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The flow of fluid moving between two parallel plates represented by the Laplace transform

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Abstract

We have checked the solution of unsteady Couette flow by Laplace transform, and also established the solution of the unsteady flow of fluid between two parallel plates suddenly set in motion with the same speed by the transform.

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1. Introduction

The Navier-Stokes equation gives a general description in the motion of viscous fluid substances, and the equation is a so-called big equation in practical uses as well as mathematics. It still remains as an open question that whether the solutions of Navier-Stokes equation in three dimensions exist or not and if they exist, they do not contain any singularity. In this equation, neglecting pressure gradients, we would obtain the simple form. This is Couette flow. In a word, the impulsive motion or periodic oscillation of the plates generate the unsteady flows, and Couette flow means the laminar(streamline) flow of a viscous fluid between two parallel plates which is moving relative to the other. Normally, in laminar flow, the motion of the fluid moves parallel to the pipe walls. In

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relation to the topic, we would like to represent the solution of unsteady Couette flow by using the Laplace transform, and this representation is based on the idea of [1, 4, 6–8].

To begin with, let us see the previous researches. Rajagopal [9] studied solutions for unsteady unidirectional flows of an incompressible second grade fluid. He established the flow between two infinite parallel plates with one oscillating in the form of $u_0 \cos wt$ and calculated the velocity field by separation of variables. Erdrogan [2, 3] studied the unsteady flows generated by impulsive motion of a boundary, and Wenchang [10] established that the governing equation for the flows of the generalized second grade fluid between two parallel plates has the form of $\rho u_t = u u_{yy} + \alpha {D_t}^{\beta}[u_{yy}]$, where ρ is the density of fluid, α is a normal stress modulus, and ${D_t}^{\beta}$ is the fractional time derivative of non-integer order. This equation is the governing equation for the flows of the generalized second grade fluid between two parallel plates, and the simplification makes $u_t = \nu u_{yy}$ for ν is a viscosity coefficient [3].

2. The flow of fluid moving between two parallel plates represented by the Laplace transform

We would like to consider the flow of fluid moving between two parallel plates represented by the Laplace transform. First, let us consider the case of the flow between two parallel plates that one plate is moved suddenly and the other is held stationary. It is assumed that the fluid is bounded by two parallel plates at $y = 0$ and $y = d$, and the fluid is initially at rest. Then the fluid sets in motion by the velocity of the upper plate in its own plane, with the lower plate being held stationary. The governing equation is the unsteady Navier-Stokes equation, and the equation reduces to

$$u_t = \nu u_{yy} \tag{1}$$

with the conditions

$$u(0, t) = 0,\ u(d, t) = u_0,\ u(y, 0) = 0 \tag{2}$$

for $0 < y < d$, where, d is the distance between the parallel plates, ν is a viscosity coefficient and u_0 is the steady velocity.

Theorem 2.1. The equation (1) with the conditions (2) has a solution u of the form

$$£^{-1}\left[\frac{u_0}{s(e^{\sqrt{s/\nu}\,d} - e^{-\sqrt{s/\nu}\,d})}(e^{\sqrt{s/\nu}\,y} - e^{-\sqrt{s/\nu}\,y})\right]$$

for $U(y, s)$ is the Laplace transform of $u(y, t)$.

Proof. Taking the Laplace transform on both sides of the equation (1), we have

$$sY - u(y, 0) = \nu \frac{\partial^2}{\partial y^2} U(y, s) - su(y, 0) - u'(y, 0)$$

for $\pounds[u(y,t)] = U(y,s)$. Since the given fluid only depends on the upper plate, this equations become

$$\frac{\partial^2 U}{\partial y^2} - \frac{s}{\nu} Y = 0$$

for $\pounds(y) = Y$. Hence, the Laplace transform of $u(y,t)$ is

$$U(y,s) = A(s)e^{\sqrt{s/\nu}\,y} + B(s)e^{-\sqrt{s/\nu}\,y}. \tag{3}$$

Since $U(0,s) = \pounds[u(0,t)] = \pounds(0) = 0$, by the equation (3), $B(s) = -A(s)$. Similarly, since $U(d,s) = \pounds[u(d,t)] = \pounds(u_0) = u_0/s$,

$$A(s)(e^{\sqrt{s/\nu}\,d} - e^{-\sqrt{s/\nu}\,d}) = \frac{u_0}{s}.$$

Organizing this equality, we have the $A(s)$. Thus, we have the solution $u(y,t)$ of the form

$$u(y,t) = \pounds^{-1}[U(y,s)] = \pounds^{-1}\left[\frac{u_0}{s(e^{\sqrt{s/\nu}\,d} - e^{-\sqrt{s/\nu}\,d})}(e^{\sqrt{s/\nu}\,y} - e^{-\sqrt{s/\nu}\,y})\right]$$

for $U(y,s)$ is the Laplace transform of $u(y,t)$. ■

Next, let us consider the case of the flow between two rigid boundaries with one moving suddenly and the other free. The fluid motion is due to the plate at $y = d$ being brought suddenly to the steady velocity u_0, with the plate at $y = 0$ as free surface. Then the boundaries conditions are

$$u(d,t) = u_0,\ u_y(0,t) = 0,\ u(y,t) = 0 \tag{4}$$

for $0 < y < d, t \le 0$. Here, we note that the free surface means $\partial u/\partial y = 0$.

Theorem 2.2. The equation (1) with the conditions (4) has a solution u of the form

$$\pounds^{-1}\left[\frac{u_0}{s(e^{\sqrt{s/\nu}\,d} - e^{2\sqrt{s/\nu}\,d})}(e^{\sqrt{s/\nu}\,y} - e^{2\sqrt{s/\nu}\,y})\right]$$

for $U(y,s) = \pounds[u(y,t)]$.

Proof. From the $U(y,s) = \pounds[u(y,t)] = \pounds(0) = 0$ and the equation (3), we have

$$A(s)e^{\sqrt{s/\nu}\,y} + B(s)e^{-\sqrt{s/\nu}\,y} = 0,$$

and so,

$$B(s) = -A(s)\,e^{2\sqrt{s/\nu}\,y}. \tag{5}$$

Since $U(d,s) = \pounds[u(d,t)] = \pounds(u_0) = u_0/s$, by (3) and (5), we have

$$A(s)(e^{\sqrt{s/\nu}\,d} - e^{2\sqrt{s/\nu}\,d}) = u_0/s.$$

Thus

$$U(y,s) = \frac{u_0}{s(e^{\sqrt{s/v}\,d} - e^{2\sqrt{s/v}\,d})}(e^{\sqrt{s/v}\,y} - e^{2\sqrt{s/v}\,y})$$

for $U(y,s) = £[u(y,t)]$.

Conversely, let us consider the case where the lower plate moves. If $y = 0$ is initially at rest and is brought suddenly to the steady velocity u_0 and the plate at $y = d$ at rest, and the other conditions are unchanged, then the boundary and initial conditions of Theorem 2.1 are changed to

$$u(0,t) = u_0,\ u(d,t) = 0,\ u(y,t) = 0$$

for $0 < y < d$. Similarly, the conditions of Theorem 2.2 are changed to

$$u(0,t) = u_0,\ u_y(d,t) = 0,\ u(y,t) = 0$$

for $0 < y < d$. Since the approach is almost similar, we would like to omit the details. ■

Finally, let us consider the case of the flow between two infinite parallel plates with one oscillating, and the other at rest or being free. If the other at rest, then the boundary conditions can be expressed as

$$u(0,t) = 0, u(d,t) = u_0\ \cos wt \tag{6}$$

from the viewpoint of [9]. Otherwise, that is, if the other being free, then the conditions are

$$u_y(0,t) = 0, u(d,t) = u_0\ \cos wt. \tag{7}$$

Theorem 2.3. The equation (1) with the conditions (6) has a solution u of the form

$$£^{-1}\left[\frac{u_0 s\ (e^{\sqrt{s/v}\,y} - e^{-\sqrt{s/v}\,y})}{(e^{\sqrt{s/v}\,d} - e^{-\sqrt{s/v}\,d})(s^2 + w^2)}\right]$$

for $U(y,s)$ is the Laplace transform of $u(y,t)$.

Proof. From $U(0,s) = £[u(0,t)] = £(0) = 0$, we have $B(s) = -A(s)$. Thus

$$U(y,s) = A(s)(e^{\sqrt{s/v}\,y} - e^{-\sqrt{s/v}\,y}).$$

Since

$$U(d,s) = £[u(d,t)] = £(u_0\ \cos wt) = u_0\frac{s}{s^2 + w^2},$$

we have

$$A(s)(e^{\sqrt{s/v}\,d} - e^{2\sqrt{s/v}\,d}) = u_0\frac{s}{s^2 + w^2}.$$

Thus

$$U(y,s) = \frac{u_0 s\ (e^{\sqrt{s/v}\,y} - e^{-\sqrt{s/v}\,y})}{(e^{\sqrt{s/v}\,d} - e^{-\sqrt{s/v}\,d})(s^2 + w^2)}$$

for $U(y,s) = £[u(y,t)]$. ■

Theorem 2.4. The equation (1) with the conditions (7) has a solution u of the form

$$£^{-1}\left[\frac{u_0 s\ (e^{\sqrt{s/v}\ y} + e^{-\sqrt{s/v}\ y})}{(e^{\sqrt{s/v}\ d} + e^{-\sqrt{s/v}\ d})(s^2+w^2)}\right]$$

for $U(y,s)$ is the Laplace transform of $u(y,t)$.

Proof. Note that $U(y,s) = A(s)e^{\sqrt{s/v}\ y} + B(s)e^{-\sqrt{s/v}\ y}$. From

$$U_y(y,s) = \sqrt{s/v}\ (A(s)e^{\sqrt{s/v}\ y} - B(s)e^{-\sqrt{s/v}\ y})$$

and $U_y(0,s) = £[u_y(0,t)] = £(0) = 0$, we have $U_y(0,s) = \sqrt{s/v}(A(s) - B(s)) = 0$, and so, $A(s) = B(s)$. From

$$U(d,s) = £[u(d,t)] = £(u_0\ \cos wt) = u_0\frac{s}{s^2+w^2},$$

we get the equation

$$A(s)(e^{\sqrt{s/v}\ d} + e^{2\sqrt{s/v}\ d}) = u_0\frac{s}{s^2+w^2}.$$

Thus, we have the Laplace transform of $u(y,t)$ as

$$U(y,s) = \frac{u_0 s\ (e^{\sqrt{s/v}\ y} + e^{-\sqrt{s/v}\ y})}{(e^{\sqrt{s/v}\ d} + e^{-\sqrt{s/v}\ d})(s^2+w^2)}.$$

■

Similarly, in Navier-Stokes equation, dropping the pressure term, we would obtain Burger's equation

$$u_t + uu_y = \nu u_{yy}$$

for ν is a viscosity coefficient. We call it viscous Burger's equation, and in case of $\nu = 0$, we call it inviscid one [6]. In a similar way, the solution of this equation can be represented by the Laplace transform as well.

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