $i_Y j_Y-\delta$ open set in Y. Since Y $\subset$ X, we have U $\subset$ X. This implies that A $\subset$ X. Therefore A is $ij-\delta$ semi open set in X.

Theorem 3.17 Let A and B be the subsets of bitopological spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) respectively. If A $\in$ $ij-\delta SO(X)$ and B $\in$ $ij-\delta SO(Y)$, then $A \times B \in ij-\delta SO(X \times Y)$.

Proof. Let (x, y) $\in$ A x B, then x $\in$ A and y $\in$ B. Since A $\in$ $ij-\delta SO(X)$ and B $\in$ $ij-\delta SO(Y)$, then there exists $ij-\delta$ open set U in X and $ij-\delta$ open set V in Y such that U $\subseteq$ A and V $\subseteq$ B. Therefore $(x,y) \in U \times V \subseteq A \times B$. Since $\times B = ij-\delta int_x(A) \times ij-\delta int_y(B) = ij-\delta \mathrm{int}_{x\times y}(\mathrm{A} \times \mathrm{B})$. Hence $A \times B \in ij-\delta$ open set in $X \times Y$. Since every $ij-\delta$ open set is $ij-\delta$ semi open set. Therefore $A \times B \in ij-\delta$ semi open set in $X \times Y$.

Theorem 3.18 Let A and B be the subsets of bitopological spaces (X,τ_1,τ_2) and (Y,σ_1,σ_2) respectively. If A $\in$ $ij-\delta SC(X)$ and B $\in$ $ij-\delta SC(Y)$, then A x B $\in$ $ij-\delta SC(X \text{ x } Y)$.

Proof. The proof is similar to that of theorem 3.17.

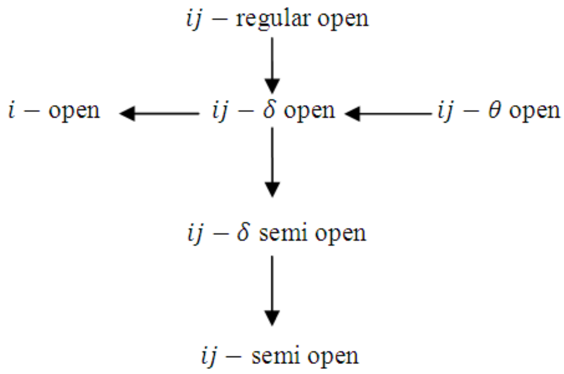
Theorem 3.19 Every $ij-\delta$ semi open in (X,τ_1,τ_2) is $\tau_{1s}\tau_{2s}$ $-$semi open in (X,τ_{1s},τ_{2s}).

Proof. Obvious.

Theorem 3.20 Every $ij-\delta$ semi open set in (X,τ_1,τ_2) is $ij-$ semi open set in (X,τ_1,τ_2).

Proof. Obvious.

Remark 3.21 The following implication shows the relationship among some bitopological sets,



4. $ij-\delta s$ Continuous and Irresolute functions

Definition 4.1 A function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be $ij-\delta s$ continuous, if $f^{-1}(V)$ is $ij-\delta$ semi open set in X for every $\sigma_i -$ open set V in Y.

Example 4.2 Let $X=\{a,b,c\}$, $\tau_1=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$, $\tau_2=\{\emptyset,X,\{c\},\{d\},\{c,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$,$\sigma_1=\{\emptyset,X,\{a\},\{a,b\},\{a,c,d\}\}$, $\sigma_2=\{\emptyset,X,\{a\},\{d\},\{a,d\}\}$. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a function defined by $f(a)=b$, $f(b)=a$, $f(c)=c$. Then f is $ij-\delta s$ continuous.

Theorem 4.3 Every $ij-\delta$ continuous function is $ij-\delta s$ continuous.

Proof. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be $ij-\delta$ continuous function. Let U be a $ij-$ regular open set in Y. Then $f^{-1}(U)$ is $ij-\delta$ open set in X. Since every $ij-\delta$ open set is $ij-\delta$ semi open, we have f is $ij-\delta s$ continuous.

Theorem 4.4 Every $ij-$ super continuous function is $ij-\delta s$ continuous.
Proof. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be $ij-$ super continuous function. Let U be a $i-$ open set in Y. Then $f^{-1}(U)$ is $ij-\delta$ open set in X. Since every $ij-\delta$ open set is $ij-\delta$ semi open, we have f is $ij-\delta s$ continuous.

Theorem 4.5 Every $ij-\delta s$ continuous function is $ij-$ semi continuous.

Proof. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a $ij-\delta s$ continuous function. Let V be a $\sigma_i -$ open set in Y. Since f is $ij-\delta s$ continuous, then $f^{-1}(V)$ is $ij-\delta$ semi open set in X. Since every $ij-\delta$ semi open set is $ij-$ semi open set. Therefore $f^{-1}(V)$ is $ij-$ semi open set in X. Thus f is $ij-$ semi continuous.

Remark 4.6 The notions of $ij-\delta s$ continuous and $ij-$ continuous are independent.

Theorem 4.7 The following are equivalent for a function is $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$,

(a) f is $ij - \delta s$ continuous.

(b) $f^{-1}(U)$ is $ij - \delta$ semi closed for each σ_i $-$ closed set U in Y.

Proof. (a) $\Rightarrow$ (b) Suppose that f is $ij - \delta s$ continuous. Let A be a σ_i $-$ closed in Y. Then A^C is σ_i $-$ open in Y. Since f is $ij - \delta s$ continuous, we have $f^{-1}(A^C)$ is $ij - \delta$ semi open in X. Consequently, $f^{-1}(A)$ is $ij - \delta$ semi closed in X.
(b) $\Rightarrow$ (a) Suppose that $f^{-1}(U)$ is $ij - \delta$ semi closed for each σ_i $-$ closed set U in Y. Let V be σ_i $-$ open set in Y. Then V^C is σ_i $-$ closed in Y. Therefore by our assumption, $f^{-1}(V^C)$ is $ij - \delta$ semi closed in X. Hence $f^{-1}(V)$ is $ij - \delta$ semi open in X.

Theorem 4.8 The following property holds for a function $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, if f is $ij - \delta s$ continuous, then $f: (X, \tau_{1_s}, \tau_{2_s}) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_{1_s}\tau_{2_s}$ $-$ semi continuous.

Proof. Suppose that f is $ij - \delta s$ continuous. Let U be σ_i $-$ open set in Y. Then $f^{-1}(U)$ is $ij - \delta$ semi open in (X, τ_1, τ_2). But every $ij - \delta$ semi open in (X, τ_1, τ_2) is $\tau_{1_s}\tau_{2_s}$ $-$semi open in $(X, \tau_{1_s}, \tau_{2_s})$. Therefore $f^{-1}(U)$ is $\tau_{1_s}\tau_{2_s}$ $-$semi open in $(X, \tau_{1_s}, \tau_{2_s})$. Thus $f: (X, \tau_{1_s}, \tau_{2_s}) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_{1_s}\tau_{2_s}$ $-$ semi continuous.

Theorem 4.9 Let $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function and $g : (X, \tau_1, \tau_2) \longrightarrow (X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$ be the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is $ij - \delta s$ continuous if and only if f is $ij - \delta s$ continuous.

Proof. Necessary Part: Let $x \in X$ and V be a σ_1 $-$ open set in Y containing $f(x)$. Then, we have $(x) = (x, f(x)) \in X \times V$, $X \times V$ is $\tau_1 \times \sigma_1 -$ open set in $X \times Y$ containing $g(x)$. Since g is $ij - \delta s$ continuous, there exists a $ij - \delta$ semi open U in X containing x, such that $g(U) \subset X \times V$ and $f(U) \subset V$. Thus f is $ij - \delta s$ continuous.
Sufficient Part: Let $x \in X$ and W be $\tau_1 \times \sigma_1 -$ open set in XxY containing $g(x)$. There exist τ_1 $-$ open set U_1 and σ_1 $-$ open set V such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is $ij - \delta s$ continuous, there exists $ij - \delta$ semi open U_2 in X such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then $x \in U$ is $ij - \delta$ semi open set in X and $g(U) \subset U_1 \times V \subset W$. Thus g is $ij - \delta s$ continuous.

Definition 4.10 A function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is said to be $ij-\delta s$ irresolute, if $f^{-1}(V)$ is $ij-\delta$ semi open set in X for every $ij-\delta$ semi open set V in Y.

Example 4.11 Let $X=\{a,b,c\}$, $\tau_1=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$, $\tau_2=\{\emptyset,X,\{c\},\{d\}\{c,d\},\{a,b\},\{a,b,c\},\{a,b,d\}\}$,$\sigma_1=\{\emptyset,X,\{a\},\{a,b\},\{a,c,d\}\}$, $\sigma_2=\{\emptyset,X,\{a\},\{d\},\{a,d\}\}$. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a function defined by $f(a)=b$, $f(b)=a$, $f(c)=c$. Then f is $ij-\delta s$ irresolute.

Theorem 4.12 The following are equivalent for a function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$,

(a) f is $ij-\delta s$ irresolute.

(b) For each $x\in X$ and each $ij-\delta$ semi open set V in Y containing $f(x)$, there exists a $ij-\delta$ semi open set U of x such that $f(U)\subset V$.

(c) $f^{-1}(V)\subset\ j-cl(ij-\delta int(f^{-1}(V)))$ for every $ij-\delta$ semi open set V in Y.

(d) $f^{-1}(F)$ is $ij-\delta$ semi closed in X for every $ij-\delta$ semi closed set F in Y.

Proof.

(a) $\Rightarrow$ (b) Let $x\in X$ and V be any $ij-\delta$ semi open set V in Y. Since f is $ij-\delta s$ irresolute, $U=f^{-1}(V)$ is $ij-\delta$ semi open in X containing x and $f(U)\subset V$.

(b) $\Rightarrow$ (c) Let V be any $ij-\delta$ semi open set V in Y and $x\in f^{-1}(V)$. By (b), there exists a $ij-\delta$ semi open set U in X containing x such that $f(U)\subset V$. Therefore, we have $x\in U\subset\ j-cl(ij-\delta int(U))\subset\ j-cl(ij-\delta int(f^{-1}(V)))$ and hence $f^{-1}(V)\subset\ j-cl(ij-\delta int(f^{-1}(V)))$.

(c) $\Rightarrow$ (d) Let F be any $ij-\delta$ semi closed subset of Y. Set V = Y-F, then V is $ij-\delta$ semi open in Y. By (c), we have $f^{-1}(V)\subset\ j-cl(ij-\delta int(f^{-1}(V)))$ and hence $f^{-1}(F)=X-(f^{-1}(Y-F))=X-f^{-1}(V)$ is $ij-\delta$ semi closed in X.

(d) $\Rightarrow$ (a) Let V be any $ij-\delta$ semi closed subset of Y. Since $f^{-1}(Y-V)=X-f^{-1}(V)$ is a subset of X. By (d), we have there exists a $ij-\delta$ semi closed in X, such that ,

$$X-j-cl\left(ij-\delta int(f^{-1}(V))\right)$$
$$=j-int(ij-\delta cl(X-f^{-1}(V)))$$
$$=j-int(ij-\delta cl(f^{-1}(Y-V))$$
$$\subset\ f^{-1}(f(j-int(ij-\delta cl(f^{-1}(Y-V))))$$
$$\subset\ f^{-1}(Y-V)=X-f^{-1}(V).$$

Therefore $f^{-1}(V) \subset j-cl(ij-\delta int(f^{-1}(V)))$. Hence $f^{-1}(V)$ is $ij-\delta$ semi open in X. Thus f is $ij-\delta s$ irresolute.

Theorem 4.13 Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ be a function and $g:(X,\tau_1,\tau_2) \longrightarrow (X\times Y,\tau_1\times\sigma_1,\tau_2\times\sigma_2)$ be the graph function defined by $g(x)=(x,f(x))$ for every $x\in X$. Then f is $ij-\delta s$ irresolute if g is $ij-\delta s$ irresolute.

Proof. Let $x\in X$ and V be any $ij-\delta$ semi open set in Y containing $f(x)$. Then, we have $(x)=(x,f(x))\in X\times V$, $X\times V$ is $ij-\delta$ semi open set in XxY containing $g(x)$. Since g is $ij-\delta s$ irresolute, there exists a $ij-\delta$ semi open U in X containing x, such that $g(U)\subset X\times V$ and $(U)\subset V$. Thus f is $ij-\delta s$ irresolute.

Theorem 4.14 If $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is $ij-\delta s$ irresolute and A is a $ij-\delta$ open subset of X, then the restriction $f_{/A}: A\longrightarrow Y$ is $ij-\delta s$ irresolute.

Proof. Let V be any $ij-\delta$ semi open set in Y. Since f is $ij-\delta s$ irresolute, then $f^{-1}(V)$ is $ij-\delta$ semi open in X. Since A is $ij-\delta$ open in X, $(f_{/A})^{-1}(V)=A\cap f^{-1}(V)$ is $ij-\delta$ semi open in X. Hence $f_{/A}$ is $ij-\delta s$ irresolute.

Theorem 4.15 If a function $f:X\longrightarrow \prod Y_\alpha$ is a $ij-\delta s$ irresolute, then $P_\alpha\circ f: X\longrightarrow Y_\alpha$ is $ij-\delta s$ irresolute, for each $\alpha\in\Lambda$, where P_α is the projection of $\prod Y_\alpha$ onto Y_α.

Proof. Let V_α be any $ij-\delta$ semi open in Y_α. Since P_α is continuous and open, it is pairwise irresolute and hence $P_\alpha^{-1}(V_\alpha)$ is $ij-\delta$ semi open in Y_α. Since f is $ij-\delta s$ irresolute, then $f^{-1}\left(P_\alpha^{-1}(V_\alpha)\right)=(P_\alpha\circ f)^{-1}(V_\alpha)$ is $ij-\delta$ semi open set in X. Hence $P_\alpha\circ f$ is $ij-\delta s$ irresolute for each $\alpha\in\Lambda$.

Theorem 4.16 Let $\{(X_\alpha,\tau_{1\alpha},\tau_{2\alpha}),\alpha\in\Lambda\}$ and $\{(Y_\alpha,\sigma_{1\alpha},\sigma_{2\alpha}),\alpha\in\Lambda\}$ be two arbitrary family of bitopological spaces with the same indeces. Let $f_\alpha: X_\alpha \longrightarrow Y_\alpha$ be a map for all $\alpha\in\Lambda$. Then the product function $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$, where $X=\prod X_\alpha$ and Y$=\prod Y_\alpha$, defined by $f(x)=f(x_\alpha)=f_\alpha(x_\alpha)$ is $ij-\delta s$ irresolute if and only if f_α is $ij-\delta s$ irresolute, for all $\alpha\in\Lambda$.

Proof. Let f be a $ij-\delta s$ irresolute. Let $\alpha_0\in\Lambda$ be arbitrary chosen index and U_{α_0} be an arbitrary $ij-\delta$ semi open set in Y_{α_0}. Since $U_{\alpha_0}\times\prod_{\alpha_0\neq\beta}Y_\beta$ is $ij-\delta$ semi open set in Y, then $f^{-1}[U_{\alpha_0}\times\prod_{\alpha_0\neq\beta}Y_\beta]=f^{-1}(U_{\alpha_0})\times\prod_{\alpha_0\neq\beta}f_\beta^{-1}(Y_\beta)$ is $ij-\delta$ semi open set in X. Hence $f_{\alpha_0}^{-1}(U_{\alpha_0})$ is $ij-\delta$ semi open set in X_{α_0} . Thus f_{α_0} is $ij-\delta s$ irresolute and α_0 being an arbitrary index. Therefore f_α is $ij-\delta s$ irresolute for every $\in \Lambda$.

Conversely, let f_α is $ij-\delta s$ irresolute for every $\in \Lambda$. Now let U be a $ij-\delta$ semi open set in $\prod X_\alpha$. That is $=\prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\alpha_j\neq\beta} Y_\beta$, where U_{α_j} is $ij-\delta$ semi open set in Y_{α_j} for $j=1,2,\dots\dots n$. Now $f^{-1}(U)=\prod_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j})\times\prod_{\alpha_j\neq\beta} f_\beta^{-1}(Y_\beta)=\prod_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j})\times\prod_{\alpha_j\neq\beta} X_\beta$. Since f_{α_j} is $ij-\delta s$ irresolute, so $f_{\alpha_j}^{-1}(U_{\alpha_j})$ is $ij-\delta$ semi open set in X_{α_j} for $=1,2,\dots\dots n$. Hence $f^{-1}(U)$ is $ij-\delta$ semi open set in X. Thus f is $ij-\delta$ irresolute.

Theorem 4.17 Let (Z,η_1,η_2), (X,τ_1,τ_2) and (Y,σ_1,σ_2) be bitopological spaces and let $(X\times Y,\tau_1\times\sigma_1,\tau_2\times\sigma_2)$ be the product space. Let $f: Z\longrightarrow X\times Y$ be a $ij-\delta s$ irresolute, then $f_1 : Z\longrightarrow X$ and $f_2 : Z\longrightarrow Y$ are $ij-\delta$ semi irresolute, where $f(x)=(x,y)$ and $f_1(x)=x$, $f_2(x)=y$.

Proof. If U be a $ij-\delta$ semi open set in X , then $f_1^{-1}(U)=f^{-1}(U\times Y)$ and if V be a $ij-\delta$ semi open set in Y, then $f_2^{-1}(V)=f^{-1}(X\times V)$. Hence, if f is $ij-\delta$ semi irresolute, then f_1 and f_2 are $ij-\delta s$ irresolute.

5. Pairwise $\delta s-T_k$ spaces

Definition 5.1 A bitopological space (X,τ_1,τ_2) is called weakly pairwise $\delta s-T_0$ ($WP-\delta s-T_0$) if for each distinct points $x,y\in X$, there exists an $ij-\delta$ semi open set containing x but not y or an an $ji-\delta$ semi open set containing y but not x.

Definition 5.2 A bitopological space (X,τ_1,τ_2) is called $ij-\delta s-T_0$ if for each distinct points $x,y\in X$, there exists an $ij-\delta$ semi open set containing x but not y or an an $ij-\delta$ semi open set containing y but not x. If (X,τ_1,τ_2) is $12-\delta s-T_0$ and $21-\delta s-T_0$, then it is called pairwise $\delta s-T_0$ ($P-\delta s-T_0$).

Remark 5.3 $P-\delta s-T_0 \Longrightarrow ij-\delta s-T_0 \Longrightarrow WP-\delta s-T_0$

Theorem 5.4 Let a bitopological space (X,τ_1,τ_2) be $P-\delta s-T_0$ if and only if for any points $x,y\in X$ such that $x\neq y$, $ji-\delta scl(\{x\})\neq ij-\delta scl(\{y\})$.

Proof. Let X be $P-\delta s-T_0$ and $x,y\in X$ such that $x\neq y$. Then there exists an $ij-\delta$ semi open set U such that $x\in U$ and $y\notin U$. Thus $\{y\}\cap U=\emptyset$, this means that $x\notin ij-\delta scl(\{y\})$. Since $x\in ji-\delta scl(\{x\})$, then we have

$ji-\delta scl(\{x\}) \neq ij-\delta scl(\{y\})$. On the other hand suppose that $x,y \in X$ and $x \neq y$. Then $ji-\delta scl(\{x\}) \neq ij-\delta scl(\{y\})$. Thus either $y \notin ji-\delta scl(\{x\})$ or $x \notin ij-\delta scl(\{y\})$. If $y \notin ji-\delta scl(\{x\})$, then there exists a $ij-\delta$ semi-open U such that $y \in U$ and $\{x\} \cap U = \emptyset$, i.e.$x \notin U$. If $x \notin ij-\delta scl(\{y\})$, then there exists an $ij-\delta$ semi-open V such that $x \in V$ and $\{y\} \cap V = \emptyset$ or $y \notin V$. In two cases X is $P-\delta s-T_0$.

Theorem 5.5 Every weakly pairwise $\delta-T_0$ space is weakly pairwise $\delta s-T_0$.

Proof. Let $x,y \in X$ such that $x \neq y$ in (X,τ_1,τ_2). Now for $x \neq y$, there exists a subset which is either $ij-\delta$ open set or $ji-\delta$ open set containing one of the points but not other. Since every $ij-\delta$ open set is $ij-\delta$ semi open set and every $ji-\delta$ open set is $ji-\delta$ semi open set. Then there exists a $ij-\delta$ semi open set containing x but not y or there exists a $ji-\delta$ semi open set containing y but not x. Therefore (X,τ_1,τ_2) is weakly pairwise $\delta s-T_0$.

Definition 5.6 A function $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is called strongly $ij-\delta s$ open if $f(V)$ is $ij-\delta$ semi open in Y for every $ij-\delta$ semi open set V in X.

Theorem 5.7 If $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is a bijection strongly $ij-\delta s$ open and X is $ij-\delta s-T_0$ space, then Y is also $ij-\delta s-T_0$.

Proof. Let $x_1,x_2 \in X$ with $x_1 \neq x_2$. Since f is bijective, there exist $y_1,y_2 \in Y$ with $y_1 \neq y_2$ such that $f(x_1)=y_1$ and $f(x_2)=y_2$. Since X is $ij-\delta s-T_0$ space, there exists $ij-\delta$ semi open set G in X such that $x_1 \in G, x_2 \notin G$. Since f is strongly $ij-\delta s$ open, $f(G)$ is also $ij-\delta$ semi open set in Y. Then $y_1 \in f(G), y_2 \notin f(G)$. Therefore Y is $ij-\delta s-T_0$.

Definition 5.8 A bitopological space (X,τ_1,τ_2) is called weakly pairwise $\delta s-T_1$ ($WP-\delta s-T_1$) if for each distinct points $x,y \in X$, there exists an $ij-\delta$ semi open set U and $ji-\delta$ semi open set V such that either $x \in U\backslash V$ and $y \in V\backslash U$ or $y \in U\backslash V$ and $x \in V\backslash U$.

Theorem 5.9 For a bitopological space $(X,\tau_1,\tau_2)$the following are equivalent:

(i) (X,τ_1,τ_2) is $WP-\delta s-T_1$.

(ii) $12-\delta scl(\{x\}) \cap 21-\delta scl(\{x\}) = \{x\}$ for every $x \in X$.

(iii) For every $x \in X$, the intersection of all $12-\delta$ semi open nbds and all $12-\delta$ semi open nbds of x is x.

Proof. $(i) \Longrightarrow (ii)$: Let $x \in X$ and $y \in 12-\delta scl(\{x\}) \cap 21-\delta scl(\{x\})$ where $y \neq x$. Since X is $WP-\delta s-T_1$, there exists a $12-\delta$ semi open set U such that $y \in U, x \notin U$ or there exists a $21-\delta$ semi open set V such that $y \in V, x \notin V$. In either case $y \notin 12-\delta scl(\{x\}) \cap 21-\delta scl(\{x\})$. Hence $12-\delta scl(\{x\}) \cap 21-\delta scl(\{x\}) = \{x\}$.

$(ii) \Longrightarrow (iii)$: If $x, y \in X$ such that $x \neq y$, then $x \notin 12-\delta scl(\{y\}) \cap 21-\delta scl(\{y\})$, so there is a $12-\delta$ semi open set or a $21-\delta$ semi open set containing x but not y. Therefore y does not belong to the intersection of all $12-\delta$ semi open nbds and all $21-\delta$ semi open nbds of x.

$(iii) \Longrightarrow (i)$: Let x and y be two distinct points of X. By (iii), y does not belong to a $12-\delta$ semi nbd or a $21-\delta$ semi nbd of x. Therefore there exists a $12-\delta$ sem open or a $21-\delta$ semi open set containing x but not y. Hence X is a $WP-\delta s-T_1$ space.

Definition 5.10 A bitopological space (X, τ_1, τ_2) is called $ij-\delta s-T_1$ if for each distinct points $x, y \in X$, there exist two $ij-\delta$ semi open sets U and V such that $x \in U\backslash V$ and $y \in V\backslash U$. If X is $12-\delta s-T_1$ and $21-\delta s-T_1$, then it is called pairwise $\delta s-T_1$ $(P-\delta s-T_1)$.

Theorem 5.11 Every pairwise $\delta-T_1$ space is pairwise $\delta s-T_1$.

Proof. Let (X, τ_1, τ_2) is a pairwise $\delta-T_1$. Then there exist two $ij-\delta$ open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. Since every $ij-\delta$ open set is $ij-\delta$ semi open set. Therefore U and V are $ij-\delta$ semi open sets. This implies $x \in U, y \notin U$ and $x \notin V, y \in V$. Thus X is pairwise $\delta s-T_1$.

Theorem 5.12 A bitopological space (X, τ_1, τ_2) is pairwise $\delta s-T_1$ if and only of every singleton is pairwise $\delta-$ semi closed.

Proof. Let (X, τ_1, τ_2) be pairwise $\delta s-T_1$. Since for every singleton $\{x\}$ we have $\{x\} \subset ij-\delta scl(\{x\})$, then there exists an $ij-\delta$ semi open set U containing x but $y \notin U$. Thus $\{x\} \cap U \neq \phi$ and $y \notin ij-\delta scl(\{x\})$. Then $\{x\} = ij-\delta scl(\{x\})$ and hence $\{x\}$ is $ij-\delta$ semi closed. Now, for every $x \neq y$, we have $y \in X\backslash\{x\}$. So there exists a $ji-\delta$ semi open set V_y such that $y \in V_y$ but $\notin V_y$. Therefore, $y \in V_y \subset X\backslash\{x\}$. Hence $X\backslash\{x\}$ is $ji-\delta$ semi open and $\{x\}$ is $ji-\delta$ semi closed.

Conversely, let $\{x\} = ji-\delta scl(\{x\})$ and $x, y \in X$ such that $x \neq y$. Then $X\backslash ji-\delta scl(\{x\})$ is a $ji-\delta$ semi open set containing y but not x. Similarly, if

$\{y\} = ij - \delta scl(\{y\})$. Then $X \backslash ij - \delta scl(\{y\})$ is an $ij - \delta$ semi open set containing x but not y. Thus X is pairwise $\delta s - T_1$.

Theorem 5.13 If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - \delta s$ irresolute injective and Y is pairwise $\delta s - T_1$, then X is pairwise $\delta s - T_1$.

Proof. Let x_1, x_2 be pair of distinct points in X. since f is $ij - \delta s$ irresolute injective, there exist distinct two points y_1, y_2 in Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is pairwise $\delta s - T_1$, there exist two $ij - \delta$ semi open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Since f is $ij - \delta s$ irresolute, then $f^{-1}(U)$, $f^{-1}(V)$ are $ij - \delta$ semi open sets in X. Thus for any two distinct points x_1, x_2 of X, there exist two $ij - \delta$ semi open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Therefore X is pairwise $\delta s - T_1$.

Definition 5.14 A bitopological space (X, τ_1, τ_2) is called weakly pairwise $\delta s - T_2$ if for any two distinct points x and y of X, there exist an $ij - \delta$ semi open set U and a disjoint $ji - \delta$ semi open set V such that $x \in U$ and $y \in V$ or $x \in V$ and $y \in U$.

Definition 5.15 A bitopological space (X, τ_1, τ_2) is called $ij - \delta s - T_2$ if for any two distinct point x and y of X, there exist two disjoint $ij - \delta$ semi open sets U and V such that $x \in U$ and $y \in V$. If X is $12 - \delta s - T_2$ and $21 - \delta s - T_2$, then it is called pairwise $\delta s - T_2$.

Theorem 5.16 Every pairwise $\delta s - T_2$ space is pairwise $\delta s - T_1$.

Proof. Let (X, τ_1, τ_2) is a pairwise $\delta s - T_2$. Then for any two distinct point x and y of X, there exist two disjoint $ij - \delta$ semi open sets U and V such that $x \in U$ and $y \in V$. This implies $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. Thus X is pairwise $\delta s - T_1$.

Theorem 5.17 Every pairwise $\delta - T_2$ space is pairwise $\delta s - T_2$.

Proof. Let (X, τ_1, τ_2) is a pairwise $\delta - T_2$. Then there exist two $ij - \delta$ open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. Since every $ij - \delta$ open set is $ij - \delta$ semi open set. Therefore U and V are $ij - \delta$ semi open sets. This implies $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$ also U and V are disjoint. Thus X is pairwise $\delta s - T_2$.

Theorem 5.18 If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - \delta s$ irresolute injective and Y is pairwise $\delta s - T_2$, then X is pairwise $\delta s - T_2$.

Proof. Let x_1, x_2 be pair of distinct points in X. since f is $ij - \delta s$ irresolute injective, there exist distinct two points y_1, y_2 in Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is pairwise $\delta s - T_2$, there exist two disjoint $ij - \delta$ semi open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Since f is $ij - \delta s$ irresolute, then $f^{-1}(U)$, $f^{-1}(V)$ are $ij - \delta$ semi open sets in X. Thus for any two distinct points x_1, x_2 of X, there exist two $ij - \delta$ semi open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Also $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Therefore X is pairwise $\delta s - T_2$.

Definition 5.19 A bitopological space (X, τ_1, τ_2) is called strongly pairwise $\delta s - T_2$ if for any distinct points x and y of X, there exist a $ij - \delta$ semi open set U and a disjoint $ij - \delta$ semi open set V such that $x \in U$ and $y \in V$.

Theorem 5.20 If $f: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $ij - \delta s$ irresolute injective and Y is strongly pairwise $\delta s - T_2$, then X is strongly pairwise $\delta s - T_2$.

Proof. Let x_1, x_2 be pair of distinct points in X. since f is $ij - \delta s$ injective, there exist distinct two points y_1, y_2 in Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is strongly pairwise $\delta s - T_2$, there exist two disjoint $ij - \delta$ semi open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Since f is $ij - \delta s$ irresolute, then $f^{-1}(U)$, $f^{-1}(V)$ are $ij - \delta$ semi open sets in X. Thus for any two distinct points x_1, x_2 of X, there exist two $ij - \delta$ semi open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Also $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint. Therefore X is strongly pairwise $\delta s - T_2$.

Remark 5.21 Strongly pairwise $\delta s - T_2 \Rightarrow$ weakly pairwise $\delta s - T_2$ and pairwise $\delta s - T_2 \Rightarrow ij - \delta s - T_2$.

6. Pairwise $\delta s - R_k$ spaces

Definition 6.1 For a subset A of a bitopological space (X, τ_1, τ_2), we define $A^{\delta s \Lambda_{ij}}$ as follows: $A^{\delta s \Lambda_{ij}} = \cap \{U : A \subset U, U \in ij - \delta SO(X)\}$.

Theorem 6.2 Let A be a subset of a space X . Then $A^{\delta s\Lambda_{ij}} = (A^{\delta s\Lambda_{ij}})^{\delta s\Lambda_{ij}}$

Proof. We have $(A^{\delta s\Lambda_{ij}})^{\delta s\Lambda_{ij}} = \cap\{U: U \in ij - \delta SO(X), A^{\delta s\Lambda_{ij}} \subset U\} = \cap\{U: U \in ij - \delta SO(X), (\cap\{V: V \in ij - \delta SO(X), A \subset V\}) \subset U\} \subset \cap\{U: U \in ij - \delta SO(X), A \subset U\} = A^{\delta s\Lambda_{ij}}$

. This means $(A^{\delta s\Lambda_{ij}})^{\delta s\Lambda_{ij}} \subset A^{\delta s\Lambda_{ij}}$. On the other hand, $A \subset A^{\delta s\Lambda_{ij}}$ for each subset A. Then $A^{\delta s\Lambda_{ij}} \subset (A^{\delta s\Lambda_{ij}})^{\delta s\Lambda_{ij}}$. Therefore $A^{\delta s\Lambda_{ij}} = (A^{\delta s\Lambda_{ij}})^{\delta s\Lambda_{ij}}$.

Definition 6.3 A bitopological space (X, τ_1, τ_2) is called an $ij - \delta s - R_0$ space if for every $ij - \delta$ semi open set U, $x \in U$ implies $ij - \delta scl(\{x\}) \subseteq U$.

Theorem 6.4 Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent:

(i) (X, τ_1, τ_2) is $ij - \delta s - R_0$ space.

(ii) For any $x \in X, ij - \delta scl(\{x\}) \subset \{x\}^{\delta s\Lambda_{ij}}$.

(iii) For any $x, y \in X, y \in \{x\}^{\delta s\Lambda_{ij}}$ if and only if $x \in \{y\}^{\delta s\Lambda_{ij}}$.

(iv) For any $x, y \in X, y \in ij - \delta scl(\{x\})$ if and only if $x \in ij - \delta scl(\{y\})$.

(v) For any $ij - \delta$ semi closed set F and a point $x \notin F$, there exists a $ij - \delta$ semi open set U such that $x \notin U, F \subset U$.

(vi) Each $ij - \delta$ semi closed F can be expressed as $F = \cap\{U : F \subset U, U \text{ is } ij - \delta \text{ semi open}\}$

(vii) Each $ij - \delta$ semi open set U can be expressed as the union of $ij - \delta$ semi closed sets contained in U.

(viii) For each $ij - \delta$ semi closed set F, $x \notin F$ implies $ij - \delta scl(\{x\}) \cap F = \phi$.

Proof. $(i) \Rightarrow (ii)$: By definition 5.1, for any $x \in X$ we have $\{x\}^{\delta s\Lambda_{ij}} = \cap\{U : \{x\} \subset U, U \text{ is } ij - \delta \text{ semi open}\}$. Since X is $ij - \delta s - R_0$ space, then each $ij - \delta$ semi open set U containing x contains $ij - \delta scl(\{x\})$. Hence $ij - \delta scl(\{x\}) \subset \{x\}^{\delta s\Lambda_{ij}}$.

$(ii) \Rightarrow (iii)$: For any $x, y \in X$, if $y \in \{x\}^{\delta s\Lambda_{ij}}$, then $x \in ij - \delta scl(\{y\})$. By (ii) since $ij - \delta scl(\{y\}) \subset \{y\}^{\delta s\Lambda_{ij}}$, we have $x \in \{y\}^{\delta s\Lambda_{ij}}$.

$(iii) \Rightarrow (iv)$: For any $x, y \in X$ if $y \in ij - \delta scl(\{x\})$, then $x \in \{y\}^{\delta s\Lambda_{ij}}$. Then by (iii) $y \in \{x\}^{\delta s\Lambda_{ij}}$, and so $x \in ij - \delta scl(\{y\})$.

$(iv) \Rightarrow (v)$: Let F be an $ij - \delta$ semi closed set and a point $x \notin F$. Then for any $y \in F$, $ij - \delta scl(\{y\}) \subset F$ and $x \notin ij - \delta scl(\{y\})$. By (iv), $x \notin ij - \delta scl(\{y\})$ and $y \notin ij - \delta scl(\{x\})$. Hence there exists a $ij - \delta$ semi open set U_y such that $y \in U_y$ and

$x \notin U_y$. Let $U = \bigcup_{y \in F}\{U_y : y \in U_y \text{ and } x \notin U_y, U_y \text{ is } ij-\delta \text{ semi open}\}$. Then U is a $ij-\delta$ semi open set such that $x \notin U$ and $F \subset U$.

$(v) \Rightarrow (vi)$: Let F be $ij-\delta$ semi closed set and suppose that $H = \cap\{U : F \subset U, U \text{ is } ij-\delta \text{ semi open}\}$. Then $F \subset H$ and we show that $H \subset F$. Let $x \notin F$. Then by (v) there exists a $ij-\delta$ semi open set U such that $x \notin U$ and $F \subset U$ and hence $x \notin H$. Therefore $H \subset F$ and so $F = H$.

$(vi) \Rightarrow (vii)$: Obvious.

$(vii) \Rightarrow (viii)$: Let F be an $ij-\delta$ semi closed set and $x \notin F$. Then $X \backslash F = U$ is an $ij-\delta$ semi open set containing x. Then by (vii), there exists a $ij-\delta$ semi closed set H such that $x \in H \subset U$ and so $ij-\delta scl(\{x\}) \subset U$. Thus $ij-\delta scl(\{x\}) \cap F = \phi$.

$(viii) \Rightarrow (i)$: Let U be an $ij-\delta$ semi open set and $x \in U$. Then $x \notin X\backslash U$ which is $ij-\delta$ semi closed set and by $(viii)$, $ij-\delta scl(\{x\}) \cap X\backslash U = \phi$. Thus $ij-\delta scl(\{x\}) \subset U$. Hence X is $ij-\delta s-R_0$.

Definition 6.5 A bitopological space (X,τ_1,τ_2) is called an pairwise $\delta s-R_0$ space if for every $ij-\delta$ semi open set U, $x \in U$ implies $ji-\delta scl(\{x\}) \subset U$.

Theorem 6.6 Let (X,τ_1,τ_2) be a bitopological space. Then the following statements are equivalent:

(i) (X,τ_1,τ_2) is pairwise $\delta s-R_0$ space.

(ii) For any $x \in X, ij-\delta scl(\{x\}) \subset \{x\}^{\delta s\Lambda_{ji}}$.

(iii) For any $x,y \in X, y \in \{x\}^{\delta s\Lambda_{ij}}$ if and only if $x \in \{y\}^{\delta s\Lambda_{ji}}$.

(iv) For any $x,y \in X, y \in ij-\delta scl(\{x\})$ if and only if $x \in ji-\delta scl(\{y\})$.

(v) For any $ij-\delta$ semi closed set F and a point $x \notin F$, there exists a $ji-\delta$ semi open set U such that $x \notin U, F \subset U$.

(vi) Each $ij-\delta$ semi closed F can be expressed as $F = \cap\{U : F \subset U, U \text{ is } ji-\delta \text{ semi open}\}$

(vii) Each $ij-\delta$ semi open set U can be expressed as the union of $ji-\delta$ semi closed sets contained in U.

(viii) For each $ij-\delta$ semi closed set F, $x \notin F$ implies $ji-\delta scl(\{x\}) \cap F = \phi$.

Proof. $(i) \Rightarrow (ii)$: By definition 5.1, for any $x \in X$ we have $\{x\}^{\delta s\Lambda_{ji}} = \cap\{U : \{x\} \subset U, U \text{ is } ji-\delta \text{ semi open}\}$. Since X is pairwise $\delta s-R_0$ space, then each $ji-\delta$ semi open set U containing x contains $ij-\delta scl(\{x\})$. Hence $ij-\delta scl(\{x\}) \subset \{x\}^{\delta s\Lambda_{ji}}$.

$(ii) \Rightarrow (iii)$: For any $x,y \in X$, if $y \in \{x\}^{\delta s\Lambda_{ij}}$, then $x \in ij-\delta scl(\{y\})$. By (ii) since $ij-\delta scl(\{y\}) \subset \{y\}^{\delta s\Lambda_{ji}}$, we have $x \in \{y\}^{\delta s\Lambda_{ji}}$.

$(iii) \Rightarrow (iv)$: For any $x, y \in X$ if $y \in ij-\delta scl(\{x\})$, then $x \in \{y\}^{\delta s\Lambda_{ij}}$. Then by (iii) $y \in \{x\}^{\delta s\Lambda_{ji}}$, and so $x \in ji-\delta scl(\{y\})$.

$(iv) \Rightarrow (v)$: Let F be an $ij-\delta$ semi closed set and a point $x \notin F$. Then for any $y \in F$, $ij-\delta scl(\{y\}) \subset F$ and $x \notin ij-\delta scl(\{y\})$. By (iv), $x \notin ij-\delta scl(\{y\})$ and $y \notin ij-\delta scl(\{x\})$. Hence there exists a $ji-\delta$ semi open set U_y such that $y \in U_y$ and $x \notin U_y$. Let $U = \cup_{y\in F}\{U_y : y \in U_y \text{ and } x \notin U_y, U_y \text{ is } ji-\delta \text{ semi open}\}$. Then U is a $ji-\delta$ semi open set such that $x \notin U$ and $F \subset U$.

$(v) \Rightarrow (vi)$: Let F be $ij-\delta$ semi closed set and suppose that $H = \cap\{U : F \subset U, U \text{ is } ji-\delta \text{ semi open}\}$. Then $F \subset H$ and we show that $H \subset F$. Let $x \notin F$. Then by (v) there exists a $ji-\delta$ semi open set U such that $x \notin U$ and $F \subset U$ and hence $x \notin H$. Therefore $H \subset F$ and so $F = H$.

$(vi) \Rightarrow (vii)$: Obvious.

$(vii) \Rightarrow (viii)$: Let F be an $ij-\delta$ semi closed set and $x \notin F$. Then $X\backslash F = U$ is an $ij-\delta$ semi open set containing x. Then by (vii), there exists a $ji-\delta$ semi closed set H such that $x \in H \subset U$ and so $ji-\delta scl(\{x\}) \subset U$. Thus $ji-\delta scl(\{x\}) \cap F = \phi$.

$(viii) \Rightarrow (i)$: Let U be an $ij-\delta$ semi open set and $x \in U$. Then $x \notin X\backslash U$ which is $ij-\delta$ semi closed set and by $(viii)$, $ji-\delta scl(\{x\}) \cap X\backslash U = \phi$. Thus $ji-\delta scl(\{x\}) \subset U$. Hence X is pairwise $\delta s-R_0$.

Definition 6.7 A bitopological space (X, τ_1, τ_2) is said to be pairwise $\delta s-R_1$ (briefly $P-\delta s-R_1$) if for each distinct points $x, y \in X$ such that $ij-\delta scl(\{x\} \neq ij-\delta scl(\{y\})$, there exist $U \in ji-\delta SO(X)$ and $V \in ij-\delta SO(X)$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

Theorem 6.8 Every bitopological pairwise $\delta s-R_1$ space is pairwise $\delta s-R_0$.

Proof. Let G be any $ij-\delta$ semi open set and $x \in G$. For each point $y \in X\backslash G$, $ji-\delta scl(\{x\}) \neq ij-\delta scl(\{y\})$ so there exist an $ij-\delta$ semi open set U_y and a $ji-\delta$ semi open set V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. If $A = \cup\{V_y : y \in X\backslash G\}$ then $X\backslash G \subset A$ and $x \notin A$. Since A is a $ji-\delta$ semi open set, then $ji-\delta scl(\{x\}) \subset X\backslash A \subset G$. Hence X is pairwise $\delta s-R_0$.

Theorem 6.9 A bitopological space (X, τ_1, τ_2) is pairwise $\delta s-R_1$ if and only if for every pair of distinct points x and y of X such that $ji-\delta scl(\{x\}) \neq ij-\delta scl(\{y\})$, there exist an $ij-\delta$ semi open set U and a $ji-\delta$ semi open set V such that $ij-\delta scl(\{x\} \subset V, ji-\delta scl(\{y\} \subset U$ and $U \cap V = \phi$.

Proof. Let (X, τ_1, τ_2) be pairwise $\delta s - R_1$ space and $x, y \in X$ such that $ij - \delta scl(\{x\} \neq ji - \delta scl(\{y\})$. Then there exist an $U \in ij - \delta SO(X)$ and $V \in ji - \delta SO(X)$ such that $x \in V, y \in U$ and $U \cap V = \phi$. By theorem 5.8, $x \in V$ implies $ij - \delta scl(\{x\}) \subset V$ and $y \in U$ implies $ji - \delta scl(\{x\}) \subset U$. The converse is obvious.

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