Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4749-4761

http://www.ripublication.com

Approximate solutions of Abel integral equations using homotopy analysis transform method

A. Y. Almousa[1], E. Edfawy[1, 2] and K. El. Rashidy[3, 4]

[(1)]Department of Mathematics and Statistics, Faculty of Science, Taif University, 21974, Kingdom of Saudi Arabia,
[(2)]Department of Mathematics and Statistics, Faculty of Science, Assuit University, Egypt,
[(3)]Mathematics Department, College of arts and Science, Taif University, Ranyah, Saudi Arabia,
[(4)]Mathematics Department, Faculty of Science, Beni-Suef University, Egypt.
** Corresponding author. Email address: m_s_mohamed2000@yahoo.com*

Abstract

The main aim of the present work is to propose a new and simple algorithm for Abel's integral equations. Abel's integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. In this paper, by combined optimal homotopy analysis method (OHAM) and Laplace transform method, we produce a new powerful method and named OHATM. By using this method, we solve second kind singular integral equations of Abel type. Also, the convergence of the proposed method is proved. The numerical results show that the presented method is effective and accurate. Also, by plotting the h-curve, we show the convergence region of the examples.

Keywords: integral equation; Abel's integral equation; Optimal homotopy analysis transform method; Laplace transform.

1 Introduction

An integral equation is defined as an equation in which the unknown function $y(x)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics and applied sciences. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics [1-4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], Li and Zhao [12] by applying various kinds of analytical and numerical methods.

The main aim of this article is to present analytical and approximate solution of integral equations by using new mathematical tool like optimal homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science [13-18]. In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [19-27].

A typical form of an integral equation in $y(x)$ is of the form:

$$\mathbf{y(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)y(t)dt}, \qquad (1.\ 1)$$

where $K(x,t)$ is called the kernel of the integral equation (1. 1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $y(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x,t)$ and the function $f(x)$ in equation (1. 1) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $y(x)$ that will satisfy equation (1. 1) using a number of solution techniques.

This paper is organized as follows: At first, in Section 2, wepresent the optimal homotopy analysis transform method (OHATM). Next, we prove the convergence theorem in Section 3 and inSection 4, by solving examples we explain advantages ofOHATM and illustrate the regionof convergency by plotting the *h*-curves. Finally, Section 5 isconclusion.

2 Preliminaries and notations

In order to elucidate the solution procedure of the optimal homotopy analysis transform method, We consider the following integral equations of second kind:

$$\mathbf{y(x) = f(x) + \int_0^x K(x,t)y(t)dt, 0 \le x \le 1} \qquad (2.\ 1)$$

Now operating the Laplce transform on both side in Eq. (2.1), we get

$$\mathbf{L[y(x)] = L[f(x)] + L\left[\int_0^x K(x,t)y(t)dt\right]} \qquad (2.\ 2)$$

We define the nonlinear operator

$$\mathbf{N}[\phi(\mathbf{x};\mathbf{q})]=\mathbf{L}[\phi(\mathbf{x};\mathbf{q})]-\mathbf{L}[\mathbf{f}(\mathbf{x})]-\mathbf{L}\left[\int_0^x \mathbf{K}(\mathbf{x},\mathbf{t})\phi(\mathbf{x};\mathbf{q})\mathbf{dt}\right] \qquad (2.\ 3)$$

where $q\in[0,1]$ be an embedding parameter and $\phi(\mathbf{x};\mathbf{q})$ is the real function of x and q. By means of generalizing the traditional homotopy methods, the great mathematician Liao [13-14] constract the zero order deformation equation

$$(1-\mathbf{q})\mathbf{L}[\phi(\mathbf{x};\mathbf{q})-\mathbf{y}_0(\mathbf{x})]=\hbar\mathbf{q}\mathbf{H}(\mathbf{x})\mathbf{N}[\phi(\mathbf{x};\mathbf{q})], \qquad (2.\ 4)$$

where is a nonzero auxiliary parameter, $\mathbf{H}(\mathbf{x})\neq 0$ an auxiliary function, $\mathbf{y_0}(\mathbf{x})$ is an initial guess of $\mathbf{y}(\mathbf{x})$ and $\phi(\mathbf{x};\mathbf{q})$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in HATM. Obviously, when $q=0$ and $q=1$, it holds

$$\phi(\mathbf{x};\mathbf{0})=\mathbf{y}_0(\mathbf{x}),\ \phi(\mathbf{x};\mathbf{1})=\mathbf{y}(\mathbf{x}) \qquad (2.\ 5)$$

respectively. Thus, as q increases from 0 to 1, the solution varies from the initial guess to the solution. Expanding $\phi(x;q)$ in Taylor's series with respect to q, we have

$$\phi(\mathbf{x};\mathbf{q})=\mathbf{y}_0(\mathbf{x},\mathbf{t})+\sum_{\mathbf{m=1}}^{\infty}\mathbf{q}^m\mathbf{y}_m(\mathbf{x}), \qquad (2.\ 6)$$

where

$$\mathbf{y}_m(\mathbf{x})=\frac{1}{m!}\frac{\partial^m\phi(\mathbf{x};\mathbf{q})}{\partial q^m}\Big|_{q=0} \qquad (2.\ 7)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (2.6) converges at $q=1$, we have

$$\mathbf{y}(\mathbf{x})=\mathbf{y}_0(\mathbf{x})+\sum_{\mathbf{m=1}}^{\infty}\mathbf{y}_m(\mathbf{x}), \qquad (2.\ 8)$$

which must be one of the solutions of the original integral equations. Define the vectors

$$\overleftrightarrow{y}_n=\{\mathbf{y}_0(\mathbf{x}),\mathbf{y}_1(\mathbf{x}),\ldots\mathbf{y}_n(\mathbf{x})\}. \qquad (2.\ 9)$$

Differentiating equation (2.5) m-times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, we obtain the m^{th}-order deformation equation.

$$\mathbf{L}[\mathbf{y}_m(\mathbf{x})-\chi_m\mathbf{y}_{m-1}(\mathbf{x})]=\hbar\mathbf{q}\mathbf{H}(\mathbf{x})\mathbf{R}_m(\vec{\mathbf{y}}_{m-1},x) \qquad (2.\ 10)$$

where

$$\mathbf{R}_m(\vec{\mathbf{y}}_{m-1},x)=\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}\Big|_{\mathbf{q=0}} \qquad (2.\ 11)$$

and

$$\chi_m=\begin{cases}0 & m\leq 1,\\ 1 & m>1.\end{cases} \qquad (2.\ 12)$$

In this way, it is easily to obtain $\mathbf{y}_m(\mathbf{x})$ for $m\geq 1$, at m^{th} order, we have

$$\mathbf{y}(\mathbf{x})=\sum_{\mathbf{m=0}}^{M}\mathbf{y}_m(\mathbf{x}), \qquad (2.\ 13)$$

when M $\to \infty$ we get an accurate approximation of the original Eq (2.1).
Yabushita et. al [28] and Mohamed S. Mohamed et. al [29-31] applied the homotopy analysis method to nonlinear ODE's and suggested the so called optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region having physical meaning. Their approach is based on the square residual error. Let $\Delta(h)$ denote the square residual error of the governing equation (2. 1) and express as

$$\Delta(h) = \int_{\Omega} (N[\tilde{u}_n(t)])^2 \, d\Omega, \tag{2. 14}$$

Where

$$\tilde{u}_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t) \tag{2. 15}$$

the optimal value of h is given by a nonlinear algebraic equation as:

$$\frac{d\Delta(h)}{dh} = 0. \tag{2. 16}$$

3 Convergence analysis

Theorem1 [32]. As long as the series solution

$$\mathbf{y}(x) = \mathbf{y}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{y}_m(\mathbf{x}), \tag{3. 1}$$

converges, where $y_m(x)$ is governed by Eq. (2. 10), it must be the exact solution of the integral Eq. (1. 1).

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the homotopy analysis transform method. The simplicity and accuracy of the proposed method is illustrated through the following numerical examples by computing the absolute error,

$$E_i(x) = | u_i(x) - \tilde{u}_{im}(x) |, \qquad 1 \le i \le n, \tag{3. 2}$$

where $u_i(x)$ is the exact solution and $u_{im}(x)$ is the approximate solution of the problem. To demonstrate the effectiveness of the HATM algorithm discussed above, several examples of variation problems will be studied in this section. Here all the results are calculated by using the symbolic calculus software Mathematica 9.

4 Solving Abel's integral equations by the optimal homotopy analysis transform method (OHATM)

In this section, we shall illustrate the optimal homotopy analysis transform technique. To demonstrate the effectiveness of the OHATM algorithm discussed above, several examples of variational problems will be studied in this section. Alsographs of comparison between exact and approximate solutionswere plotted for different values of h. Furthermore, weplot the graphs of absolute error function and *h*-curve.

Example (4. 1). Let us Consider the Abel integral equation:

$$\mathbf{y(x)} = \sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x} - \int_0^x \frac{\mathbf{y(t)}}{\sqrt{\mathbf{x-t}}}\mathbf{dt}, \quad 0 \le x \le 1, \tag{4. 1. 1}$$

with the initial condition

$$\mathbf{y}(0) = \sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x}, \tag{4. 1. 2}$$

with the exact solution

$$\mathbf{y(x)} = \sqrt{x}, \tag{4. 1. 3}$$

where

$$\ell[\phi(x;q)] = L[\phi(x;q)], \tag{4. 1. 4}$$

with the property that

$\ell[c] = 0$, c is constants,

which implies that

$$\ell^{-1}(\bullet) = \int_0^t (\bullet)dt.$$

Taking Laplace transform of equation (4.1.1) both of sides subject to the initial condition, we get

$$\mathbf{L[y(x)]} - \mathbf{L}[\sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x}] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L[y(x)]}) = \mathbf{0}. \tag{4. 1. 5}$$

We now define the nonlinear operator as

$$\mathbf{N}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})] = \mathbf{L}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})] - \mathbf{L}[\sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x}] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}\mathbf{L}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})] = \mathbf{0}, \tag{4. 1. 6}$$

and then the m th-order deformation equation is given by

$$\mathbf{L}[\mathbf{y}_m(\mathbf{x}) - \boldsymbol{\chi}_m \mathbf{y}_{m-1}(\mathbf{x})] = \hbar\mathbf{H(x)R}_m(\overleftrightarrow{\mathbf{y}}_{m-1}). \tag{4. 1. 7}$$

Taking inverse Laplace transform of Eq. (4.1.7), we get

$$\mathbf{y}_m(\mathbf{x}) = \boldsymbol{\chi}_m \mathbf{y}_{m-1} + \hbar\mathbf{L}^{-1}[\mathbf{H(x)R}_m(\overleftrightarrow{\mathbf{y}}_{m-1})], \tag{4. 1. 8}$$

where

$$\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1}) = \mathbf{L}[\mathbf{y}_{\mathbf{m-1}}] - \mathbf{L}\left[\sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x}\right](1 - \chi_m) + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{y}_{m-1}]), \tag{4. 1. 9}$$

with assumption $\mathbf{H(x)} = \mathbf{1}$.

Let us take the initial approximation as

$$\mathbf{y}_0(x) = \sqrt{x} + \tfrac{1}{2}\boldsymbol{\pi}\,\mathbf{x},$$

the other components are given by

$$\begin{aligned} \mathbf{y}_1(x) &= \tfrac{h}{2}\pi x + \tfrac{2}{3}hx^{\frac{3}{2}}\pi, \\ \mathbf{y}_2(x) &= \tfrac{h}{2}(1+h)\pi x + \tfrac{2}{3}hx^{\frac{3}{2}}(1+2h)\pi + \tfrac{1}{4}h^2x^2\pi^2, \\ \mathbf{y}_3(x) &= \tfrac{1}{2}h(1+h)^2\pi x + \tfrac{2}{3}h(1+h)(1+3h)\pi x^{\frac{3}{2}} + \tfrac{1}{4}h^2(2+3h)x^2\pi^2 + \tfrac{4}{15}h^3\pi^2x^{\frac{5}{2}}, \end{aligned} \tag{4. 1. 10}$$

:

Proceeding in this manner, the rest of the components $y_n(x)$ for n ≥ 5 can be completely obtained and the series solutions are thus entirely determined.
The homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary h if we select $h = -1$, then

$$\begin{aligned} \mathbf{y}(x) &= \mathbf{y}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{y}_m(\mathbf{x}) \\ &= \sqrt{x} + 0.117665x^8 \end{aligned} \quad (4.\ 1.\ 11)$$

From Fig. 1 and Fig. 2 shows the graphical comperison between the exact solution and the approximate solution obtained by the OHATM. It can be seen that the solution obtained by the present method nearly identical to the exat solution. The above result is in complete agreement with [10].
To investigate the influence of *h* on the convergence of the solution series given by the HATM, we first plot theso-called *h*-curves of $y'(1)$. According to the *h*-curves, it is easy to discover the valid region of *h*. We used 15−terms in evaluating the approximate solution $y(x) = \sum_{i=0}^{15} y_i(x)$. In general, by means of the socalled *h*-curvei. e., a curve of a versus *h*. As pointed by Liao [13], the valid region of *h* is a horizontal line segment. Therefore, it isstraightforward to choose an appropriate range for *h* which ensure the convergence of the solution series. We stretchthe *h*-curve of $y'(1)$ in Fig. 3, which shows that the solution series is convergent when $-1 \leq h \leq$-0. 1.

Example (4. 2). Consider the Abel integral equation:

$$\mathbf{y}(x) = 2\sqrt{x} - \int_0^x \frac{\mathbf{y}(t)}{(x-t)^{\frac{1}{2}}} \mathbf{dt}, \quad 0 \leq x \leq 1, \quad (4.\ 2.\ 1)$$

with the initial condition,

$$y(0) = 2\sqrt{x}, \quad (4.\ 2.\ 2)$$

with the exact solution

$$y(x) = 1 - e^{\pi x} erfc\,(\sqrt{\pi x}), \quad (4.\ 2.\ 3)$$

wherethe complimentary error function defined as $erfc\,(x) = \frac{2}{\pi}\int_x^{\infty} e^{-u^2} du.$

To solve equation (4.2.1) by means of the homotopy analysis transform method we consider the following linear

$$\ell[\phi(x;q)] = L[\phi(x;q)], \quad (4.\ 2.\ 4)$$

with the property that

$$\ell[c] = 0,$$

which implies that

$$\ell^{-1}(\bullet) = \int_0^t (\bullet) dt$$

Taking Laplace transform of equation (4.2.1) both of sides subject to the initial condition, we get

$$\mathbf{L}[\mathbf{y}(\mathbf{x})] - \mathbf{L}\left[\sqrt{x}\right] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{y}(\mathbf{x})]) = \mathbf{0}. \qquad (4.\ 2.\ 5)$$

We now define the nonlinear operator as:

$$\mathbf{N}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})] = \mathbf{L}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})] - \mathbf{L}\left[\sqrt{x}\right] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\boldsymbol{\phi}(\mathbf{x};\mathbf{q})]) = \mathbf{0}, \qquad (4.\ 2.\ 6)$$

and then the mth-order deformation equation is given by

$$\mathbf{L}[\mathbf{y}_m(\mathbf{x}) - \boldsymbol{\chi}_m \mathbf{y}_{m-1}(\mathbf{x},\mathbf{n})] = \hbar \mathbf{H}(\mathbf{x})\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1}), \qquad (4.\ 2.\ 7)$$

taking inverse Laplace transform of Eq. (4.2.7), we get

$$\mathbf{y}_m(\mathbf{x},\mathbf{t}) = \boldsymbol{\chi}_m \mathbf{y}_{m-1} + \hbar \mathbf{L}^{-1}[\mathbf{H}(\mathbf{x})\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1})], \qquad (4.\ 2.\ 8)$$

where

$$\mathbf{R}_m(\overleftrightarrow{\mathbf{y}}_{m-1}) = \mathbf{L}[\mathbf{y}_{\mathbf{m-1}}] - \mathbf{L}\left[\sqrt{x}\right](1-\chi_m) + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{y}_{m-1}]), \qquad (4.\ 2.\ 9)$$

with assumption $\mathbf{H}(\mathbf{x}) = \mathbf{1}$. Let us take the initial approximation as

$y_0(x) = 2\sqrt{x},$

the other components are given by

$$\mathbf{y}_1(x) = h\pi x,$$

$$\mathbf{y}_2(x) = h(1+h)\pi x + \tfrac{4}{3}h^2 x^{\frac{3}{2}}, \qquad (4.\ 2.\ 10)$$

$$\mathbf{y}_3(x) = h(1+h)^2 \pi x + \tfrac{8}{3}h^2(1+h)\pi x^{\frac{3}{2}} + \tfrac{1}{2}h^3\pi^2 x^2,$$

:

:

Hence the solution of the Eq. (4. 2. 1) at $h = -1$ is given as

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(x) + \sum_{\mathbf{m=1}}^{\infty} \mathbf{y}_m(\mathbf{x}) = 2\sqrt{x} - \pi x + \tfrac{4}{3}h^2 x^{\frac{3}{2}} - \tfrac{1}{2}\pi^2 x^2 + \tfrac{8}{15}\pi^2 x^{\frac{5}{2}} - \tfrac{1}{6}\pi^3 x^3 + \ldots$$

$$= \sum_{\mathrm{n=1}}^{\infty} \frac{(-1)^{\mathrm{n-1}}(\pi x)^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} = 1 - e^{\pi x} erfc(\sqrt{\pi x}). \qquad (4.\ 2.\ 11)$$

This is the exact solution of the Abel integral Eq. (4. 2. 1). The homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h if we select $h = -1$, then the above result is in complete agreement with [33].

We stretch the *h*-curve of $y'(0.5)$ in Fig. 4, which shows that the solution series is convergent when $-1 \le h \le 0$.

From Fig. 1 to Fig. 6 shows the graphical comparison between the exact solution and the approximate solution obtained by the OHATM. It can be seen that the solution obtained by the present method nearly identical to the exact solution.

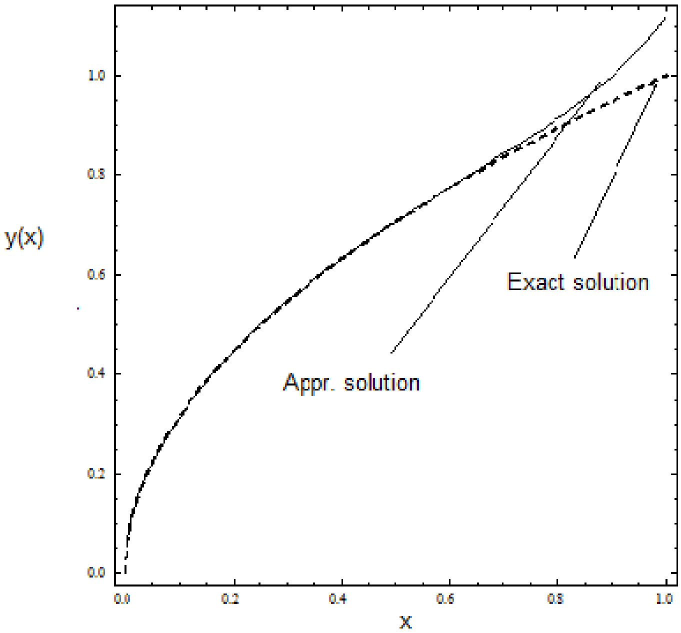

Fig. 1 The comparison between the exact solution (straight line)and the approximate solution (dotted line)of the Abel integral Eq. (4. 1. 1) at $h_{optimal} = -0.95$.

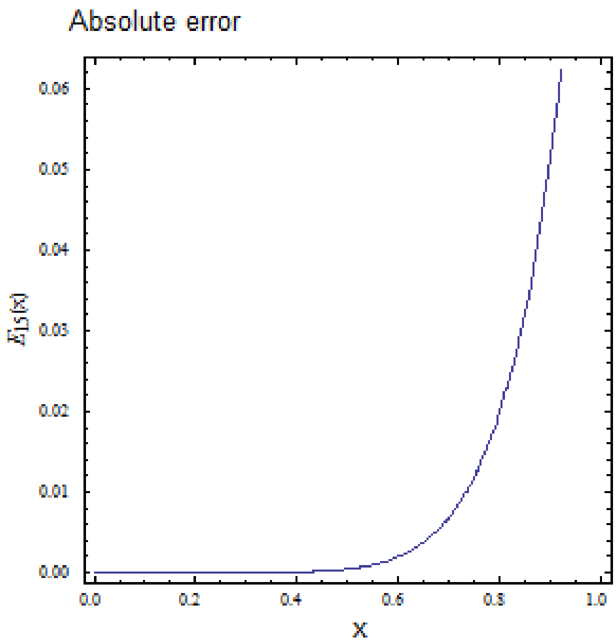

Fig. 2 The absolute error $E_{15}(y) = |y_{exact}(x) - y_{app.}(x)|$ **for Eq. (4. 1. 1)**

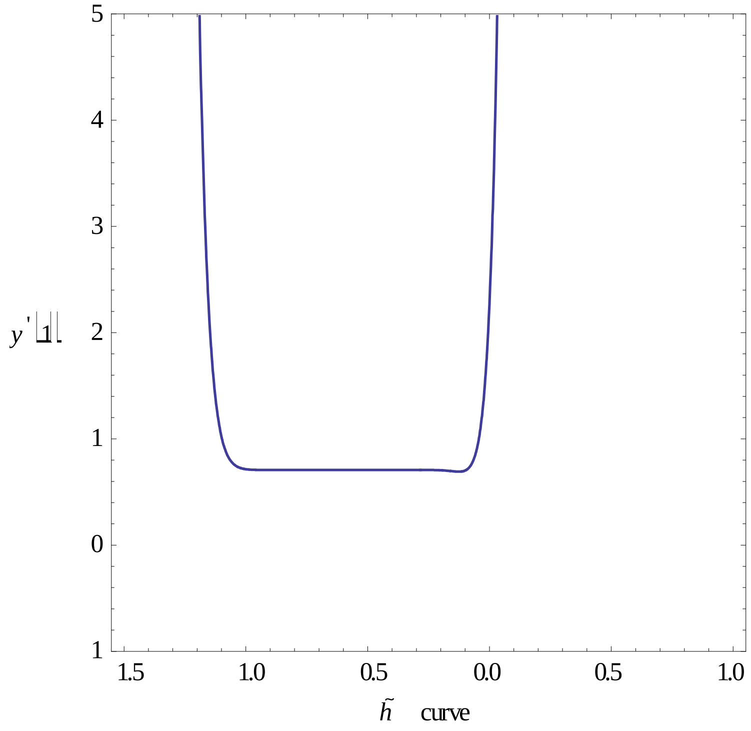

Fig. 3 The h-curve of $y'(1)$ at the 15th order of approximation when $H(x) = 1$.

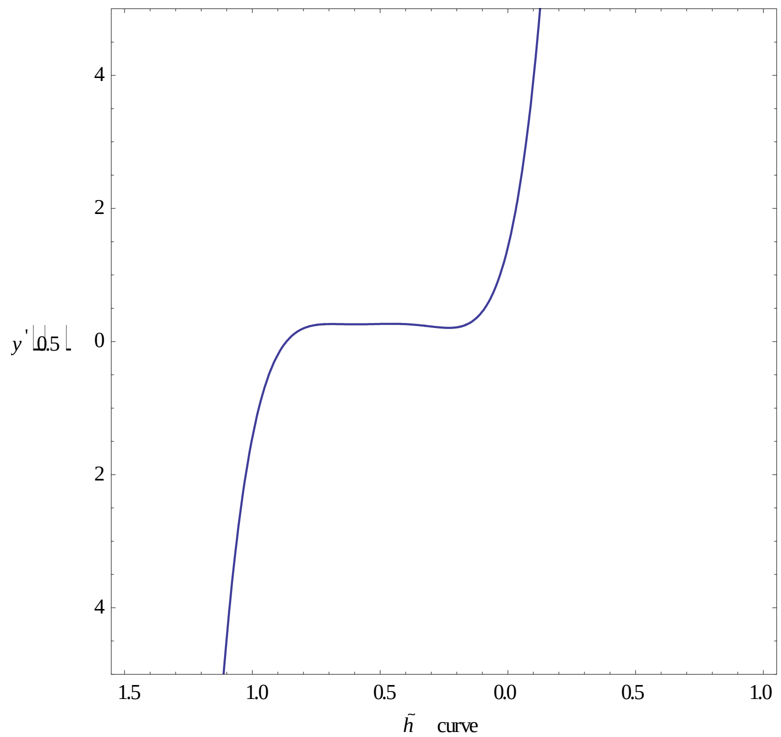

Fig. 4 The h-curve of $y'(0.5)$ at the 5th order of approximation when $H(x) = 1$.

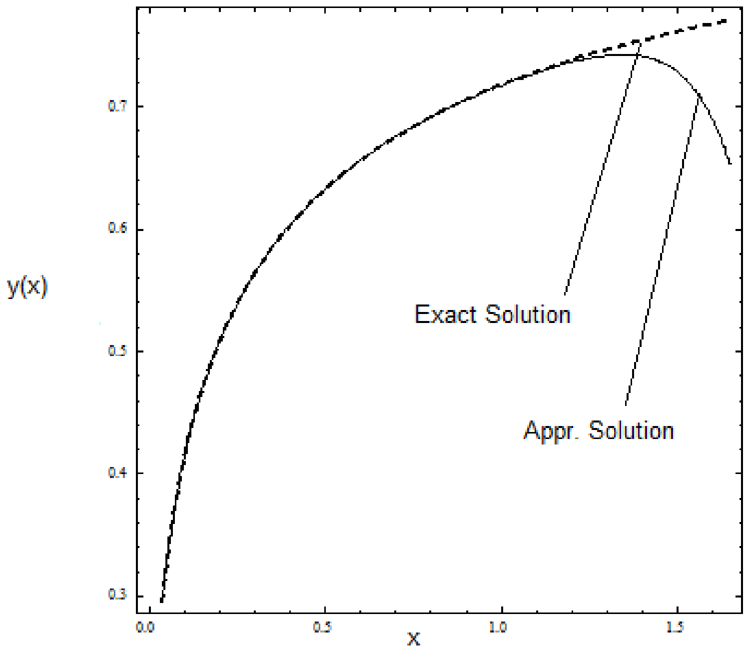

Fig. 5 The comparison between the exact solution (straight line) and the approximate solution (dotted line) of the Abel integral Eq. (4. 2. 1) at $h_{optimal} = -0.68$.

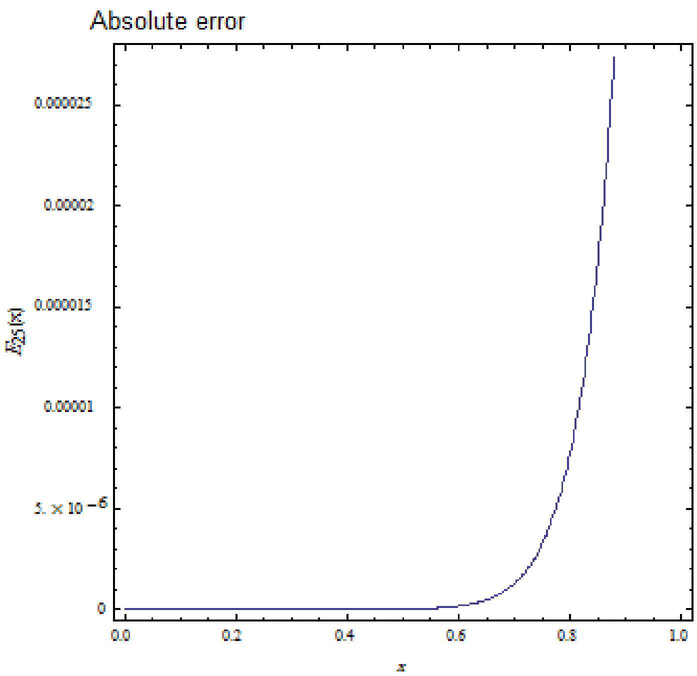

Fig. 6 The absolute error $E_{25}(y) = |y_{exact}(x) - y_{app.}(x)|$ **for Eq. (4. 2. 1)**

5 Conclusion

The main aim of this work is to provide the series solution of the Abel's integral equation by using the new optimal homotopy analysis transform method (OHATM). OHATM is coupling of homotopy analysis and Laplace transforms method. The new modification is a powerful tool to search for solutions of Abel's integral equation. The proposed method is employed without using linearization, discretization or transformation. It may be concluded that the OHATM is very powerful and efficient in finding the analytical solutions for a wide class of differential and integral equation. In (HATM) we have great choose the auxiliary linear operator L, and the auxiliary function $H(x)$ and initial function $y_0(x)$ but in other methods we have not these advantages. But most importantly, solutions given by the (HATM) contain the auxiliary parameter *h*, which provides us with a simple way to adjust and control convergence region and rate of solution series. The results show that the method is powerful and efficient techniques in finding exact and approximate solutions for integral equations.

References

[1] A. M. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications, Springer, Heidelberg, Dordrecht, London, New York, 2011.

[2] A. M. Wazwaz, A First Course in Integral Equations, World Scientific, New Jersey, 1997.

[3] R. Gorenflo, S. Vessella, Abel Integral Equations, Springer, Berlin, 1991.

[4] A. M. Wazwaz, M. S. Mehanna, The combined Laplace Adomian method for handling singular integral equation of heat transfer, Int. J. Nonlinear Sci. 10 (2010) 248-252.

[5] N. Zeilon, Sur quelques points de la theorie de l'equationintegraled'Abel, Arkiv. Mat. Astr. Fysik. 18 (1924) 1-19.

[6] R. K. Pandey, Om P. Singh, V. K. Singh, Efficient algorithms to solve singular integral equations of Abel type, Comput. Math. Appl. 57 (2009) 664-676.

[7] S. Kumar, Om P. Singh, Numerical inversion of Abel integral equation using homotopy perturbation method, Z. Naturforsch. 65a (2010) 677-682.

[8] S. Kumar, Om P. Singh, S. Dixit, Homotopy perturbation method for solving system of generalized Abel's integral equations, Appl. Appl. Math. 6 (2009) 268-283.

[9] S. Dixit, Om P. Singh, S. Kumar, A stable numerical inversion of generalized Abel's integral equation, Appl. Numer. Math. 62 (2012) 567-579.

[10] A. Saadatmandia, M. Dehghanb, A Collocation Method for Solving Abel's Integral Equations of First and Second Kinds, Verlag der Zeitschrift fu¨r Naturforschung, 752-756 (2008).

[11] M. Khan, M. A. Gondal, A reliable treatment of Abel's second kind singular integral equations, Appl. Math. Lett. 25 (11) (2012) 1666-1670.

[12] M. Li, W. Zhao, Solving Abel's type integral equation with Mikusinski's operator of fractional order, Adv. Math. Phys. (2013). Article ID 806984, 4 pp.

[13] S. J. Liao. On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147(2004): 499-513.

[14] S. J. Liao. Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169(2005): 1186-1194.

[15] S. Abbasbandy, T. Hayat, A. Alsaedi, M. M. Rashidi, Numerical and analytical solutions for Falkner-Skan flow of MHD Oldroyd-B fluid, Internat. J. Numer. Methods Heat Fluid Flow 24(2)(2014) 390-401.

[16] M. S. Mohamed. Analytical treatment of Abel integral equations by optimal homotopy analysis transform method, Journal of Information and Computing Science, 10(1) (2015) 019-028.

[17] S. Abbasbandy, R. Naz, T. Hayat, A. Alsaedi, Numerical and analytical solutions for Falkner-Skan flow of MHD Maxwell fluid, Appl. Math. Comput. 242 (2014) 569-575.

[18] M. S. Mohamed, Analytic approximate solution for some integral equations by optimal homotopy analysis transform method, Journal of Communications in Numerical Analysis. 2015 No. 1 (2015) 51-61.

[19] S. A. Khuri. A Laplace decomposition algorithm applied to a class of nonlinear differential equations, Journal of Applied Mathematics 1 (2001): 141-155.

[20] E. Yusufoglu. Numerical solution of Du¢ ng equation by the Laplace decomposition algorithm, Applied Mathematics and Computation 177 (2006): 572-580.

[21] K. Yasir. An effective modication of the Laplace decomposition method for nonlinear equations, International Journal of Nonlinear Sciences and Numerical Simulation 10 (2009): 1373-1376.

[22] Y. Khan, N. Faraz, S. Kumar, A. A. Yildirim. A coupling method of homotopy method and Laplace transform for fractional modells. UPB Sci Bull Ser A Appl Math Phys 74(1)(2012):57-68.

[23] J. Singh, D. Kumar, S. Kumar. New homotopy analysis transform algorithm to solve Volterra integral equation. Ain Shams Eng J 2013, doi:. org/10. 1016/j. asej. 2013. 07. 004.

[24] M. M. Khader, S. Kumar, S. Abbasbandy. New homotopy analysis transform method for solving the discontinued problems arising in nanotechnology. Chin Phys B 22(11)(2013):110201.

[25] A. S. Arife, S. K. Vanani, F. Soleymani. The Laplace homotopy analysis method for solving a general fractional difusion equation arising in nano-hydrodynamics. J Comput Theor Nanosci 10(2012):1-4.

[26] M. Khan, MA. Gondal, I. Hussain, SK. Vanani. A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on a semi-infinite domain. Math Comput Modell 2012;55:1143-50.

[27] ZM. Odibat. Differential transform method for solving Volterra integral equation with seperable kernel. Math Comput Modell 2008;48:1144-1149.

[28] K. Yabushita, M. Yamashita and K. Tsuboi, An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, J. Phys. A. Math. Gen., 40 (2007), 8403-8410.

[29] S. M. Abo-Dahab, Mohamed S. Mohamed and T. A. Nofal, A One Step Optimal Homotopy Analysis Method for propagation of harmonic waves in nonlinear generalized magneto-thermoelasticity with two relaxation times under inuence of rotation, Journal of in Abstract and Applied Analysis, (2013) 1-14.

[30] K. A. Gepreel, and M. S. Mohamed, An optimal homotopy analysis method nonlinear fractional differential equation, Journal of Advanced Research Dynamical and Control Systems, 6(2014), 1-10.

[31] M. S. Mohamed, Application of optimal HAM for solving the fractional order logistic equation, Applied and computational mathematics. 3(1)(2014):27-31.

[32] S. Noeiaghdam, E. Zarei and H. Barzegar Kelishami, Homotopy analysis transform method for solving Abel's integral equations of the first kind, Ain Shams Engineering Journal, (2015), http://dx. doi. org/10. 1016/j. asej. 2015. 03. 006.

[33] S. Kumar, A. Kumar, D. Kumar, J. Singh and A. Singh. Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, Journal of the Egyptian Mathematical Society, 2014.