Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4809–4823

http://www.ripublication.com/gjpam.htm

Uniform Order One Step Hybrid Block Method with two Generalized Off step Points for Solving Third Order Ordinary Differential Equations Directly

Ra'ft Abdelrahim

Department of Mathematics,
School of Quantitative Sciences,
College of Art and Sciences,
Univeristi Utara Malaysia
E-mail: Rafatshaab@yahoo.com

Zurni Omar

Department of Mathematics,
School of Quantitative Sciences,
College of Art and Sciences,
Univeristi Utara Malaysia
E-mail: zurni@uum.edu.my

Abstract

In this paper, the method of interpolation and collocation is adopted in developing a new one step hybrid block method for solving third order initial value problem of ordinary differential equations directly. In deriving this method the power series used as basis function to approximate the solution is interpolated at x_n and all off-step points while its third derivative is collocated at all points in the selected interval. The method is zero stable, consistent, convergent and having order four. The numerical results generated when the new method was applied to some third order initial value problems are better than existing methods in terms of accuracy.

AMS subject classification: 65L05, 65L06, 65L20.
Keywords: Hybrid method, Block method, Third order differential equation, power series, Two off step points.

1. Introduction

In this article, a general third order ordinary differential equation (ODEs) of the form

$$y^{'''} = f(x, y, y^{'}, y^{''}),\ x \in [a, b]\,. \tag{1.1}$$

with three initial conditions $y(a) = \eta_0,\ y^{'}(a) = \eta_1,\ y^{''}(a) = \eta_2$ is considered.

Many scholars have developed numerical methods for solving third order initial value problems directly such as [8], [3], [6] and others. In order to improve the performance of the numerical methods mentioned above, researchers adopted predictor corrector mode which has been found to be very expensive to implement in term of function to be evaluated per step. Furthermore, the development of separate predictor consumes human effort, but and this approach also needs more subroutines which lead to inefficiency of the method in term of error[2]. To overcome the setbacks mentioned, a hybrid block method which is a class of linear multistep which uses the information at the off step points is applied. The idea of this method is to exploit the advantages of both block and hybrid methods which include using information of all points in selected interval simultaneously and overcoming zero stability barrier. This approach has been proposed by [3], [9] and [2]. [5], [6] and [1] among others proposed hybrid block method for solving third order ODEs with some specific off step points which its accuracy depends on the off step points chosen. The aim of this paper is to develop a new one step hybrid block method with generalized two off-step points for solving third order ODEs directly.

2. Derivation of the Method

In this section, a hybrid one step block method with generalized two off-step points x_{n+s} and x_{n+r} for solving (1.1) is derived.

Let the approximate solution to the Equation(1) be the power series of the form

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left(\frac{x - x_n}{h}\right)^i,\ x \in \left[x_n, x_{n+1}\right] \tag{2.1}$$

where $n = 0, 1, 2, \ldots, N-1$, v denotes of the number of interpolation points which is equal to the order of differential equation, m represents the number of collocation points, $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$.

Differentiating (2.1) three times gives

$$\begin{aligned} y^{'''}(x) &= f(x, y, y^{'}, y^{''}) \\ &= \sum_{i=3}^{v+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x - x_n}{h}\right)^{i-3}. \end{aligned} \tag{2.2}$$

where $v = 3$ and $m = 4$. Equation (2.1) is interpolated at x_n, x_{n+s} and x_{n+r}, while equation (2.2) is collocated at all points i.e $x_n, x_{n+s}, x_{n+r}, x_{n+1}$ in the selected interval to obtain the following equations which can be written in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 \\ 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24s}{h^3} & \frac{60s^2}{h^3} & \frac{120s^3}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24r}{h^3} & \frac{60r^2}{h^3} & \frac{120r^3}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s} \\ y_{n+r} \\ f_n \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix} \tag{2.3}$$

Gaussian elimination method is applied in (2.3) to find the values of a_i, $i = 0(1)6$. Then, these values are substituted into equation (2.1) to give a continuous implicit scheme of the form

$$y(x) = \sum_{i=0,s,r} \alpha_i y_{n+i} + \sum_{i=0}^{1} \beta_i f_{n+i} + \sum_{i=s,r} \beta_i f_{n+i} \tag{2.4}$$

The first and second derivatives of equation (2.4) are

$$y'(x) = \sum_{i=0,s,r} \frac{\partial}{\partial x}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial}{\partial x}\beta_i(x) f_{n+i} + \sum_{i=s,r} \frac{\partial}{\partial x}\beta_i(x) f_{n+i} \tag{2.5}$$

$$y''(x) = \sum_{i=0,s,r} \frac{\partial^2}{\partial x^2}\alpha_i(x) y_{n+i} + \sum_{i=0}^{1} \frac{\partial^2}{\partial x^2}\beta_i(x) f_{n+i} + \sum_{i=s,r} \frac{\partial^2}{\partial x^2}\beta_i(x) f_{n+i} \tag{2.6}$$

respectively where

$$\alpha_0 = \frac{(x_n - x + hs)(x_n - x + hr)}{(h^2rs)}$$

$$\alpha_s = \frac{(x - x_n)(x_n - x + hr)}{(h^2s(r - s))}$$

$$\alpha_r = \frac{(x - x_n)(x - x_n - hs)}{h^2r(r - s)}$$

$$\beta_0 = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{(120h^3rs)}(h^3r^3 - 2h^3r^2s + x_n^3$$
$$-3h^3r^2 - 2h^3rs^2 - 3xx_n^2 + 12h^3rs + h^3s^3 - 3h^3s^2 + h^2r^2x$$
$$-h^2r^2x_n - 2h^2rsx + 2h^2rsx_n - 3h^2rx + 3h^2rx_n + h^2s^2x - x^3$$
$$-3h^2sx + 3h^2sx_n + hrx^2 - 2hrxx_n + hrx_n^2 + hsx^2 - 2hsxx_n$$
$$-h^2s^2x_n - 4hxx_n + 2hx_n^2 + 3x^2x_n + hsx_n^2 + 2hx^2)$$

$$\beta_s = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{120h^3s(s - 1)(r - s)}(h^3r^3 + h^3r^2s - 3h^3r^2$$
$$-3h^3rs - h^3s^3 + 2h^3s^2 + h^2r^2x - h^2r^2x_n + h^2rsx - h^2rsx_n$$
$$-3h^2rx + 3h^2rx_n - h^2s^2x - hsx^2 + 2h^2sx - 2h^2sx_n + hrx^2$$
$$-2hrxx_n + hrx_n^2 + 2hsxx_n - hsx_n^2 + 2hx^2 - 4hxx_n + 2hx_n^2$$
$$+h^3rs^2 - x^3 + 3x^2x_n - 3xx_n^2 + h^2s^2x_n + x_n^3)$$

$$\beta_r = \frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{120h^3r(r - 1)(r - s)}(h^3r^3 - h^3r^2s + x^3$$
$$-h^3rs^2 + 3h^3rs + 3h^3s^2 + h^2r^2x - h^2r^2x_n - h^2rsx - 2hx_n^2$$
$$+h^2rsx_n - 2h^2rx + 2h^2rx_n - h^2s^2x + h^2s^2x_n - 3h^2sx_n - x_n^3$$
$$+4hxx_n + hrx^2 - 2hrxx_n + hrx_n^2 + 2hsxx_n - hsx_n^2 - 2hx^2$$
$$-hsx^2 - h^3s^3 - 2h^3r^2 - 3x^2x_n + 3xx_n^2 + 3h^2sx)$$

$$\beta_1 = -\frac{(x - x_n)(x_n - x + hs)(x_n - x + hr)}{120h^3(s - 1)(r - 1)}(-2h^3r^2s + h^3r^3$$
$$-2h^3rs^2 + h^3s^3 + h^2r^2x - h^2r^2x_n - 2h^2rsx + 2h^2rsx_n - x^3$$
$$-h^2s^2x_n + hrx^2 - 2hrxx_n + hrx_n^2 + hsx^2 - 2hsxx_n + 3x^2x_n$$
$$+h^2s^2x + hsx_n^2 - 3xx_n^2 + x_n^3)$$

Equation(2.4)is then evaluated at the non-interpolating point i.e. x_{n+1} and Equations (2.5) and (2.6) are evaluated at all points to produce the discrete schemes and its derivatives. Next, the discrete scheme and its derivative at x_n are combined to obtained the following equation in the matrix form

$$A^{[2]_3}Y_m^{[2]_3} = B_1^{[2]_3}R_1^{[2]_3} + B_2^{[2]_3}R_2^{[2]_3} + B_3^{[2]_3}R_3^{[2]_3} + h^3D^{[2]_3}R_4^{[2]_3} + h^3E^{[2]_3}R_5^{[3]_3} \quad (2.7)$$

where

$$A^{[2]_3} = \begin{pmatrix} \frac{-(r-1)}{(s(r-s))} & \frac{(s-1)}{(r(r-s))} & 1 \\ \frac{-r}{hs(r-s)} & \frac{s}{hr(r-s)} & 0 \\ \frac{2}{h^2(rs-s^2)} & \frac{2}{h^2(rs-s^2)} & 0 \end{pmatrix}, Y_m^{[2]_3} = \begin{pmatrix} y_{n+s} \\ y_{n+r} \\ y_{n+1} \end{pmatrix},$$

$$B_1^{[2]_3} = \begin{pmatrix} 0 & 0 & \frac{(r-1)(s-1)}{rs} \\ 0 & 0 & \frac{-(r+s)}{(hrs)} \\ 0 & 0 & \frac{2}{(h^2rs)} \end{pmatrix}, \; R_1^{[2]_3} = \begin{pmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

$$B_2^{[2]_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; R_2^{[2]_3} = \begin{pmatrix} y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix},$$

$$B_3^{[2]_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; R_3^{[2]_3} = \begin{pmatrix} y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix},$$

$$D^{[2]_3} = \begin{pmatrix} 0 & 0 & D_{13}^{[2]_3} \\ 0 & 0 & D_{23}^{[2]_3} \\ 0 & 0 & D_{33}^{[2]_3} \end{pmatrix}, \; R_4^{[2]_3} = \begin{pmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix},$$

$$E^{[2]_3} = \begin{pmatrix} E_{11}^{[2]_3} & E_{12}^{[2]_3} & E_{13}^{[2]_3} \\ E_{21}^{[2]_3} & E_{21}^{[2]_3} & E_{23}^{[2]_3} \\ E_{31}^{[2]_3} & E_{32}^{[2]_3} & E_{33}^{[2]_3} \end{pmatrix}, \; R_5^{[2]_3} = \begin{pmatrix} f_{n+s} \\ f_{n+r} \\ f_{n+1} \end{pmatrix}$$

with

$$D_{13}^{[2]_3} = \frac{h^3(r-1)(s-1)}{120rs}(r^3 - 2r^2s - 2r^2 - 2rs^2 + 10rs \\ -2r + s^3 - 2s^2 - 2s + 1)$$

$$D_{23}^{[2]_3} = \frac{h^2}{120}(r^3 - 2r^2s - 3r^2 - 2rs^2 + 12rs + s^3 - 3s^2)$$

$$D_{33}^{[2]_3} = -\frac{(h}{60rs}(r^4 - 2r^3s - 3r^3 - 2r^2s^2 + 12r^2s - 2rs^3 \\ +12rs^2 + s^4 - 3s^3)$$

$$E_{11}^{[2]_3} = \frac{h^3(r-1)}{120s(r-s)}(r^3 + r^2s - 2r^2 + rs^2 - 2rs - 2r - s^3 \\ +s^2 + s + 1)$$

$$E_{12}^{[2]_3} = \frac{h^3(s-1)}{120r(r-s)}(r^3 - r^2s - r^2 - rs^2 + 2rs - r - s^3 + 2s^2 \\ +2s - 1)$$

$$E_{13}^{[2]_3} = \frac{-h^3}{120}(r^3 - 2r^2s + r^2 - 2rs^2 - 2rs + r + s^3 + s^2 + s - 1)$$

$$E_{21}^{[2]_3} = \frac{h^2r}{120(r-s)(s-1)}(r^3 + r^2s - 3r^2 + rs^2 - 3rs - s^3 + 2s^2)$$

$$E_{22}^{[2]_3} = \frac{h^2s}{120(r-s)(r-1)}(r^3 - r^2s - 2r^2 - rs^2 + 3rs - s^3 + 3s^2)$$

$$E_{23}^{[2]_3} = \frac{-h^2rs(r+s)(r^2 - 3rs + s^2)}{120(r-1)(s-1)}$$

$$E_{31}^{[2]_3} = -\frac{h}{60s(r-s)(s-1)}(r^4 + r^3s - 3r^3 + r^2s^2 - 3r^2s + rs^3 \\ -3rs^2 - s^4 + 2s^3)$$

$$E_{32}^{[2]_3} = -\frac{h}{60r(r-s)(r-1)}(r^4 - r^3s - 2r^3 - r^2s^2 + 3r^2s - rs^3 \\ +3rs^2 - s^4 + 3s^3)$$

$$E_{33}^{[2]_3} = \frac{h}{60(r-1)(s-1)}(r^4 - 2r^3s - 2r^2s^2 - 2rs^3 + s^4)$$

Multiplying equation (2.7) by $(A^{[2]_3})^{-1}$ gives hybrid block method of the form.

$$\begin{aligned} I^{[2]_3} Y_m^{[2]_3} &= \bar{B}_1^{[2]_3} R_1^{[2]_3} + \bar{B}_2^{[2]_3} R_2^{[2]_3} + \bar{B}_3^{[2]_3} R_3^{[2]_3} \\ &\quad + h^3 \bar{D}^{[2]_3} R_4^{[2]_3} + h^3 \bar{E}^{[2]_3} R_5^{[2]_3} \end{aligned} \tag{2.8}$$

where

$$I^{[2]_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \bar{B}_1^{[2]_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{B}_2^{[2]_3} = \begin{pmatrix} 0 & 0 & hs \\ 0 & 0 & hr \\ 0 & 0 & h \end{pmatrix}, \ \bar{B}_3^{[2]_3} = \begin{pmatrix} 0 & 0 & \dfrac{h^2 s^2}{2} \\ 0 & 0 & \dfrac{h^2 r^2}{2} \\ 0 & 0 & \dfrac{h^2}{2} \end{pmatrix}$$

$$\bar{D}^{[2]_3} = \begin{pmatrix} 0 & 0 & \bar{D}_{13}^{[2]_3} \\ 0 & 0 & \bar{D}_{23}^{[2]_3} \\ 0 & 0 & \bar{D}_{33}^{[2]_3} \end{pmatrix}, \ \bar{E}^{[2]_3} = \begin{pmatrix} \bar{E}_{11}^{[2]_3} & \bar{E}_{12}^{[2]_3} & \bar{E}_{13}^{[2]_3} \\ \bar{E}_{21}^{[2]_3} & \bar{E}_{21}^{[2]_3} & \bar{E}_{23}^{[2]_3} \\ \bar{E}_{31}^{[2]_3} & \bar{E}_{32}^{[2]_3} & \bar{E}_{33}^{[2]_3} \end{pmatrix}$$

and the non-zero elements of $\bar{D}^{[2]_3}$ and $\bar{E}^{[2]_3}$ are given by

$$\begin{aligned} \bar{D}_{13}^{[2]_3} &= -\frac{s^3(3s - 15r + 3rs - s^2)}{120r} \\ \bar{D}_{23}^{[2]_3} &= \frac{r^3(15s - 3r - 3rs + r^2)}{120s} \\ \bar{D}_{33}^{[2]_3} &= \frac{(15rs - 3s - 3r + 1)}{120rs} \\ \bar{E}_{11}^{[2]_3} &= \frac{s^3(2s - 5r + 2rs - s^2)}{120(s-1)(r-s)} \\ \bar{E}_{12}^{[2]_3} &= -\frac{s^5(s-3)}{120r(r-1)(r-s)} \end{aligned}$$

$$\bar{E}_{13}^{[2]_3} = \frac{s^5(3r-s)}{120(r-1)(s-1)}$$

$$\bar{E}_{21}^{[2]_3} = \frac{r^5(r-3)}{120s(s-1)(r-s)}$$

$$\bar{E}_{21}^{[2]_3} = \frac{r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}$$

$$\bar{E}_{23}^{[2]_3} = -\frac{r^5(r-3s)}{120(s-1)(r-1)}$$

$$\bar{E}_{31}^{[2]_3} = -\frac{(3r-1)}{120s(s-1)(r-s)}$$

$$\bar{E}_{32}^{[2]_3} = \frac{(3s-1)}{120r(r-1)(r-s)}$$

$$\bar{E}_{33}^{[2]_3} = \frac{(5rs-2s-2r+1)}{120(s-1)(r-1)}$$

Equation (2.8) gives

$$\begin{aligned}
y_{n+s} &= y_n + hsy_n' + \frac{h^2s^2}{2}y_n'' - \frac{h^3s^3(3s-15r+3rs-s^2)}{120r}f_n \\
&\quad + \frac{h^3s^3(2s-5r+2rs-s^2)}{120(s-1)(r-s)}f_{n+s} + \frac{h^3s^5(3r-s)}{120(r-1)(s-1)}f_{n+1} \\
&\quad - \frac{h^3s^5(s-3)}{120r(r-1)(r-s)}f_{n+r} \\
y_{n+r} &= y_n + hry_n' + \frac{h^2r^2}{2}y_n'' + \frac{h^3r^3(15s-3r-3rs+r^2)}{120s}f_n \\
&\quad + \frac{h^3r^3(5s-2r-2rs+r^2)}{120(r-1)(r-s)}f_{n+r} - \frac{h^3r^5(r-3s)}{120(s-1)(r-1)}f_{n+1} \\
&\quad + \frac{h^3r^5(r-3)}{120s(s-1)(r-s)}f_{n+s} \\
y_{n+1} &= y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3(15rs-3s-3r+1)}{120rs}f_n \\
&\quad + \frac{h^3(3s-1)}{120r(r-1)(r-s)}f_{n+r} + \frac{h^3(5rs-2s-2r+1)}{120(s-1)(r-1)}f_{n+1} \\
&\quad - \frac{h^3(3r-1)}{120s(s-1)(r-s)}f_{n+s}
\end{aligned} \tag{2.9}$$

The first and second derivative of (2.9) are given by

$$
\begin{aligned}
y'_{n+s} &= y'_n + hsy''_n - \frac{h^2s^2(5s-20r+5rs-2s^2)f_n}{60r} \\
&+ \frac{h^2s^4(5r-2s)}{60(s-1)(r-1)}f_{n+1} - \frac{h^2s^4(2s-5)}{60r(r-1)(r-s)}f_{n+r} \\
&+ \frac{h^2s^2(5s-10r+5rs-3s^2)}{60(s-1)(r-s)}f_{n+s} \\
y'_{n+r} &= y'_n + hry''_n - \frac{h^2r^2(20s-5r-5rs+2r^2)}{60s}f_n \\
&+ \frac{h^2r^4(2r-5)}{60s(s-1)(r-s)}f_{n+s} - \frac{h^2r^4(2r-5s)}{60(r-1)(s-1)}f_{n+1} \\
&- \frac{h^2r^2(10s-5r-5rs+3r^2)}{60(r-1)(r-s)}f_{n+r} \\
y'_{n+1} &= y'_n + hy''_n - \frac{h^2(20rs-5s-5r+2)}{60rs}f_n \\
&- \frac{h^2(5r-2)}{60s(s-1)(r-s)}f_{n+s} + \frac{h^2(5s-2)}{60r(r-1)(r-s)}f_{n+r} \\
&+ \frac{h^2(10rs-5s-5r+3)}{60(s-1)(r-1)}f_{n+1} \\
y''_{n+s} &= y''_n + \frac{hs^3(2r-s)}{12(s-1)(r-1)}f_{n+1} - \frac{hs(2s-6r+2rs-s^2)}{(12r)}f_n \\
&- \frac{hs^3(s-2)}{12r(r-1)(r-s)}f_{n+r} + \frac{hs(4s-6r+4rs-3s^2)}{(12(s-1)(r-s))}f_{n+s} \\
y''_{n+r} &= y''_n + \frac{hr(6s-2r-2rs+r^2)}{(12s)}f_n + \frac{hr^3(r-2)}{12s(s-1)(r-s)}f_{n+s} \\
&+ \frac{hr(6s-4r-4rs+3r^2)}{12(r-1)(r-s)}f_{n+r} \\
&- \frac{hr^3(r-2s)}{(12(s-1)(r-1)}f_{n+1} \\
y''_{n+1} &= y''_n + \frac{h(6rs-2s-2r+1)}{(12rs)}f_n - \frac{h(2r-1)}{12s(s-1)(r-s)}f_{n+s} \\
&+ \frac{h(2s-1)}{12r(r-1)(r-s)}f_{n+r} + \frac{h(6rs-4s-4r+3)}{12(s-1)(r-1)}f_{n+1} \qquad (2.10)
\end{aligned}
$$

3. Properties of Method

3.1. Order of Methhod

The linear difference operator L associated with (2.8) is defined as

$$L[y(x);h] = IY_m - \bar{B}_1^{[2]_2} R_1^{[2]_3} - \bar{B}_2^{[2]_3} R_2^{[2]_3} - \bar{B}_3^{[2]_3} R_3^{[2]_3} - h^3 \left[\bar{D}^{[2]_3} R_4^{[2]_3} + \bar{E}^{[2]_3} R_5^{[2]_3} \right] \quad (3.1)$$

where $y(x)$ is an arbitrary test function continuously differentiable on [a,b]. $Y_m^{[2]_3}$ and $R_3^{[2]_3}$ components are expanded in Taylor’s series respectively and their terms are collected in powers of h to give

$$L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \bar{C}_2 h y''(x) + \cdots \quad (3.2)$$

Definition 3.1. Hybrid block method (2.8) and associated linear operator (3.1) are said to be of order p, if $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_{p+2} = 0$ with error vector constants $\bar{C}_{p+3} \neq 0$.

Expanding the functions of $y$and$f$- in (2.8) gives

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{(s)^j h^j}{j!} y_n^j - y_n - shy_n' - \frac{h^2 s^2}{2} y_n'' + \frac{s^3 h^3 (3s - 15r + 3rs - s^2)}{120r} y_n''' \\ - \frac{s^5(3r-s)}{120(r-1)(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} + \frac{s^5(s-3)}{120r(r-1)(r-s)} \sum_{j=0}^{\infty} \frac{r^j h^{j+3}}{j!} y_n^{j+3} \\ - \frac{s^3(2s - 5r + 2rs - s^2)}{120(s-1)(r-s)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \\ \sum_{j=0}^{\infty} \frac{(r)^j h^j}{j!} y_n^j - y_n - rhy_n' - \frac{h^2 r^2}{2} y_n'' - \frac{r^3 h^3 (15s - 3r - 3rs + r^2)}{120s} y_n''' \\ + \frac{r^5(r-3s)}{120(s-1)(r-1)} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\ - \frac{r^3(5s - 2r - 2rs + r^2)}{120(r-1)(r-s)} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+3}}{j!} y_n^{j+2} - \frac{r^5(r-3)}{120s(s-1)(r-s)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - hy_n' - \frac{h^2}{2} y_n'' - \frac{h^3(15rs - 3s - 3r + 1)}{120rs} y_n''' \\ - \frac{(5rs - 2s - 2r + 1)}{120(s-1)(r-1)} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+3} \\ - \frac{(3s-1)}{120r(r-1)(r-s)} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+3}}{j!} y_n^{j+2} + \frac{(3r-1)}{120s(s-1)(r-s)} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j yields the order of the method which is $[4, 4, 4]^T$

with error constant

$$\bar{C}_7 = \begin{bmatrix} \dfrac{s^5(7s - 21r + 7rs - 3s^2)}{20160} \\ -\dfrac{r^5(21s - 7r - 7rs + 3r^2)}{20160} \\ -\dfrac{(21rs - 7s - 7r + 3)}{20160} \end{bmatrix} \tag{3.3}$$

for all $s, r \in (0, 1) \setminus \{\{r = \dfrac{3s^2 - 7s}{7s - 21}\} \cup \{s = \dfrac{7r - 3r^2}{21 - 7r}\} \cup \{s = \dfrac{7r - 3}{21r - 7}\}\}$.

3.2. Zero Stability

Definition 3.2. The hybrid block method (2.8) is said to be zero stable if the first characteristic polynomial $\pi(z)$ having roots such that $|z_r| \leq 1$, and if $|z_r| = 1$ then the multiplicity of z_r must not greater than three.

In finding the zero-stability of the block (2.8), we only put into consideration the first characteristic polynomial according to Definition(3.2). Therefore we have

$$\begin{aligned} \Pi(z) &= |z\, I^{[2]_3} - \bar{B}_1^{[3]_3}| = \left| z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right| \\ &= z^2(z - 1) \end{aligned}$$

which implies $z = 0, 0, 1$.
Hence, our method is zero stable for all $s, r \in (0, 1)$.

3.3. Consistency

Definition 3.3. The one step hybrid block method (2.8) is said to be consistent if order of the method greater than or equal one i.e. $P \geq 1$.

The block method (2.8) is consistent because it satisfies the condition stated in Definition (3.3).

3.4. Convergence

Theorem 3.4. [Henrici, 1962] Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.

Following the theorem above, the new block method proposed is convergent since it is consistent and zero stable.

3.5. Region of Absolute Stability

In this article, the locus method was adopted to determine the region of absolute stability. The method (2.8) is said to be absolutely stable if for a given h, all roots of the characteristic polynomial $\pi(z, h) = \rho(z) - h^3\sigma(z)$, satisfies $|z_t| < 1$. The test equation $y = -\lambda^3 y$

is substituted in (2.8) where $\bar{h} = -\lambda^3 h^3$ and $\lambda = \frac{df}{dy}$. Substituting $r = \cos\theta - i\sin\theta$ and considering real part yields

$$\bar{h}(\theta, h) = \frac{(172800\cos\theta - 172800)}{(10r^2s^2 - 6r^2s^3 - 6r^3s^2 + 3r^3s^3 + r^3s^3\cos\theta)} \tag{3.4}$$

3.6. Specific Case

Substituting $s = \frac{1}{5}, r = \frac{3}{5}$ into equation (10)-(12) the following block of one step with two hybrid points and its derivatives are obtained

$$\begin{aligned}
y_{n+\frac{1}{5}} &= y_n + \frac{h}{5}y_n' + \frac{h^2}{50}y_n'' + \frac{101h^3}{112500}f_n - \frac{7h^3}{90000}f_{n+\frac{3}{5}} + \frac{h^3}{2000}f_{n+\frac{1}{5}}.\\
&\quad + \frac{30223145490365 72875h^3}{226673591177742970257408}f_{n+1}\\
y_{n+\frac{3}{5}} &= y_n + \frac{3h}{5}y_n' + \frac{9h^2}{50}y_n'' + \frac{27h^3}{2500}f_n + \frac{9h^3}{10000}f_{n+\frac{3}{5}} + \frac{243h^3}{10000}f_{n+\frac{1}{5}}.\\
&\quad - \frac{125h^3}{75557863725914323419136}f_{n+1}\\
y_{n+1} &= y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{36}f_n + \frac{5h^3}{144}f_{n+\frac{3}{5}} + \frac{5h^3}{48}f_{n+\frac{1}{5}}\\
&\quad - \frac{588233h^3}{14167099448608935641088 0000}f_{n+1} \qquad (3.5)\\
y_{n+\frac{1}{5}}' &= y_n' + \frac{h}{5}y_n'' + \frac{131h^2}{11250}f_n + \frac{16370870473948104375h^2}{75557863725914323419136}f_{n+1}\\
&\quad - \frac{23h^2}{18000}f_{n+\frac{3}{5}} + \frac{113h^2}{12000}f_{n+\frac{1}{5}}.\\
y_{n+\frac{3}{5}}' &= y_n' + \frac{3h}{5}y_n'' + \frac{21h^2}{625}f_n - \frac{30600934808995 2798125h^2}{226673591177742970257408}f_{n+1}\\
&\quad + \frac{39h^2}{2000}f_{n+\frac{3}{5}} + \frac{513h^2}{4000}f_{n+\frac{1}{5}}.
\end{aligned}$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{18} f_n + \frac{1475739525896764128197747h^2}{14167099448608935641088000} f_{n+1}$$
$$+ \frac{25h^2}{144} f_{n+\frac{3}{5}} + \frac{25h^2}{96} f_{n+\frac{1}{5}}.$$
$$y''_{n+\frac{1}{5}} = y''_n + \frac{h}{12} f_n + \frac{196765270119568531 25h}{944473296573929042739 2} f_{n+1}$$
$$- \frac{h}{80} f_{n+\frac{3}{5}} + \frac{61h}{480} f_{n+\frac{1}{5}}.$$
$$y''_{n+\frac{3}{5}} = y''_n + \frac{3h}{100} f_n - \frac{1062532458645670468 75h}{944473296573929042739 2} f_{n+1}$$
$$- \frac{3h}{16} f_{n+\frac{3}{5}} + \frac{63h}{160} f_{n+\frac{1}{5}}.$$
$$y''_{n+1} = y''_n + \frac{h}{12} f_n + \frac{1598717819721494473304 33h}{1180591620717411303424000} f_{n+1}$$
$$+ \frac{25h}{48} f_{n+\frac{3}{5}} + \frac{25h}{96} f_{n+\frac{1}{5}}. \tag{3.6}$$

Replacing $s = \frac{1}{5}, r = \frac{3}{5}$ into Equation(3.3) gives the order of the method to be $[4, 4, 4]^T$ with vector of error constants

$$\bar{C}_7 = \begin{bmatrix} -1.663492e^{-7} \\ -9.257143e^{-7} \\ 3.968254e^{-6} \end{bmatrix}$$

After substituting the values of s and r in (3.4), the stability interval $(-4411614, 0)$ is obtained.

3.6.1 Numerical Examples

Two Numerical Examples were used to ascertain the accuracy of the method. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Problem 1: $y''' + y = 0,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1.$
Exact solution: $y(x) = e^{-x}$ with $h = 0.1$.

Table1: Comparison of the new method with[2] for solving Problem 1

x	Exact solution	Computed solution	Error in our method, p=4	Error in [2], p=7
0.1	0.90483741803595952	0.90483741803633722	$3.7769e^{-13}$	$2.4525e^{-13}$
0.2	0.81873075307798182	0.81873075306295096	$1.5030e^{-11}$	$6.2109e^{-11}$
0.3	0.74081822068171777	0.74081822060941671	$7.2301e^{-11}$	$1.5746e^{-10}$
0.4	0.67032004603563933	0.67032004584064453	$1.9499e^{-10}$	$3.1477e^{-9}$
0.5	0.60653065971263342	0.60653065930829053	$4.0434e^{-10}$	$6.1617e^{-9}$
0.6	0.54881163609402650	0.54881163537464328	$7.1938e^{-10}$	$9.1732e^{-9}$
0.7	0.49658530379140953	0.49658530263435252	$1.1570e^{-9}$	$1.3329e^{-8}$
0.8	0.44932896411722162	0.44932896238495096	$1.7322e^{-9}$	$1.6378e^{-8}$
0.9	0.40656965974059917	0.40656965728267613	$2.4579e^{-9}$	$1.7134e^{-8}$
1.0	0.36787944117144233	0.36787943782653731	$3.3449e^{-9}$	$7.4405e^{-9}$

Problem 2: $y''' = 3\sin x,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,$

Exact solution: $y(x) = 3\cos x + \frac{x^2}{2} - 2$ with $h = 0.1$.

Table2: Comparison of the new method with [1] for solving Problem 2

x	Exact solution	Computed solution	Error in our method, p=4	Error in [1], p=6
0.1	0.99001249583407702	0.99001249583401840	$5.8619e^{-14}$	$2.5934e^{-12}$
0.2	0.96019973352372512	0.96019973352470489	$9.7977e^{-13}$	$1.1857e^{-11}$
0.3	0.91100946737681809	0.91100946738433841	$7.5203e^{-12}$	$2.6224e^{-11}$
0.4	0.84318298200865538	0.84318298203673059	$2.8075e^{-11}$	$4.7034e^{-10}$
0.5	0.75774768567111828	0.75774768574629991	$7.5181e^{-10}$	$7.2700e^{-11}$
0.6	0.65600684472903525	0.65600684489431116	$1.6527e^{-10}$	$1.0437e^{-10}$
0.7	0.53952656185346548	0.53952656217199491	$3.1852e^{-10}$	$1.4049e^{-10}$
0.8	0.41012012804149611	0.41012012860014274	$5.5864e^{-10}$	$1.8197e^{-10}$
0.9	0.26982990481199343	0.26982990572462090	$9.1262e^{-10}$	$2.2736e^{-10}$
1.0	0.12090691760441930	0.12090691901491930	$1.4105e^{-9}$	$2.7729e^{-10}$

4. Conclusion

A new one step hybrid block method with generalized two off-step points for the direct solution of third order ordinary differential equation has been developed successfully.

The developed method is consistent, zero-stable and also convergent. When solving the same problems, the numerical results confirm that the new method produces better accuracy if compared to the existing methods.

References

[1] A. Adesanya1, D. Udoh, A. Ajileye, A new hybrid block method for the solution of general third order initial value problems of ordinary differential equations, In: *International Journal of Pure and Applied Mathematics*, **86** (2013), 365–375.

[2] Awoyemi, D. Kayode, S. Adoghe, L. A four-point fully implicit method for the numerical integration of third-order ordinary differential equations, In: *Internal Journal of Physical Sciences*, **9** (2014), 7–12.

[3] T. A. Anake, D. O. Awoyemi and A. A. Adesanya, A one step method for the solution of general second order ordinary differential equations, *International Journal of science and Technology*, (2012b), **2**(4), 159–163.

[4] T. A Anake , D. O. Awoyemi and A. A Adesanya, One-step implicit hybrid block method for the direct solution of general second order ordinary differential equations, *IAENG International Journal of Applied Mathematics*, (2012a), **42**(4), 224–228.

[5] K.M. Fasasi, A. O. Adesanya and S. O. Adee, One Step Continuous Hybrid Block Method for the Solution of $y''' = f(x, y, y', y'')$, *Journal of Natural Sciences Research*, (2014), **4**(10), 55–62.

[6] S. N. Jator, Solving second order initial value problems by a hybrid multistep method without predictors, *Applied Mathematics and Computation*, (2010), **217**(8), 4036–4046.

[7] J.D. Lambert, *Computational methods in ordinary differential equations*, (1973).

[8] Z. Omar and J.O. Kuboye, Developing Block Method of Order Seven for Solving Third Order Ordinary Differential Equations Directly using Multistep Collocation Approach, *International Journal of Applied Mathematics and Statistics*, (2015), **53**(3), 165–173.

[9] A. Sagir, An accurate computation of block hybrid method for solving stiff ordinary differential equations, *Journal of Mathematics*, (2012), **4**, 18–21.