Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4729–4747

http://www.ripublication.com/gjpam.htm

Triangular reaction-diffusion systems with compact result

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Abstract

The goal of this work is to study the global existence in time of solutions for some coupled systems of reaction diffusion which homogeneous Dirichlet boundary conditions.

We consider a triangular matrix of diffusion coefficients and we show the global existence of the solutions.

AMS subject classification: 35K57, 35K40, 35K55.
Keywords: Global solution, semigroups, local solution, reaction-diffusion systems.

1. Introduction

In this study, we are interested in global existence of classical solutions to the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = f(u, v), & \text{in }]0, +\infty[\times \Omega, \\ \dfrac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u, v) & \text{in }]0, +\infty[\times \Omega, \end{cases} \tag{1.1}$$

where Ω is a regular and bounded domain of $\mathbb{R}^n$, $(n \geq 1)$, with boundary $\partial\Omega$, $u = u(t, x)$, $v = v(t, x)$, $x \in \Omega$, $t > 0$ are real valued functions, Δ denotes the Laplacian operator, and the constants of diffusion a, c, and d are assumed to be nonnegative such that $a > d$.

System (1.1) is subjected to the following boundary conditions

$$u(t, x) = v(t, x) = 0, \quad \text{in }]0, +\infty[\times \partial\Omega, \tag{1.2}$$

and the initial data

$$u(0, .) = u_0, \; v(0, .) = v_0 \quad \text{in } \Omega \tag{1.3}$$

which are assumed to be continuous and nonnegative.

The above system (1.1)-(1.3) arises in physics, chemistry and various biological processes including population dynamics. (See [4, 17] and references therein).

Concerning the problem (1.1)-(1.3), we assume the following hypothesis:

$$u_0, \; v_0 \text{ are nonnegative functions in } L^1(\Omega) \tag{\textbf{H1}}$$

$$f(0, v) \geq 0, \quad \forall v \geq 0. \tag{\textbf{H2}}$$

$$\exists C \geq 0 : f(u, v) + g(u, v) \leq C(u + v + 1), \quad \forall u, v \geq 0. \tag{\textbf{H3}}$$

$$\exists \hat{C} \geq 0 : f(u, v) \leq \hat{C}(u + v + 1), \quad \forall u, v \geq 0. \tag{\textbf{H4}}$$

$$f(u, v) \leq \frac{a - d}{c} g(u, v), \quad \forall u, v \geq 0. \tag{\textbf{H5}}$$

The main question we want to address is the existence of global solutions for system (1.1)-(1.3). In fact, the subject of the global existence of reaction-diffusion systems has received a lot of attention in the last decades and several outstanding results have been proved by some of the major experts in the field. see [2, 3, 12].

For $c = 0$, this question has been investigated by many authors by considering special forms of the nonlinear terms f and g. Note that, Alikakos [1], treated the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = f(u, v), & \text{in }]0, +\infty[\times \Omega, \\ \dfrac{\partial v}{\partial t} - d\Delta v = g(u, v), & \text{in }]0, +\infty[\times \Omega, \end{cases} \tag{1.4}$$

with the same boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad \text{in }]0, +\infty[\times \partial\Omega, \tag{1.5}$$

and with initial condition (1.3), where $f(u, v) = -g(u, v) = -uv^{\sigma}$, and gave a positive answer to the problem of the global existence of a system (1.4)-(1.5)-(1.3) under the assumption

$$1 < \sigma < \sigma_0$$

where

$$\sigma_0 = 1 + \frac{2}{n}. \tag{1.6}$$

The method used in [1] is based on some Sobolev embedding theorems.

Note that the exponent σ_0 given in (1.6) is exactly the critical exponents given by Fujita [5] for the parabolic problem

$$\begin{cases} u_t = \Delta u + u^{\sigma}, \\ u(0, x) = u_0(x), \end{cases} \tag{1.7}$$

where u_0 in (1.7) is a nonnegative. Fujita proved that if $1 < \sigma < \sigma_0$, then (1.7) possesses no global nonnegative solutions while if $\sigma > \sigma_0$, both global and non global nonnegative solutions exist, depending on the nature of the initial energy.

In [16], Moumeni and Barrouk obtained a global existence result. By combining the compact semigroup methods and some L^1 estimates, we show that global solutions exist for a large class of the functions f and g.

In [15], Masuda obtained a global existence result for a large class of the parameter σ. In fact, by using some L^p estimates, he showed that the solution of the problem (1.1)-(1.5)-(1.3) exists globally in time if $\sigma > 1$.

The same result in [15] was obtained by Hollis, Martin and Pierre [9] by exploiting the duality of arguments in L^p techniques, allowing to derive the uniform boundeness of the solution.

Following Masuda's approach, Haraux and Youkana [7] established a global existence result of the system (1.1)-(1.5)-(1.3) for a large class of the function f and g. More precisely, they showed that for

$$f(u, v) = g(u, v) = -u\Psi(v)$$

the problem (1.1)-(1.5)-(1.3) admits a global solution provided that the following condition holds:

$$\lim_{v \to +\infty} \frac{\left[\log(1 + \Psi(v))\right]}{v} = 0.$$

In the general case, that is to say for

$$f(u, v) = -g(u, v) \tag{1.8}$$

the positivity of the function $g(u,v)$ together with the maximum principle of the heat operator give the following uniform estimate of the solution in $L^{\infty}(\Omega)$

$$\|u(t)\|_{\infty} \leq \|u_0\|_{\infty}, \quad \forall t \in [0, T_{\max}[$$

where $T_{\max}$ is the maximal time of existence. See Pazy [19] for more details.

Based on the Lyapunov functional method and for f and g satisfying (1.8), Kouachi [11] proves that the solution of the problem (1.1)-(1.5)-(1.3) exists globally in time if

$$\lim_{v \to +\infty} \frac{\left[\log(1 + f(u,v))\right]}{v} \leq \frac{8\alpha\beta}{n(\alpha-\beta)^2 \|u_0\|_{\infty}}$$

Recently, Moumeni and Salah Derradji [17] has established the existence of global solution using an approach that involves the Lyapunov's functional for the system (1.1)-(1.5)-(1.3) where the functions f and g are assumed to satisfy the condition

$$\sup\left(|f(r,s)|, |g(r,s)|\right) \leq C(r+s+1)^m, \quad \forall r,s \geq 0$$

where C is a positive constant and $m \geq 1$.

If $a \neq d$, an important particular case is that when $f \leq 0$, which means that the first substance is absorbed by the reaction. In this case, the problem of the global existence of a system (1.4) is reduced to obtaining a uniform estimate for v, since by the maximal principle we have

$$u(t,x) \leq \|u_0\|_{\infty}.$$

The global existence when $a > d$ has been treated by Kanel and Kirane [10] for a bounded domain and by Martin and Pierre [14] for whole space $\mathbb{R}^n$.

Still for the case $a \neq d$, but without assuming $a > d$, the answer is again positive to the problem of the global existence of the system (1.4) under condition (1.9) and a polynomial growth assumption on g:

$$g(u,v) \leq C(u+v+1)^{\gamma}, \quad \text{for all } u, v \geq 0 \text{ and some } \gamma \geq 1,$$

see [9] for more details.

If the diffusion coefficients are the same, that is, if $a = d$, then system (1.4) has a global solution under the condition

$$f(u,v) + g(u,v) \leq 0, \tag{1.9}$$

which is known as the mass dissipative structure condition. Indeed if $a = d$, then the solution (u,v) of (1.4) satisfies (by summing up the two equations in (1.4))

$$\frac{\partial}{\partial t}(u+v) - a(u+v) = f + g \leq 0.$$

Then the maximal principle implies that:

$$0 \leq u + v \leq \|u_0\|_{\infty} + \|v_0\|_{\infty}.$$

Therefore, the global existence follows.

In the present work we consider the problem (1.1)-(1.3) by using a technique based on L^1-estimate we establish a global existence result of the solution.

1.1. Formulation of the result

The existence of global solutions for the system (1.1)-(1.3) is to equivalence to existence a (u, v) true for the following theorem:

Theorem 1.1. Suppose that the hypotheses **(Hi), i=$\overline{\mathbf{1,5}}$** are satisfied, so it exists (u, v) solution of:

$$\begin{cases} u, v \in C\left([0, +\infty[\,, L^1(\Omega)\right) \\ f(u, v), g(u, v) \in L^1(Q) \text{ where } Q = (0, T) \times \Omega \text{ for all } T > 0, \\ u(t, x) = S_1(t)\, u_0 + \displaystyle\int_0^t S_1(t-s)\, f(u(s), v(s))\, ds \qquad \forall t \in [0, T[\\ v(t, x) = S_3(t)\left(v_0 - \dfrac{c}{a-d} u_0\right) + \dfrac{c}{a-d} S_1(t)\, u_0 + \\ \dfrac{c}{a-d} \displaystyle\int_0^t (S_1(t-s) - S_3(t-s))\, f(u(s), v(s))\, ds + \\ \qquad \displaystyle\int_0^t S_3(t-s)\, g(u(s), v(s))\, ds \qquad \forall t \in [0, T[\end{cases} \tag{1.10}$$

where $S_1(t)$ and $S_3(t)$ are the Semigroups of contractions in $L^1(\Omega)$ generated by $a\Delta$ and $d\Delta$, with homogeneous Dirichlet boundary conditions.

To prove this theorem we will rely on studying a single system through which it is more convenient to derive the evidence.

2. Main results

Let A m-dissipative operator of the dense domain in the Banach space X and $S(t)$ a Semigroup engendered by A, f a function locally Lipchitz, so $\forall u_0 \in X$ it exists $T(u_0) = T_{\max}$ such that the problem

$$\begin{cases} u \in C([0, T], D(A)) \cap C^1([0, T], X), \\ \dfrac{du}{dt} - Au = F(u(s)), \\ u(0) = u_0. \end{cases} \tag{2.1}$$

admits a unique solution u verifying

$$u(t) = S(t)\, u_0 + \int_0^t S(t-s)\, F(u(s))\, ds, \ \forall t \in [0, T_{\max}].$$

3. Compactness of the solution

In this section we will give a compactness result of operator L defining the solution of the problem (2.1) in the case where the initial value equals zero $[u(0) = 0]$ i.e.

$$L(F)(t) = u(t) = \int_0^t S(t-s) F(u(s))\, ds, \ \forall t \in [0, T]$$

Theorem 3.1. If for all $t > 0$, the operators $S(t)$ are compact, then L are compact of $L^1([0, T], X)$ in $L^1([0, T], X)$.

Proof.
Step 1: We show that $S(\lambda) L : F \to S(\lambda) L(F)$ is compact in $L^1([0, T], X)$ i.e. show that: the set $\{S(\lambda) L(F)(t);\ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0, T], X)$, $\forall t \in [0, T]$.

Since $S(t)$ is compact then, the application: $t \to S(t)$ is continuous of $]0, +\infty[$ in $£(X)$ therefore:

$$\forall \varepsilon > 0, \forall \delta > 0,\ \exists \eta > 0.\ \forall 0 \leq h \leq \eta,\ \forall t \geq \delta,\ \|S(t+h) - S(t)\|_{£(X)} \leq \varepsilon$$

we choose $\lambda = \delta$, we have for $0 \leq t \leq T - h$ $S(\lambda) u(t+h) - S(\lambda) u(t)$

$$\begin{aligned} &= \int_0^{t+h} S(\lambda + t + h - s) F(u(s))\, ds - \int_0^t S(\lambda + t - s) F(u(s))\, ds \\ &= \int_t^{t+h} S(\lambda + t + h - s) F(u(s))\, ds \\ &\quad + \int_0^t (S(\lambda + t + h - s) - S(\lambda + t - s)) F(u(s))\, ds \end{aligned}$$

where from

$$\|S(\lambda) u(t+h) - S(\lambda) u(t)\|_X \leq \int_t^{t+h} \|F(u(s))\|_X\, ds + \varepsilon \int_0^t \|F(u(s))\|_X\, ds$$

we definite $v(t)$ by

$$v(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq T \\ 0 & \text{if no} \end{cases}$$

therefore:

$$\|S(\lambda) v(t+h) - S(\lambda) v(t)\|_1 \leq (h + \varepsilon T) \|F(u(s))\|_1$$

which implies that all $\{S(\lambda)\,v;\ \|F\|_1 \leq 1\}$ is equi-integrable, then it is conventional that all $\{S(\lambda)\,L\,(F)\,(t)\,;\ \ \|F\|_1 \leq 1\}$ is relatively compact in $L^1\,([0,T]\,,X)$, this way $S(\lambda)\,L$ is compact.

Step 2: We show that $S(\lambda)\,L$ converge towards L when λ goes towards 0, in $L^1\,([0,T]\,,X)$.

We have:

$$S(\lambda)\,u(t) - u(t) = \int_0^t S(\lambda + t - s)\,F(u(s))\,ds - \int_0^t S(t-s)\,F(u(s))\,ds.$$

So for $t \geq \delta$ we have:

$$\|S(\lambda)\,u(t) - u(t)\| \leq \int_\delta^t \|S(\lambda + s) - S(s)\|_{\pounds(X)}\,\|F(u(s))\|\,ds + 2\int_{t-\delta}^t \|F(u(s))\|\,ds$$

we choose $0 < \lambda < \eta$ then:

$$\|S(\lambda)\,u(t) - u(t)\| \leq \varepsilon \int_\delta^t \|F(u(s))\|\,ds + 2\int_{t-\delta}^t \|F(u(s))\|\,ds$$

and for $0 \leq t < \delta$ we have:

$$\|S(\lambda)\,u(t) - u(t)\| \leq 2\int_0^t \|F(u(s))\|\,ds$$

as $F \in L^1\,(0,T,X)$ where from:

$$\|S(\lambda)\,u(t) - u(t)\| \leq (\varepsilon T + 2\delta)\,\|F(u(s))\|_1$$

so if $\lambda \to 0$ then

$$S(\lambda)\,u \to u$$

into

$$L^1\,([0,T]\,,X)$$

where the operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1\,([0,T]\,,X)$. ■

Remark 3.2. The Semigroup $S(t)$ generated by the operator Δ is compact in $L^1\,(\Omega)$.

4. Study of a particular system

for all $n > 0$, we define the functions u_{n_0} and v_{n_0} by:

$$u_{n_0} = min(u_0, n) \geq 0, \qquad \text{and} \qquad v_{n_0} = min(v_0, n) \geq 0$$

it is clear that u_{n_0} and v_{n_0} verify **(H1),** i.e.

$$\begin{aligned} u_{n_0} &\in L^1(\Omega), \quad u_{n_0} \geq 0 \\ v_{n_0} &\in L^1(\Omega), \quad v_{n_0} \geq 0 \end{aligned}$$

Let us consider the following system:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - a\Delta u_n = f(u_n, v_n) & \text{in } [0, T[\times \Omega \\ \dfrac{\partial v_n}{\partial t} - c\Delta u_n - d\Delta v_n = g(u_n, v_n) & \text{in } [0, T[\times \Omega \\ u_n(t, x) = v_n(t, x) = 0 & \text{in } [0, T[\times \partial\Omega \\ u_n(0, x) = u_{n_0}(x), v_n(0, x) = v_{n_0}(x) & \text{in } \Omega, \end{cases} \tag{P$_n$}$$

4.1. Existence of a local solution and its positivity of the solution of the system ($\mathbf{P_n}$)

We convert the system (P_n) to an abstract first order system in the Banach space $X = L^1(\Omega) \times L^1(\Omega)$ of the form

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = Aw_n + F(w_n), \quad t > 0, \\ w_n(0) = w_{n_0} = \left(u_{n_0}, v_{n_0}\right) \in X. \end{cases} \tag{S$_n$}$$

Here

$$w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix};$$

the operator A is defined as

$$A = \begin{pmatrix} a\Delta & 0 \\ c\Delta & d\Delta \end{pmatrix}$$

where

$$D(A) := \left\{ w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix} \in X : \begin{pmatrix} \Delta u_n \\ \Delta v_n \end{pmatrix} \in X \right\}.$$

The function F is defined as

$$F(w_n(t)) = \begin{pmatrix} f(u_n(t), v_n(t)) \\ g(u_n(t), v_n(t)) \end{pmatrix}.$$

so the system (S_n) can be returned to the shape of the system (2.1), thus, if (u_n, v_n) is a solution of (S_n) so it verifies the integral equations:

$$\begin{cases} u_n(t,x) = S_1(t)\,u_{n_0} + \displaystyle\int_0^t S_1(t-s)\, f(u_n(s), v_n(s))\,ds \\ v_n(t,x) = S_3(t)\left(v_{n_0} - \dfrac{c}{a-d}u_{n_0}\right) + \dfrac{c}{a-d}S_1(t)\,u_{n_0} + \\ \dfrac{c}{a-d}\displaystyle\int_0^t (S_1(t-s) - S_3(t-s))\, f(u_n(s), v_n(s))\,ds + \\ \qquad \displaystyle\int_0^t S_3(t-s)\, g(u_n(s), v_n(s))\,ds \end{cases} \tag{4.1}$$

where $S_1(t)$ is the semigroup generated by the operator $a\Delta$, and $S_3(t)$ is the semigroup generated by the operator $d\Delta$.

Theorem 4.1. It exists $T_M > 0$ and (u_n, v_n) a local solution of (S_n) for all $t \in [0, T_M]$.

Proof. We know that $S_1(t)$, $S_3(t)$ are Semigroups of contraction and as F is locally Lipschitz in w_n in the space X, so we have $\exists T_M > 0$ and (u_n, v_n) is a local solution of (S_n) on $[0, T_M]$. ■

Lemma 4.2. Let (u_n, v_n) be the solution of the problem $(\mathrm{P_n})$ such that

$$u_{n_0}(x) \geq 0, \qquad \forall x \in \Omega,$$

and

$$v_{n_0}(x) \geq \frac{c}{a-d}u_{n_0}(x) \geq 0, \qquad \forall x \in \Omega,$$

Then

$$u_n(t,x) \geq 0 \text{ and } v_n(t,x) \geq \frac{c}{a-d}u_n(t,x) \geq 0, \qquad \forall (t,x) \in (0,T) \times \Omega.$$

Proof. Let $\bar{u}_n(t,x) = 0$ in $(0,T) \times \Omega \implies \dfrac{\partial \bar{u}_n}{\partial t} = 0$ and $\Delta \bar{u}_n = 0$.

Then

$$\frac{\partial u_n}{\partial t} - a\Delta u_n - f(u_n, v_n) = 0 \geq \frac{\partial \bar{u}_n}{\partial t} - a\Delta \bar{u}_n - f(\bar{u}_n, v_n)$$

and

$$u_n(0,x) = u_{n_0}(x) \geq 0 = \bar{u}_n(0,x).$$

Hence, by the comparison theorem we obtain

$$u_n(t,x) \geq \bar{u}_n(t,x)$$

where from:

$$u_n(t,x) \geq 0.$$

To prove that $v_n(t,x) \geq \frac{c}{a-d} u_n(t,x)$, let us write the integral formula for $u_n(t)$ and $v_n(t)$; By (4.1) we obtain

$$u_n(t) = S_1(t)\, u_{n_0} + \int_0^t S_1(t-s)\, f(u_n(s), v_n(s))\, ds$$

$$v_n(t) = S_3(t)\left[v_{n_0} - \frac{c}{a-d} u_{n_0}\right]$$

$$+\frac{c}{a-d}\left[S_1(t)\, u_{n_0} + \int_0^t S_1(t-s)\, f(u_n(s), v_n(s))\, ds\right]$$

$$+\int_0^t S_3(t-s)\left[\frac{-c}{a-d} f(u_n(s), v_n(s)) + g(u_n(s), v_n(s))\right] ds$$

$$= S_3(t)\left[v_{n_0} - \frac{c}{a-d} u_{n_0}\right] + \frac{c}{a-d} u_n(t)$$

$$+\int_0^t S_3(t-s)\left[\frac{-c}{a-d} f(u_n(s), v_n(s)) + g(u_n(s), v_n(s))\right] ds$$

by **(H5)**

$$\int_0^t S_3(t-s)\left[\frac{-c}{a-d} f(u_n(s), v_n(s)) + g(u_n(s), v_n(s))\right] ds \geq 0$$

and

$$v_{n_0} \geq \frac{c}{a-d} u_{n_0} \Rightarrow S_3(t)\left[v_{n_0} - \frac{c}{a-d} u_{n_0}\right] \geq 0$$

then

$$v_n(t) \geq \frac{c}{a-d} u_n(t) \geq 0.$$

■

4.2. Global existence of the solution of the system ($\mathrm{P_n}$)

To prove the global existence of the solution of the system ($\mathrm{P_n}$) for all non-negative t, it is enough to find an estimate of the solution for everything $t \geq 0$, according to Haraux and Kirane [6], Henry [8] and Routh [20].

For this we give the following lemma according to us shows the existence of an estimate of the solution of ($\mathrm{P_n}$) in $L^1(\Omega)$.

Lemma 4.3. Let (u_n, v_n) the solution of the system ($\mathrm{P_n}$), so it exists $M(t)$ which depends only of t, such that for all $0 \leq t \leq T_M$, we have:

$$\|u_n(t) + v_n(t)\|_{L^1(\Omega)} \leq M(t)$$

Proof. Of the first and second equation of ($\mathrm{P_n}$) we obtain:

$$\frac{\partial}{\partial t}(u_n + v_n) - \Delta(au_n + cu_n + dv_n) = f(u_n, v_n) + g(u_n, v_n)$$

By taking into account of **(H3)** we have:

$$\frac{\partial}{\partial t}(u_n + v_n) - \Delta((a + c)u_n + dv_n) \leq C(u_n + v_n + 1)$$

Let us integrate on Ω and apply the formula of Green, we find:

$$\frac{\partial}{\partial t}\int_\Omega (u_n + v_n)\,dx \leq C\int_\Omega (u_n + v_n + 1)\,dx$$

so

$$\frac{\frac{\partial}{\partial t}\int_\Omega (u_n + v_n)\,dx}{\int_\Omega (u_n + v_n + 1)\,dx} \leq C$$

integrate on $[0, t]$, we find:

$$\ln \int_\Omega (u_n + v_n + 1)\,dx \Big|_0^t \leq Ct$$

thus

$$\ln \frac{\int_\Omega (u_n + v_n + 1)\,dx}{\int_\Omega \left(u_{n_0} + v_{n_0} + 1\right)dx} \leq Ct$$

which implies:

$$\frac{\int_\Omega (u_n + v_n + 1)\,dx}{\int_\Omega \left(u_{n_0} + v_{n_0} + 1\right)dx} \leq \exp(Ct)$$

$$\Rightarrow \int_{\Omega} (u_n + v_n + 1)\, dx \leq \exp(Ct) \int_{\Omega} \left(u_{n_0} + v_{n_0} + 1\right) dx$$

$$\Rightarrow \int_{\Omega} (u_n + v_n)\, dx \leq \exp(Ct) \int_{\Omega} \left(u_{n_0} + v_{n_0}\right) dx$$

$$\Rightarrow \int_{\Omega} (u_n + v_n)\, dx \leq \exp(Ct) \int_{\Omega} (u_0 + v_0)\, dx \text{ as if } u_{n_0} \leq u_0,\ v_{n_0} \leq v_0.$$

Let us put:

$$M(t) = \exp(Ct) \|u_0 + v_0\|_{L^1(\Omega)}$$

as u_n, v_n are positives, then:

$$\|u_n + v_n\|_{L^1(\Omega)} \leq M(t),\ 0 \leq t \leq T_M$$

■

We can conclude from this estimate that the solution (u_n, v_n) given by the **Theorem 4.1** is a global solution.

5. Global existence of the solution of the system (1.1)-(1.3)

We give the following lemma which shows the existence of estimate of the solution (u_n, v_n) of system $(\mathrm{P_n})$ in $L^1(Q)$.

Lemma 5.1. For any solution (u_n, v_n) of $(\mathrm{P_n})$, there is a constant $K(t)$ which depends only of t, such that:

$$\|u_n(t) + v_n(t)\|_{L^1(Q)} \leq K(t) \left(\|u_0 + v_0\|_{L^1(\Omega)} + 1\right)$$

Proof. To prove this lemma, we use the following results: (see Hollis, Martin and Pierre [9] and Bonafede, Schmitt [2]).

So, we introduce $\theta \in C_0^{\infty}(Q)$, $\theta \geq 0$ and $\Phi \in C^{2,1}(Q)$ a nonnegative solution of the following system

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} - d_1 \Delta \Phi = \theta & \text{on } Q \\ \Phi = 0 & \text{on } [0, T] \times \partial\Omega \\ \Phi(T, .) = 0 & \text{on } \Omega, \end{cases} \tag{S}$$

According to Ladyzenskaya and Solonnikov [13] (S) possesses a unique nonnegative solution. Moreover, for all $q \in]1, +\infty[$, there exists a nonnegative constant c independent of θ, such that,

$$\|\Phi\|_{L^q(Q)} \leq c \|\theta\|_{L^q(Q)}$$

We have according to Bonafede and Schmitt [2]**:**

$$\int_Q S_1(t)\,u_{n_0}(x)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_\Omega u_{n_0}(x)\,\Phi(0,x)\,dx \tag{5.1}$$

$$\int_Q S_3(t)\,u_{n_0}(x)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_\Omega u_{n_0}(x)\,\Phi(0,x)\,dx \tag{5.2}$$

$$\int_Q S_3(t)\,v_{n_0}(x)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_\Omega v_{n_0}(x)\,\Phi(0,x)\,dx$$

and that:

$$\int_Q\left(\int_0^t S_1(t-s)\,f(u_n,v_n)\,ds\right)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_Q f(u_n,v_n)\,\Phi(s,x)\,dxds \tag{5.3}$$

$$\int_Q\left(\int_0^t S_3(t-s)\,f(u_n,v_n)\,ds\right)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_Q f(u_n,v_n)\,\Phi(s,x)\,dxds \tag{5.4}$$

$$\int_Q\left(\int_0^t S_3(t-s)\,g(u_n,v_n)\,ds\right)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=\int_Q g(u_n,v_n)\,\Phi(s,x)\,dxds$$

(5.1)-(5.2) give

$$\int_Q (S_1(t)-S_3(t))\,u_{n_0}(x)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=0$$

and (5.3)-(5.4) give

$$\int_Q\left(\int_0^t (S_1(t-s)-S_3(t-s))\,f(u_n,v_n)\,ds\right)\left(-\frac{\partial\Phi}{\partial t}-d_1\Delta\Phi\right)dxdt=0$$

where from

$$\int_Q \left(S_1(t)\,u_{n_0}(x)\right)\theta dxdt=\int_\Omega u_{n_0}(x)\,\Phi(0,x)\,dx \tag{5.5}$$

$$\int_Q\left(\int_0^t S_1(t-s)\,f(u_n,v_n)\,ds\right)\theta dxdt=\int_Q f(u_n,v_n)\,\Phi(s,x)\,dxds \tag{5.6}$$

and

$$\int_Q \left(S_3(t)\, v_{n_0}(x) + \frac{c}{a-d} \left(S_1(t) - S_3(t) \right) u_{n_0}(x) \right) \theta dxdt$$

$$= \int_\Omega v_{n_0}(x)\, \Phi(0,x)\, dx \qquad (5.7)$$

$$\int_Q \left(\int_0^t S_3(t-s)\, g(u_n, v_n)\, ds + \frac{c}{a-d} \int_0^t \left(S_1(t-s) - S_3(t-s) \right) f(u_n, v_n)\, ds \right) \theta dxdt$$

$$= \int_Q g(u_n, v_n)\, \Phi(s,x)\, dxds \qquad (5.8)$$

Let us multiply the first equation of (4.1) by θ, and let us integrate on Q, by using (5.5) and (5.6), we obtain:

$$\begin{aligned} \int_Q u_n \theta dxdt &= \int_Q S_1(t)\, u_{n_0}(x)\, \theta dxdt + \int_Q \left(\int_0^t S_1(t-s)\, f(u_n, v_n)\, ds \right) \theta dxdt \\ &= \int_\Omega u_{n_0}(x)\, \Phi(0,x)\, dx + \int_Q f(u_n, v_n)\, \Phi(s,x)\, dxds \end{aligned}$$

multiplying the second equation of (4.1) by θ, and let us integrate on Q, by using (5.7) and (5.8), we find:

$$\int_Q v_n \theta dxdt = \int_\Omega v_{n_0}(x)\, \Phi(0,x)\, dx + \int_Q g(u_n, v_n)\, \Phi(s,x)\, dxds$$

Therefore:

$$\begin{aligned} \int_Q (u_n + v_n)\, \theta dxdt &= \int_\Omega \left(u_{n_0}(x) + v_{n_0}(x) \right) \Phi(0,x)\, dx \\ &\quad + \int_Q \left(f(u_n, v_n) + g(u_n, v_n) \right) \Phi(s,x)\, dxds \\ &\leq \int_\Omega \left(u_0(x) + v_0(x) \right) \Phi(0,x)\, dx \\ &\quad + \int_Q C\left(u_n + v_n + 1 \right) \Phi(s,x)\, dxds \end{aligned}$$

Using Holder inequality we deduce

$$\begin{aligned}\int_Q (u_n + v_n)\,\theta dxdt &\leq \|u_0 + v_0\|_{L^1(\Omega)} \cdot \|\Phi(0,x)\|_{L^\infty(Q)} \\ &\quad + C\,\|u_n + v_n + 1\|_{L^1(Q)} \cdot \|\Phi\|_{L^\infty(Q)} \\ &\leq k_1 \left(\|u_0 + v_0\|_{L^1(\Omega)} + \|u_n + v_n\|_{L^1(Q)} + 1\right) \cdot \|\theta\|_{L^\infty(Q)}\end{aligned}$$

For θ is arbitrary in $C_0^\infty(Q)$ this implies

$$\|u_n + v_n\|_{L^1(Q)} \leq k_1 \left(\|u_0 + v_0\|_{L^1(\Omega)} + \|u_n + v_n\|_{L^1(Q)} + 1\right)$$

we take $k = \dfrac{k_1(t)}{1 - k_1(t)}$ we find:

$$\|u_n + v_n\|_{L^1(Q)} \leq k(t) \left(\|u_0 + v_0\|_{L^1(\Omega)} + 1\right)$$

■

Proof. [Proof of Theorem 1.1]. Let us define the application L by:

$$L : (w_0, h) \to S_d(t)\,w_0 + \int_0^t S_d(t-s)\,h(s)\,ds$$

where $S_d(t)$ the semigroup of contraction generated by the operator $d\Delta$, according to the previous result **Theorem 3.1** and as $S_d(t)$ is compact, then the application L, is adding two compact applications in $L^1(Q)$, so it was that L is compact $L^1(Q) \times L^1(Q)$ in $L^1(Q)$.

Therefore, there is a subsequence $\left(u_{n_j}, v_{n_j}\right)$ of (u_n, v_n) and (u, v) of $L^1(Q) \times L^1(Q)$, such that:

$$\left(u_{n_j}, v_{n_j}\right) \text{ converges towards } (u, v)$$

Let us now show that $\left(u_{n_j}, v_{n_j}\right)$ is a solution of (4.1).

We have:

$$\begin{cases} u_{n_j}(t,x) = S_1(t)\,u_{n_0} + \displaystyle\int_0^t S_1(t-s)\,f\left(u_{n_j}(s), v_{n_j}(s)\right) ds \\ v_{n_j}(t,x) = S_3(t)\left(v_{n_0} - \dfrac{c}{a-d} u_{n_0}\right) + \dfrac{c}{a-d} S_1(t)\,u_{n_0} + \\ \dfrac{c}{a-d} \displaystyle\int_0^t \left(S_1(t-s) - S_3(t-s)\right) f\left(u_{n_j}(s), v_{n_j}(s)\right) ds + \\ \qquad \displaystyle\int_0^t S_3(t-s)\,g\left(u_{n_j}(s), v_{n_j}(s)\right) ds \end{cases} \tag{P$_j$}$$

so it is enough to show that (u, v) verifies (1.10).

It is clear that if $j \to +\infty$, we have the following limits:

$$\begin{array}{l} f\left(u_{n_j}, v_{n_j}\right) \to f(u, v) \text{ a.e} \\ g\left(u_{n_j}, v_{n_j}\right) \to g(u, v) \text{ a.e} \end{array} \tag{5.9}$$

and

$$\begin{array}{rcl} u_{n_0} & \to & u_0 \\ v_{n_0} & \to & v_0 \end{array}$$

Thus to show that (u, v) verifies (1.10), it remains to show that:

$$\begin{array}{rcl} f\left(u_{n_j}, v_{n_j}\right) & \to & f(u, v) \\ g\left(u_{n_j}, v_{n_j}\right) & \to & g(u, v) \end{array}$$

in $L^1(Q)$ when $j \to +\infty$.

We integrate the first and second equations of $(\mathrm{P_n})$ on Q by taking into account that:

$$-a \int_Q \Delta u_{n_j} dxdt = 0 \qquad - c \int_Q \Delta u_{n_j} dxdt = 0 \qquad - d \int_Q \Delta v_{n_j} dxdt = 0$$

we have:

$$\begin{array}{rcl} \displaystyle\int_\Omega u_{n_j} dx - \int_\Omega u_{n0} dx & = & \displaystyle\int_Q f\left(u_{n_j}, v_{n_j}\right) dxdt \\ \displaystyle\int_\Omega v_{n_j} dx - \int_\Omega v_{n0} dx & = & \displaystyle\int_Q g\left(u_{n_j}, v_{n_j}\right) dxdt \end{array}$$

where from:

$$-\int_Q f\left(u_{n_j}, v_{n_j}\right) dxdt \le \int_\Omega u_0 dx \tag{5.10}$$

$$-\int_Q g\left(u_{n_j}, v_{n_j}\right) dxdt \le \int_\Omega v_0 dx. \tag{5.11}$$

Let us put

$$\begin{array}{rcl} N_n & = & C\left(u_{n_j} + v_{n_j} + 1\right) - f\left(u_{n_j}, v_{n_j}\right) \\ M_n & = & C\left(u_{n_j} + v_{n_j} + 1\right) - f\left(u_{n_j}, v_{n_j}\right) - g\left(u_{n_j}, v_{n_j}\right) = N_n - g\left(u_{n_j}, v_{n_j}\right) \end{array}$$

it is clear that N_n and M_n are positives according to (**H3**) and (**H4**), of (5.10) and (5.11) we obtain:

$$\begin{aligned}\int_Q N_n dxdt &\leq C\int_Q \left(u_{n_j}+v_{n_j}+1\right)dxdt+\int_\Omega u_0 dx\\ \int_Q M_n dxdt &\leq C\int_Q \left(u_{n_j}+v_{n_j}+1\right)dxdt+\int_\Omega (u_0+v_0)\,dx\end{aligned}$$

the **Lemma 5.1** gives us:

$$\int_Q N_n dxdt < +\infty$$
$$\int_Q M_n dxdt < +\infty$$

which implies:

$$\int_Q \left|f\left(u_{n_j},v_{n_j}\right)\right| dxdt \leq C\int_Q \left(u_{n_j}+v_{n_j}+1\right)dxdt+\int_Q N_n dxdt < +\infty$$

$$\int_Q \left|g\left(u_{n_j},v_{n_j}\right)\right| dxdt \leq \int_Q M_n dxdt+\int_Q N_n dxdt < +\infty$$

let

$$\begin{aligned}h_n &= N_n+C\left(u_{n_j}+v_{n_j}+1\right)\\ \Psi_n &= N_n+M_n\end{aligned}$$

h_n and Ψ_n are in $L^1(Q)$ and positives and furthermore

$$\begin{aligned}\left|f\left(u_{n_j},v_{n_j}\right)\right| &\leq h_n \text{ a.e}\\ \left|g\left(u_{n_j},v_{n_j}\right)\right| &\leq \Psi_n \text{ a.e.}\end{aligned}$$

Let us combine this result with (5.9) and we apply the theorem of convergence dominated by Lebesgue.

We obtain:

$$\begin{matrix} f\left(u_{n_j},v_{n_j}\right)\to f(u,v)\\ g\left(u_{n_j},v_{n_j}\right)\to g(u,v)\end{matrix} \quad \text{in } L^1(Q)$$

by passing in the limit $j \to +\infty$ of $(\mathrm{P_j})$ in $L^1(Q)$ we find:

$$\begin{cases} u(t,x) = S_1(t)u_0 + \int_0^t S_1(t-s) f(u(s), v(s))\,ds \\ v(t,x) = S_3(t)\left(v_0 - \frac{c}{a-d}u_0\right) + \frac{c}{a-d}S_1(t)u_0 + \\ \frac{c}{a-d}\int_0^t [S_1(t-s) - S_3(t-s)] f(u(s), v(s))\,ds + \\ \qquad \int_0^t S_3(t-s)\, g(u(s), v(s))\,ds \end{cases}$$

Then (u, v) verify (1.10) consequently (u, v) is the solution of (1.1)-(1.3). ■

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