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Morphisms Between Semi Magic Squares And Magic Squares

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Abstract:

Some advanced mathematical properties of semi magic squares and magic squares are discussed in this paper.

Key Words: Semi Magic Square, Magic Square, Magic Constant, Abelian Group, Commutative Ring, Field

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A magic square of order 'n' is an n^{th} order matrix such that the sum of elements in every row/column/diagonal remains the same. The common sum is known as 'magic constant' or 'magic number'. If the above condition is valid only for the sum of elements of rows and columns and not for the diagonal elements, then that array is known as a semi magic square. 'Cornelius Agrippa' (1486 B.C. to 1535 B.C.) of China is believed to be the first to take up construction of magic squares. There it was called 'Loh Shu'. Interest in magic squares spread from China to Japan, India and the middle East. They were introduced to Europe in Byzantine times. The first magic square of order 4 in the first century was introduced in India by a mathematician named 'Nagarjuna'. All magic squares are semi magic squares. In real life situations, some problems related to division of objects equal in numbers and value can be easily solved by constructing a semi magic square or a magic square in accordance with the given conditions.

Apart from the recreational aspect of semi magic squares, it is found that they posses several advanced mathematical properties. A few among them are discussed here. The main aim is to construct rings from the set of all semi magic squares and define a ring isomorphism.

Mathematical Preliminaries

$Magic\ Square$: A magic square of order 'n' is an n^{th} order matrix $[a_{ij}]$ such that

$\sum_{j=1}^{n} a_{ij} = k$, for $i = 1,2,3, \dots \dots \dots, n$ (1)

$\sum_{j=1}^{n} a_{ji} = k$, for $i = 1,2,3, \dots \dots \dots n$ (2)

$\sum_{i=1}^{n} a_{ii} = k$ and $\sum_{i=1}^{n} a_{i,n-i+1} = k$, (3)

Where $'k'$ is a constant and the above mentioned a_{ij}'s and a_{ji}'s are the row and column elements and a_{ii}'s & $a_{i,n-i+1}$'s are the left and right diagonal elements of the magic square respectively. $Magic\ Constant$: The constant $'k'$ in the above definition is known as the magic constant or magic number. Magic constant of the magic square A is denoted as $\rho(A)$.

For example, the below given magic squares A and A′ are of order 3 *and* $\rho(A) = 21$ & $\rho(A') = 15$

$A =$

4	9	8
11	7	3
6	5	10

$A' =$

4	3	8
9	5	1
2	7	6

Semi Magic Square:

In the definition of a magic square, if only conditions (1) and (2) are satisfied, then that square array is known as a semi magic square.

For example, the below given array B is a semi magic square of order 3 *with* $\rho(B) = 30$

$B =$

8	18	4
16	2	12
6	10	14

Here sum of elements of each row/column $= 30$. Sum of the left diagonal elements = $8 + 2 + 14 = 24$ and sum of the right diagonal elements $= 12$.

Notations

We use $(i)\Re$ to denote the set of all real numbers. (ii) S_{J_a} to denote the set of all n^{th} order magic squares of the form $[a_{ij}]$, where $a_{ij} = a,$ for $i, j = 1,2,3, \dots \dots, n,$ $a \in \Re. \rho([a_{ij}]) = na$. For convenience, $A \in S_{J_a}$ can be represented as $[a]$.(iii) S_a to denote the set of all n^{th} order semi magic squares of the form $[a_{ij}]$, where

$$a_{ij} = \begin{cases} a, \text{for } i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases} \quad a \in \Re$$

(Here we are not excluding the case $a = 0. \rho([a_{ij}]) = a$).

Propositions and Theorems

***Theorem* 1:** *$\langle S_{J_a}, +, . \rangle$ forms a field, where $' + '$ and $'.'$ denote the addition and multiplication of matrices respectively.*

Proof: By [2], (Theorem 3.3, Theorem 3.2 and Proposition 3.2).

Theorem 2: *$\langle S_a, +, . \rangle$ forms a field, where $' + '$ and $'.'$ denote the addition and multiplication of matrices respectively.*

Proof: By [3], (Theorem 3.3, Theorem 3.2 and Proposition 3.2).

***Theorem* 3**: *The function $f: S_{J_a} \to S_a$, defined by $f(A) = A' = [a_{ij}]$, where*

$$a_{ij} = \begin{cases} \rho(A), for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases}$$ *, is a ring isomorphism.*

Proof: First we have to show that $f: S_{J_a} \to S_a$, defined by $f(A) = A'$ is a bijection from S_{J_a} to S_a . For $A = [a], B = [b] \in S_{J_a}$ let $f(A) = f(B)$. Then by the given data, $\rho(A) = \rho(B)$. i.e., $na = nb \Rightarrow a = b$. Then we must have $A = B$, due to the construction of elements in S_{J_a}. Hence f is one-one. Now for any $A' = [a_{ij}] \in S_a$, where $a_{ij} = \begin{cases} a, for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases}$, $\exists\ A = \left[\frac{a}{n}\right]$ in S_{J_a} with $\rho(A) = n.\frac{a}{n} = a$. Then, $f(A) = A'$. So f is onto. Therefore, f is a bijection from S_{J_a} to S_a.

Now, for $A = [a], B = [b] \in S_{J_a}$, $A + B = [a + b] \in S_{J_a}$ and let $f(A + B) = C' = [c_{ij}]$, then, $c_{ij} = \begin{cases} \rho(A + B), for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases}$

$$= \begin{cases} \rho(A) + \rho(B), for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases}$$

$$= \begin{cases} na + nb = n(a + b), for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases}$$

[$\rho(A + B) = \rho(A) + \rho(B)$, by [4]]. Let $f(A) = A' = [a_{ij}]$ and $f(B) = B' = [b_{ij}]$. Then, $f(A) + f(B) = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$, where

$$a_{ij} = \begin{cases} \rho(A) = na, for\ i = j, i, j = 1,2, \dots \dots, n \\ 0, otherwise \end{cases},$$

$b_{ij} = \begin{cases} \rho(B) = nb, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}$ and

$a_{ij} + b_{ij} = \begin{cases} na + nb = n(a+b), for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases} = c_{ij}$

$\Rightarrow \left[a_{ij} + b_{ij}\right] = [c_{ij}]$. Hence $f(A+B) = f(A) + f(B)$.

Now, $AB = [nab]$ & let $f(AB) = D' = \left[d_{ij}\right]$, then,

$d_{ij} = \begin{cases} \rho(AB) = n(nab) = n^2ab, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}$.

$f(A)f(B) = \left[a_{ij}\right]\left[b_{ij}\right] = \left[a_{ij}.b_{ij}\right]$, due to the construction of elements in S_a. Then,

$a_{ij}.b_{ij} = \begin{cases} na.nb = n^2ab, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases} = d_{ij}$.

Hence $f(AB) = f(A)f(B)$. This completes the proof.

Note: If we are taking two elements A & A' from S_{J_a} and S_a respectively, where $A = [a]$ and $A' = \left[a_{ij}\right], where, a_{ij} = \begin{cases} ka, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}, k, a \in \Re$, then,

***Proposition* 1**: *The function $f: S_{J_a} \to S_a$, defined by $f(A) = A'$, where A & A' are taken as in the above note is a ring isomorphism if and only if $k = n$, where n is the order of both A & A'.*

Proof: Assume that $k = n$. Then $f: S_{J_a} \to S_a$, defined by $f(A) = A'$ is an isomorphism by Theorem 3.

Now assume that $f: S_{J_a} \to S_a$, defined by $f(A) = A'$ is a ring isomorphism. i.e., f is a bijection from S_{J_a} to S_a, $f(A+B) = f(A) + f(B)$ and $f(AB) = f(A)f(B)$. For any value of k, we can show that $f(A+B) = f(A) + f(B)$.

Let $A = [a]$ and $B = [b], f(A) = A' = [a_{ij}]$ and $f(B) = B' = [b_{ij}]$ then $a_{ij} = \begin{cases} ka, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}$, & $b_{ij} = \begin{cases} kb, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}, k, a, b \in \Re$.

$a_{ij}.b_{ij} = \begin{cases} k^2ab, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}$ (1)

Now, $AB = [nab]$ and if $f(AB) = D' = \left[d_{ij}\right]$, then,

$d_{ij} = \begin{cases} knab, for\ i = j, i, j = 1,2, \ldots \ldots, n \\ 0, otherwise \end{cases}$ (2)

If $f(AB) = f(A)f(B)$, then from (1) and (2) we must have, $k^2ab = knab \Rightarrow k = n$. Hence the proof.

Here we have seen a ring isomorphism from $S_{J_a} \to S_a$ through the function f.

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