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W-Convex subsets of a Pseudo Ordered Set

Shashirekha B Rai
Depatrment of Mathematics,
NMAMIT Nitte, Karkal taluk,
Karnataka State, INDIA.
E-mail: sbrai8757@gmail.com

Prashantha Rao
Department of Mathematics,
Vivekananda College of Engineering and Technology,
Nehru nagar, Puttur-574203,
Karnataka State, INDIA.
E-mail: prashanthvcet@gmail.com

Abstract

In this paper the notion of w-convex (weakly convex) subsets of a pseudo ordered set is introduced and it is proved that for any pseudo ordered set A, lattice of all w-convex subsets $WCS(A)$ is lower semi modular. Also it is proved that w-convex homomorphism maps atoms of $WCS(A)$ to atoms of $WCS(A^1)$. Concept of path preserving mapping is introduced in a pseudo ordered set and it is proved that every mapping of a pseudo ordered set A to itself is path preserving if and only if A is a cycle.

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1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A, \trianglelefteq\rangle$ is called a *pseudo-ordered set* or a *psoset*.For $a, b \in A$ if $a \trianglelefteq b$ and $a \neq b$, then we write $a \lhd b$. For a subset B of A, the notions of a lower bound, an upper bound, the greatest lower bound (GLB or meet, denoted by $\wedge B$), the least upper bound (LUB or join, denoted by $\vee B$) are defined analogous to the corresponding notions in a poset (refer [1]).

It is shown in [3] that any psoset can be regarded as a digraph(possibly infinite) in which for any pair of distinct elements u and v there is no directed line between u and v or if there is a directed line from u to v, there is no directed line from v to u. Define a relation $\sqsubseteq_B$ on a subset B of a psoset $\langle A, \trianglelefteq \rangle$ by setting $b \sqsubseteq_B b^1$for two elements b and $b^1$of B if and only if there is a directed path in B from b to b^1 say $b = b_0 \trianglelefteq b_1 \trianglelefteq ..... \trianglelefteq b_n = b^1$ for some $n \geq 0$. The relation $\sqsupseteq_B$ is defined dually. If for each pair of elements b and b^1 of B at least one of the relations $b \sqsubseteq_B b^1$ or $b^1 \sqsubseteq_B b$ holds, then B will be called a *pseudo chain* or a *p-chain*.If for each pair of elements b and b^1 of B both the relations $b \sqsubseteq_B b^1$ and $b^1 \sqsubseteq_B b$ hold, then B will be called a *cycle*. The empty set and a single element set in a psoset are cycles. A non-trivial cycle contains at least three elements. A psoset is said to be *acyclic* if it does not contain any non-trivial cycle.

2. Definitions and Results

Definition 2.1. A subset S of a psoset A is said to be a w-convex subset (weakly convex subset) of A whenever $a, b \in S$ and $c \in A$ such that $a \sqsubseteq_A c \sqsubseteq_A b$ then $c \in S$.

Set of all w-convex subsets of a psoset A is denoted by $WCS(A)$ and it forms a lattice with respect to the relation $\subseteq$.

Remark 2.2.

(1) For $H_1, H_2 \in WCS(A)$, define $H_1 \wedge H_2 = H_1 \cap H_2$ and $H_1 \vee H_2 =$ the smallest w-convex subset of A containing $H_1 \cup H_2$.

(2) $\langle WCS(A), \subseteq \rangle$ is a complete lattice as φ is the least element and A is the greatest element of $WCS(A)$.

Example 2.3. A psoset $\langle A, \trianglelefteq \rangle$ where $A = \{a, b, c, d\}$ and the lattice of all its w-convex subsetsare shown in Figure 1.

Definition 2.4. Let S be a subset of a psoset A. The w-convex hull of S denoted by $wch(S)$ is defined to be the smallest w-convex subset of A containing S.

Theorem 2.5. Let S be a subset of a psoset A. Then $wch(S) = \{q \in A | p_1 \sqsubseteq_A q \sqsubseteq_A p_2$ for some $p_1, p_2 \in S\}$ where p_1, p_2 need not be distinct.

Proof. Let $Q = \{q \in A | p_1 \sqsubseteq_A q \sqsubseteq_A p_2$ for some $p_1, p_2 \in S\}$. Clearly Q is a subset of any w-convex subset of A containing S. Then $Q \subseteq wch(S)$. Let us prove that Q itself is a w-convex subset of A. Let $q_1, q_2 \in Q$ such that $q_1 \sqsubseteq_A r \sqsubseteq_A q_2$ for some $r \in A$. Now $q_1 \in Q$ implies there exist some $p_1, p_2 \in S$ such that $p_1 \sqsubseteq_A q_1 \sqsubseteq_A p_2$. Also $q_2 \in Q$ implies there exist $p_1^1, p_2^1 \in S$ such that $p_1^1 \sqsubseteq_A q_2 \sqsubseteq_A p_2^1$. Then $p_1 \sqsubseteq_A q_1 \sqsubseteq_A r \sqsubseteq_A q_2 \sqsubseteq_A p_2^1$ which implies $r \in Q$. Therefore $Q = wch(S)$. ∎

Corollary 2.6. For any element a in a cycle C, $wch(\{a\}) = C$.

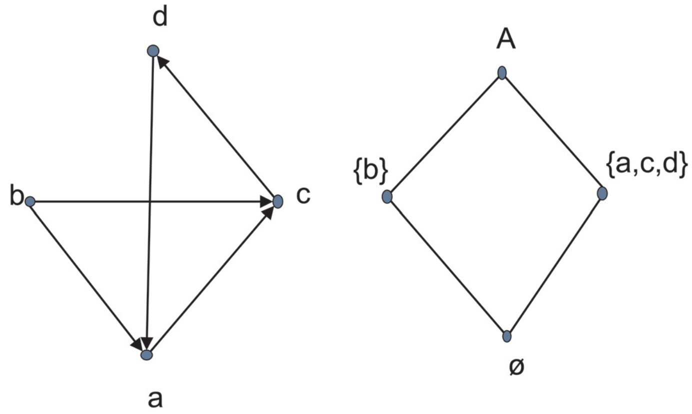

Figure 1:

A lattice L is said to be *lower semi modular* if $x \vee y$ covers x and y imply x and y cover $x \wedge y$.

Theorem 2.7. Lattice of all w-convex subsets of a psoset A is lower semi modular.

Proof. Let $S_1, S_2 \in WCS(A)$, lattice of all w-convex subsets of a psoset A. Let $P = S_1 \vee S_2$ and $Q = S_1 \wedge S_2$. Let P cover both S_1 and S_2. It suffices to prove that S_1 covers Q. Suppose there exists a w-convex subset S^1 of A such that $Q \subseteq S^1 \subseteq S_1$. Let $s_0 \in S^1 - Q$. Then $s_0 \in S_1$ and $s_0 \notin S_2$. Let $s_1 \in S_1 - S^1$. Then $s_1 \in S_1$, $s_1 \notin S_2$ and $s_1 \neq s_0$. Now $S_2 \subset S_2 \cup \{s_0\} \subset P$, as P is the smallest w-convex subset of A containing $S_1 \cup S_2$. But P covers S_2 imply $S_2 \cup \{s_0\}$ is not a w-convex subset of A. Therefore there exists a path between s_0 and an element $k_0 \in S_2$ which does not lie completely in $S_2 \cup \{s_0\}$. Assume the path in the form $k_0 \sqsubseteq_A s_0$. (similar argument holds if the path is of the form $s_0 \sqsubseteq_A k_0$).

Let $X = \{s \in S^1 - Q | k \sqsubseteq_A s$ for some $k \in S_2\}$. X is non empty as $s_0 \in X$ and $S_2 \subset S_2 \cup X$. Further as $s_1 \notin X$ (in fact $s_1 \notin S^1$), we have $S_2 \cup X \subset P$. As P covers S_2, $S_2 \cup X \notin WCS(A)$. Therefore there exists a path $m \sqsubseteq_A n$ between two elements m, n of $S_2 \cup X$ which is not contained in $S_2 \cup X$. This implies $m \sqsubseteq_A t \sqsubseteq_A n$ but $t \notin S_2 \cup X$. We can assume that $t \in P$ as $S_2 \cup X$ is not a w-convex subset of P.

In the following cases either we get a contradiction to the w-convexity of S_2 or S^1 itself is not w-convex, proving S_1 covers Q.

Case (i): Let $m, n \in S_2$. This is a contradiction to the w-convexity of S_2.

Case (ii): Let $m \in X$ and $n \in S_2$. As $m \in X$, $m \notin S_2$. By the definition of X there exists a $k \in S_2$ such that $k \sqsubseteq_A m$. Thus $k \sqsubseteq_A m$ and $m \sqsubseteq_A t \sqsubseteq_A n$ imply $k \sqsubseteq_A t \sqsubseteq_A n$, which contradicts the w-convexity of S_2.

Case (iii): Let $m, n \in X$. As $m \in X$, there exists a path $k \sqsubseteq_A m$ for some $k \in S_2$. But we have a path $m \sqsubseteq_A t$ which implies there is a path $k \sqsubseteq_A t$. But $t \notin X$, ie $t \notin S^1 - Q$.

Since $t \notin S_2$, it can not be in Q. So $t \notin S^1$. But $m, n \in X \subseteq S^1$ shows that S^1 is not a w-convex subset of A.

Case (iv): Let $m \in S_2$ and $n \in X$. As we have a path from $m \sqsubseteq_A t, t \notin X = S^1 - Q$ and since $t \notin S_2$ imply $t \notin S^1$. Now P covers S_2 and $t \notin S_2$, we must have $wch(S_2 \cup \{t\}) = P$. Thus every element of P is in S_2 or else lies on some path between t and an element of S_2. In particular consider some $s_0 \in X$, we have $k_0 \sqsubseteq_A s_0$ and $t \sqsubseteq_A n$. If $k \sqsubseteq_A s_0 \sqsubseteq_A t$ where $k \in S_2$, then we have $s_0 \sqsubseteq_A t \sqsubseteq_A n$ which proves that S^1 is not w-convex. On the other hand if $t \sqsubseteq_A s_0 \sqsubseteq_A k$ then $k_0 \sqsubseteq_A s_0 \sqsubseteq_A k$, contradicting the w-convexity of S_2.■

Theorem 2.8. If S covers S^1 in $WCS(A)$ and p, q belong to $S - S^1$ then $wch(\{p\}) = wch(\{q\})$.
Proof. As S covers S^1, $wch(S^1 \cup \{p\}) = wch(S^1 \cup \{q\}) = S$. Then p lies in a path from q to r where $r \in S^1$ and q lies in a path from p to s where $s \in S^1$. If there exist paths $p \sqsubseteq_A q$ and $q \sqsubseteq_A p$ then the proof is done. But if both paths have the same direction say $p \sqsubseteq_A q$, we have paths $p \sqsubseteq_A r$ and $s \sqsubseteq_A p$ with $r, s \in S^1$, contradicting the w-convexity of S^1. ■

Definition 2.9. Let $\langle A, \trianglelefteq \rangle$ and $\langle A^1, \trianglelefteq^1 \rangle$ be any two psosets. A mapping $f : A \longrightarrow A^1$ is called

(i) order preserving if for $a, b \in A$, $a \trianglelefteq b$ implies $f(a) \trianglelefteq^1 f(b)$.

(ii) path preserving if for $a, b \in A$, $a \sqsubseteq_A b$ implies $f(a) \sqsubseteq_{A^1} f(b)$.

Remark 2.10. Any order preserving mapping f is path preserving. The converse is not true. For example, in the psoset A of FIGURE 2, define a mapping $f : A \longrightarrow A$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Clearly f is path preserving but not order preserving.

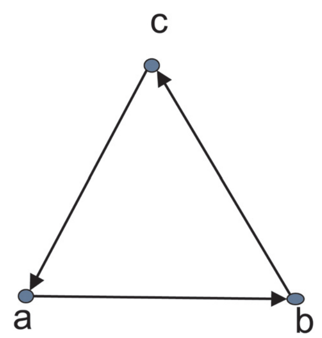

Figure 2:

Theorem 2.11. Every mapping of a psoset A to itself is path preserving if and only if A is a cycle.

Proof. If A is a cycle, then for any two elements a, b of A, both $a \sqsubseteq_A b$ and $b \sqsubseteq_A a$ hold. Therefore every mapping of A to itself is path preserving. Conversely, let us assume that A is not a cycle. Then there exists at least one pair of elements say (a, b) in A such that $a \sqsubseteq_A b$ holds but $b \sqsubseteq_A a$ does not hold. Define $f : A \longrightarrow A$ by $f(b) = a$ and $f(c) = b$ for all $c \neq b$. Then f is not path preserving as $a \sqsubseteq_A b$ but $f(a) \sqsubseteq_A f(b)$ does not hold. ■

One can easily prove the following theorem.

Theorem 2.12. Let $f : A \longrightarrow A^1$ be path preserving. If S is a w-convex subset in A then $f(S)$ is a w-convex subset in A^1.

Definition 2.13. Let $\langle A, \trianglelefteq\rangle$ and $\left\langle A^1, \trianglelefteq^1\right\rangle$ be any two psosets. A mapping $f : A \to A^1$is called a homomorphism if

(i) f is order preserving.

(ii) $a^1 \trianglelefteq^1 b^1$ in A^1 implies there exists $a \in f^{-1}(a^1)$ and $b \in f^{-1}(b^1)$ such that $a \trianglelefteq b$.

Theorem 2.14. Let $f : A \to A^1$ be a homomorphism. If S^1 is a w-convex subset of A^1 then $f^{-1}(S^1)$ is a w-convex subset of A.

Proof. Let $a, b \in f^{-1}(S^1)$ such that $a \sqsubseteq_A c \sqsubseteq_A b$ for some $c \in A$. If $c \notin f^{-1}(S^1)$ then $f(c) \notin S^1$. Now $f(a) \sqsubseteq_{A^1} f(c) \sqsubseteq_{A^1} f(b)$ and $f(c) \notin S^1$, a contradiction to the w-convexity of S^1. Hence $c \in f^{-1}(S^1)$ and $f^{-1}(S^1)$ is w-convex. ■

Remark 2.15. If $f : A \longrightarrow A^1$ is a homomorphism between two psosets A and A^1 and if S is a w-convex subset of A then $f(S)$ need not be a w-convex subset of A^1. For, in FIGURE 3 define a map $f : A \longrightarrow A^1$ by $f(a) = w$, $f(b) = y$, $f(c) = x$, $f(d) = z$. Clearly f is a homomorphism. Observe that $\{b\}$ is w-convex in A where as $f(\{b\}) = \{y\}$ is not w-convex in A^1.

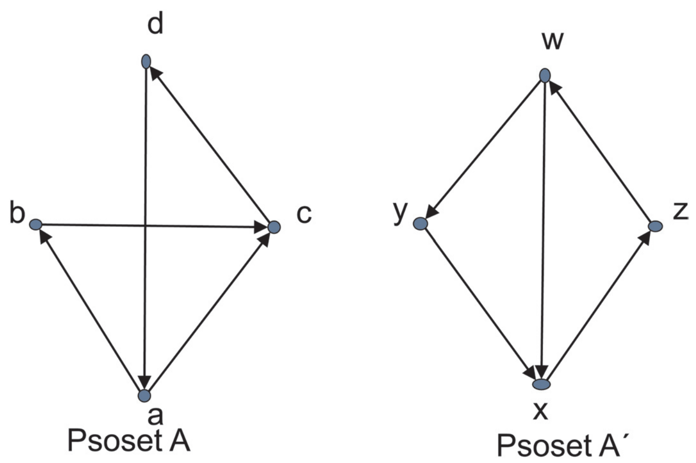

Figure 3:

Definition 2.16. A homomorphism between two psosets A and A^1 is called a w-convex homomorphism if it takes w-convex subsets of A onto w-convex subsets of A^1.

An element a of a lattice L is said to be an *atom* if for any $b \in L, 0 \leq b \leq a$ imply either $b = 0$ or $b = a$ where 0 is the least element of L.

Remark 2.17.

(1) A cycle in a psoset A is always an atom of $WCS(A)$

(2) w-convex hull of a single element in a psoset A is an atom in $WCS(A)$.

Theorem 2.18. Let $f : A \to A^1$ be a w-convex homomorphism. If S is an atom in $WCS(A)$ then $f(S)$ is an atom in $WCS(A^1)$. Conversely if S^1 is an atom in $WCS(A^1)$ then there exists an atom S in $WCS(A)$ such that $f(S) = S^1$.

Proof. If $f(S)$ is not an atom in $WCS(A^1)$ then there exists a w-convex subset S^1 in $WCS(A^1)$ such that $\varphi \subset S^1 \subset f(S)$. But then $\varphi \subset f^{-1}(S^1) \cap S \subset S$, where $f^{-1}(S^1) \cap S$ is also a w-convex subset of A, contradicting the fact that S is an atom.

Conversely, let S^1 be an atom in $WCS(A^1)$ and $S \subseteq f^{-1}(S^1)$ be an atom in $WCS(A)$. Then $\varphi \subseteq f(S) \subseteq S^1$ and since S^1 is an atom in $WCS(A^1)$, we have $f(S) = S^1$. ■

Corollary 2.19. Any w-convex homomorphism maps acyclic psosets into acyclic psosets.

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