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Computation of Multi-Choice Multi-Objective Fuzzy Probabilistic Programming Problem

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Abstract

In this paper, we propose a solution procedure for solving multi-choice multi-objective fuzzy probabilistic programming problem. Fuzzy random variables are considered as fuzzy log-normal distribution. In the first step we transform the fuzzy probabilistic constraint into its equivalent crisp model by using log-normal distribution. In the second step, we tackle the multi-choice parameter by using fitting of curves by least square approximation technique to a n-th order polynomial. In the third step, fuzzy programming method is used to solve the equivalent mathematical crisp multi-objective non-linear mixed integer programming problem. In order to solve the final mathematical model the existing methodology or software is used. To illustrate and clarify the proposed approach, a numerical examples is presented.

AMS subject classification:
Keywords: Fuzzy programming approach, Fuzzy random variable, Fuzzy stochastic programming problem, Multi-choice mathematical programming, Multi-criteria decision making.

1. Introduction

Stochastic Programming (SP) model was first formulated by Dantzig (1955) who suggested a two stage programming technique that involves conversion of SP models into their equivalent deterministic programming models. However, this technique suffers from the limitation that it does not allow any constraint to be violated even at specific probability level. This gave rise to the concept of chance constrained programming

(CCP), where constraints containing random variables are guaranteed to be satisfied with a certain probability. Charnes and Cooper (1959, 1963) developed the concept of CCP. CCP is a particular type of decision making problem in which constraints having finite probability is being violated. Luhandjula (2003) derived methodologies to solve fuzzy stochastic programming problem in the presence of fuzzy random variables. A different approach for stochastic programming problem with fuzzy data has described by Luhandjula and Gupta (1996).

An optimization problem is said to be fuzzy stochastic or fuzzy probabilistic programming problem when some or all parameters are described by random variables. The uncertainties in the parameters are represented by fuzzy probability distributions. The distribution is estimated on the basis of the available observed fuzzy random data. There are two types of uncertainties, one arising from randomness, which can be incorporated through a probability distribution and other from fuzziness, which can be characterized by fuzzy numbers.

The concept of fuzzy random variable was first introduced by Zadeh (1968) and further developed by Kwakernaak (1978); Kratschmer (2001); Buckley (2004) according to different requirements of measurability. Buckley and Eslami (2004); Buckley (2005) defined fuzzy probability using fuzzy numbers as parameters in probability density function and probability mass function. These fuzzy numbers are obtained from the set of confidence interval. The approach of fuzzy probability theory by Buckley (2005) is different from his predecessors and also comfortable for computational point of view.

Fuzzy probabilistic programming (FPP) problem is a mathematical programming problem in which both fuzziness and randomness are involved. In the recent past Iwamura (1998) extended chance constrained programming from stochastic to fuzzy environments and also derived a technique of fuzzy simulation for the chance constraints which are usually hard to be converted to their crisp equivalents. Luhandjula (1983); Buckley (2005) derived different method to solve some typical fuzzy probabilistic programming problems. All these techniques involve fuzziness and randomness in different scenarios. Recently Zarghami (2010) suggested a multi objective model to design optimum water supply and demand management under different kind of uncertainty, where the uncertainty in satisfying some constraints is tackled by using the chance constraint approach. Abdelaziz (2012) presented a solution procedure for multi-objective stochastic problems where random variables can be in both objectives and constraints parameters. Nanda et al. (2006, 2008) have discussed the deterministic equivalent of FPP in the presence of normally distributed fuzzy random variable in different scenarios.

Multi-choice programming is a mathematical programming problem in which decision maker is allowed to set multiple number of choices for a parameter. Chang (2007) has proposed a mathematical model for modeling the multi-choice aspiration level problem. He has introduced the multiplicative terms of binary variables in order to tackle with multi-choice aspiration levels for each goal. Later Chang (2008) replaced the multiplicative terms of binary variables by a continuous variable. Biswal and Acharya (2009b,a); Acharya and Biswal (2011) used binary variables in order to convert a multi-choice linear programming problem to an equivalent mathematical model. Using the concept of Chang (2007), Liao (2009) proposed the formulation of a multi-segment goal programming problem, which obtains a solution close to the decision makers multi-segment aspiration levels. Acharya and Acharya (2012) generalized the transformation technique proposed by Biswal and Acharya (2009b). Biswal and Acharya (2011) used interpolating polynomial approach to solve multi-choice linear programming problem. After using the interpolation, the formulated mathematical model was a mixed integer nonlinear programming problem. Later Acharya and Biswal (2011) investigated a probabilistic linear programming problem. in which the right hand side parameters are multi-choice in nature and the rest parameters are normally distributed independent random variables with known mean and variance in the probabilistic constraints. Patro et al. (2015) presented a paper to solve multi-choice goal programming problem by using binary variables, Vandermonde's interpolating polynomial, and linear least square approximation. In our model we have used the least square approximation technique to tackle multi-choice parameter.

In this paper we have discussed multi-choice multi-objective, FPP problem where the parameters used in objective function or in constraint or in both are independent log-normally distributed Fuzzy Random Variables (FRV). We also proposed here a solution procedure for solving the model by transforming it to a crisp equivalent non-linear mixed integer mathematical programming problem.

The organization of the paper is as follows: Introduction of the paper is discussed in Section-1 and following that basic preliminaries is discussed in Section-2. In Section-3 the mathematical model of the proposed programming problem is presented and the problem formulation and methodology for different cases are presented in subsection-4.1. In the subsection-4.2 and subsection-4.3 interpolating polynomial approach and fuzzy programming approach are presented respectively. In order to illustrate the proposed method a numerical example is presented in Section-5. Finally conclusion is presented in Section-6.

2. Basic Preliminaries

Definition 2.1. A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line R, with membership function $\mu_A : R \to [0, 1]$, satisfying the following conditions.

1. There exists exactly one interval I ∈ R such that $\mu_A(x) = 1; x \in I$:
2. The membership function μ_A is piecewise continuous.

Definition 2.2. (Lai and Hwang, 1992) A fuzzy number $\tilde{A} = \left(A^{(m)}; A^{(p)}; A^{(o)}\right)$ is said to be triangular if its membership function is strictly increasing in the interval $\left(A^{(p)}; A^{(m)}\right)$ and strictly decreasing in $\left(A^{(m)}; A^{(o)}\right)$ and $\mu_A\left(A^{(m)}\right) = 1$, where $A^{(m)}$ is core, $A^{(m)} - A^{(p)}$ is left spread and $A^{(o)} - A^{(m)}$ is right spread of the fuzzy number $\tilde{A}$.

Definition 2.3. (Buckley, 2005) α-cut of the fuzzy number $\tilde{A}$ is the set $\{x|\mu_A(x) \geq \alpha\}$ for $0 < \alpha < 1$ and denoted $\tilde{A}[\alpha]$.

Definition 2.4. A fuzzy number $\tilde{A}$ is said to be positive if it's membership function $\mu_A(x) = 0; \forall x \geq 0$. It may be stated as follow: Let $\tilde{A}[\alpha] = [A_*, A^*]$ be the α-cut of the fuzzy number $\tilde{A}$ for $0 < \alpha < 1$. $\tilde{A}$ is said to be positive if $A_* > 0$.

Definition 2.5. (Nanda and Kar, 1992) Let and $\tilde{A} = \left(A^{(m)}; A^{(p)}; A^{(o)}\right)$ and $\tilde{B} = \left(B^{(m)}; B^{(p)}; B^{(o)}\right)$ be two fuzzy numbers with α-cut $\tilde{A}[\alpha] = [A_*, A^*]$ and $\tilde{B}[\alpha] = [B_*, B^*]$ respectively. then $\tilde{A} \preceq \tilde{B}$ iff $A^* \leq B_*$.

Definition 2.6. (Buckley and Eslami, 2004) A fuzzy random variable is a random variable whose parameter is fuzzy number. Let $\tilde{X}$ be continuous random variable with fuzzy parameter $\tilde{\theta}$ and $\tilde{P}$ as fuzzy probability, then $\tilde{X}$ is said to be a continuous fuzzy random variable with density function $f(x; \tilde{\theta})$. $\tilde{P}(\tilde{X} \leq x) = \tilde{\beta}$, where $0 \leq \tilde{\beta} \leq 1$; $\tilde{\beta} = \left(\beta^{(m)}; \beta^{(p)}; \beta^{(o)}\right)$, $\beta^{(p)} \geq 0$ and $\beta^{(o)} \leq 1$.

Definition 2.7. (Buckley and Eslami, 2004) Let $E = [c, d]$ be an event. Then the probability of the event E of continuous fuzzy random variable $\tilde{X}$ is a fuzzy number whose α-cut is

$$
\begin{aligned}
&\tilde{P}[c \leq \tilde{X} \leq d][\alpha] \\
&= \left[\min : \left\{\int_c^d f(x,\theta)dx | \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f(x,\theta)dx = 1\right\},\right. \\
&\quad \left. \max : \left\{\int_c^d f(x,\theta)dx | \theta \in \tilde{\theta}[\alpha], \int_{-\infty}^{\infty} f(x,\theta)dx = 1\right\}\right] \\
&= [\beta_*[\alpha], \beta^*[\alpha]]
\end{aligned}
$$

Definition 2.8. A log-normal fuzzy random variable $\tilde{LN}(\tilde{\mu}, \tilde{\sigma}^2)$ is a log-normally distributed random variable with fuzzy parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$. In a fuzzy log-normal distribution $\tilde{X}$ the parameters denoted by $\tilde{\mu}$ and $\tilde{\sigma}$ are the fuzzy mean and standard deviation of log $\tilde{X}$, which follows fuzzy normal distribution.

Theorem 2.9. (Nanda et al., 2008) Let $\tilde{Z} = a\tilde{X} + b\tilde{Y}$ is the linear combination of the FRVs $\tilde{X}$ and $\tilde{Y}$, which is also a FRV, whose mean and variance are fuzzy numbers. We denote the mean and variance as $\mu_{\tilde{Z}}$ and $\sigma^2_{\tilde{Z}}$ respectively. Then the following results holds good:

(a) $\mu_{\tilde{Z}} = a\mu_{\tilde{X}} + b\mu_{\tilde{Y}}$

(b) $\sigma^2_{\tilde{Z}} = a^2\sigma^2_{\tilde{X}} + b^2\sigma^2_{\tilde{Y}}$

Theore 2.10. Let X be a random variable which follows normal distribution with mean μ and variance σ^2. If $Y = e^X$, then Y follows log-normal distribution. The mean and variance of Y calculated by the relation

$$E(Y) = e^{\mu + \frac{1}{2}\sigma^2} \tag{2.1}$$

$$Var(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \tag{2.2}$$

3. Problem Formulation

We present multi-objective multi-choice fuzzy probabilistic programming problem by considering fuzzyness and randomness under one umbrella either in objective function or in constraints or in both. Depending on different situation following mathematical models are formulated.

Model-I:
The model of a multi-objective multi-choice fuzzy probabilistic programming problem (MOMCFPPP) where a_{ij} is a FRV distributed log-normally is presented as follows.

$$\max : (Z_k) = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1, 2, \ldots, K \tag{3.1}$$

Subject to

$$\tilde{P}(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i) \succeq \tilde{\beta}_i, b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\}, \quad i = 1, 2, \ldots, m \tag{3.2}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{3.3}$$

where $\tilde{\beta}_i$ is a positive fuzzy number and $c_j^k \in R$, $\tilde{a}_{ij}, i = 1, 2, \cdots, m.$ $j = 1, 2, \ldots, n$ are independent FRV distributed log-normally. The right hand side of i-th constraint (b_i) has a set of h_i number of goals out of which only one goal is to be selected so as to maximize the objective function.

Model-II:
The model of a multi-objective multi-choice fuzzy probabilistic programming problem (MOMCFPPP) where a_{ij} and b_i are FRVs distributed log-normally is presented as follows.

$$\max : (Z_k) = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1, 2, \ldots, K \tag{3.4}$$

Subject to

$$\tilde{P}(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i) \succeq \tilde{\beta}_i, b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\}, \quad i = 1, 2, \ldots, m \tag{3.5}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{3.6}$$

where $\tilde{\beta}_i$ is a positive fuzzy number and $c_j^k \in R$, $\tilde{a}_{ij}, i = 1, 2, \ldots, m.$ $j = 1, 2, \ldots, n$ and $\tilde{b}_i, i = 1, 2, \ldots, m$ are independent FRVs distributed log-normally. The right hand side of i-th constraint (b_i) has a set of h_i number of goals out of which only one goal is to be selected so as to maximize the objective function.

Model-III:
The model of a multi-objective multi-choice fuzzy probabilistic programming problem (MOMCFPPP) where b_i is a FRV distributed log-normally is presented as follows.

$$\max : (Z_k) = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1, 2, \ldots, K \tag{3.7}$$

Subject to

$$\tilde{P}(\sum_{j=1}^{n} a_{ij} x_j \leq \tilde{b_i}) \succeq \tilde{\beta}_i, b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\}, \quad i = 1, 2, \ldots, m \tag{3.8}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{3.9}$$

where $\tilde{\beta_i}$ is a positive fuzzy number and $c_j^k \in R$, $\tilde{b_i}, i = 1, 2, \cdots, m$ is a independent FRV distributed log-normally. The right hand side of i-th constraint (b_i) has a set of h_i number of goals out of which only one goal is to be selected so as to maximize the objective function.

4. Solution Methodology

The solution procedure for the proposed mathematical model is completed in the following three steps.

4.1. Step-1: Transformation of fuzzy random variable

In this step fuzzy random parameters are converted to its crisp equivalent by using $\alpha - cut$ technique followed by chance constraint technique.

Theorem 4.1. If $\tilde{a}_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, 3, \ldots, n$ are independent FRVs distributed log-normally then

$$\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq b_i\right) \succeq \tilde{\beta}_i, b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\} \tag{4.1}$$

is equivalent to

$$\sum_{i=1}^{n} \ln x_j \leq n \ln\left(\frac{b_i}{n}\right) - \sum_{j=1}^{n}\left[\ln \mu^*_{a_{ij}} - \frac{1}{2}\ln\left(1 + \frac{{\sigma^*_{a_{ij}}}^2}{{\mu^*_{a_{ij}}}^2}\right)\right] + k_{\beta_i{}^*}\sqrt{\sum_{j=1}^{n}\ln\left(1 + \frac{{\sigma^*_{a_{ij}}}^2}{{\mu^*_{a_{ij}}}^2}\right)} \tag{4.2}$$

$b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\}$ where $\tilde{\mu}_{a_{ij}}$ and $\tilde{\sigma}^2_{a_{ij}}$ are mean and variance of $\tilde{a}_{ij}$, which follows fuzzy log-normal distribution.

Proof. It is assumed that $\tilde{a}_{ij}$, $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, \ldots, n$ are independent FRVs distributed log-normally. $\ln \tilde{a}_{ij}$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}_{ij}$ and $\tilde{\sigma}^2_{ij}$ respectively [2.8].

Let $y_i = \sum_{j=1}^{n} \ln a_{ij}$ is a normal random variable, whose mean and variance are μ'_{ij} and σ'^2_{ij} respectively. So $\sum_{j=1}^{n} \ln \tilde{a}_{ij}$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}'_{ij}$ and $(\tilde{\sigma}'_{ij})^2$ respectively. The α-cuts of $\tilde{y}_i$, $\tilde{\mu}'_{ij}$ and $(\tilde{\sigma}'_{ij})^2$ are $\tilde{y}_i[\alpha] = [y_{i*}, y_i{}^*]$, $\tilde{\mu}'_{ij}[\alpha] = [\tilde{\mu}'_{ij_*}, \tilde{\mu}'^*_{ij}]$, $(\tilde{\sigma}'_{ij})^2[\alpha] = [(\tilde{\sigma}'_{ij*})^2, (\tilde{\sigma}'^*_{ij})^2]$.

Now the i-th constraint of the fuzzy probabilistic programming problem with multi choice parameter can be represented as

$$\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij}x_j \leq b_i\right) \succeq \tilde{\beta}_i, b_i \epsilon \left\{ b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)} \right\} \tag{4.3}$$

Where $\tilde{\beta}_i$,(i=1,2,3,…,m) are fuzzy number and whose α-cuts are $\tilde{\beta}_i[\alpha] = [\beta_{i*}, \beta_i^*]$. The α-cuts of i-th fuzzy probabilistic constraint is:

$$\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij}x_j \leq b_i\right)[\alpha]$$

$$, \qquad = \left\{ P\left(\sum_{j=1}^{n} a_{ij}x_j \leq b_i\right) | a_{ij} \in \tilde{a}_{ij}[\alpha] \right\} \tag{4.4}$$

$$= \left\{ P\left(n \left(\prod_{j=1}^{n} a_{ij}\ x_j\right)^{1/n} \leq b_i\right) | a_{ij} \in \tilde{a}_{ij}[\alpha] \right\} \tag{4.5}$$

Taking logarithm on both sides inside the parenthesis of (4.5) yields

$$\begin{aligned}
&\left\{P\left(\ln n\left(\prod_{j=1}^{n} a_{ij}\, x_j\right)^{1/n} \le \ln b_i\right)|a_{ij}\in\tilde{a}_{ij}[\alpha]\right\}\\
&= \left\{P\left(\ln n+\frac{1}{n}\left(\sum_{j=1}^{n}\ln a_{ij}+\sum_{j=1}^{n}\ln x_j\right)\le \ln b_i\right)|a_{ij}\in\tilde{a}_{ij}[\alpha]\right\}\\
&= \left\{P\left(n\ln n+\sum_{j=1}^{n}\ln a_{ij}+\sum_{j=1}^{n}\ln x_j\le n\ln b_i\right)|a_{ij}\in\tilde{a}_{ij}[\alpha]\right\}\\
&= \left\{P\left(\sum_{j=1}^{n}\ln a_{ij}\le n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j\right)|a_{ij}\in\tilde{a}_{ij}[\alpha]\right\}\\
&= \left\{P\left(y_i\le n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j\right)|y_i\in\tilde{y}_i[\alpha]\right\}\\
&= \left\{P\left(\frac{y_i-\mu'_{ij}}{\sigma'_{ij}}\le\frac{n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j-\mu'_{ij}}{\sigma'_{ij}}\right)|K\right\} \qquad (4.6)
\end{aligned}$$

where

$$K=(y_i,\mu'_{ij},{\sigma'_{ij}}^2|y_i\ \in\ \tilde{y}_i[\alpha],\mu'_{ij}\in\tilde{\mu}'_{ij}[\alpha],{\sigma'_{ij}}^2\in\tilde{\sigma}'_{ij})^2[\alpha])$$

Now

$$\begin{aligned}
&\tilde{P}\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j\le b_i\right)[\alpha]\\
&= \left(\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\frac{n\ln b_i-n\ln n-\sum_{i=1}^{n}\ln x_j-\mu'_{ij}}{\sigma'_{ij}}} e^{\frac{-z^2}{2}}dz|K\right),\ where\ z=\left(\frac{y_i-\mu'_{ij}}{\sigma'_{ij}}\right) \qquad (4.7)\\
&= \left\{\Phi\left(\frac{n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j-\mu'_{ij}}{\sigma'_{ij}}\right)|K\right\} \qquad (4.8)
\end{aligned}$$

where Φ is the cumulative distribution function of $N(0,1)$ distribution, which is an increasing function.

$$\begin{aligned}
\min\ :\ &\left\{\Phi\left(\frac{n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j-\mu'_{ij}}{\sigma'_{ij}}\right)|K\right\}\\
=\ &\left\{\Phi\left(\frac{n\ln b_i-n\ln n-\sum_{j=1}^{n}\ln x_j-{\mu'_{ij}}^*}{{\sigma'_{ij}}^*}\right)\right\}
\end{aligned}$$

$$\begin{aligned}\max\quad : &\quad \left\{\Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - \mu'_{ij}}{\sigma'_{ij}}\right)|K\right\}\\ = &\quad \left\{\Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - \mu'_{ij_*}}{\sigma'_{ij_*}}\right)\right\}\end{aligned}$$

$$\begin{aligned}&\tilde{P}\quad\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j \leq b_i\right)[\alpha]\\ &= \left[\Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^*}{{\sigma'_{ij}}^*}\right),\right.\\ &\qquad \left.\Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - \mu'_{ij_*}}{\sigma'_{ij_*}}\right)\right]\end{aligned}$$

Using fuzzy inequality, the α-cut of the fuzzy probabilistic constraint it is expressed as:

$$\begin{aligned}\tilde{P}(\sum_{j=1}^{n}\tilde{a}_{ij}x_j \leq b_i[\alpha] &\succeq \tilde{\beta}_i[\alpha], b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}\\ \Rightarrow \Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^*}{{\sigma'_{ij}}^*}\right) &\geq {\beta_i}^*, b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}\\ \Rightarrow \Phi\left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^*}{{\sigma'_{ij}}^*}\right) &\geq \Phi(-k_{\beta_i{}^*}), b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}\end{aligned}$$

$$\begin{aligned}\Rightarrow \left(\frac{n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^*}{{\sigma'_{ij}}^*}\right) &\geq (-k_{\beta_i{}^*}), b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}\\ \Rightarrow n\ln b_i - n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^* &\geq -k_{\beta_i{}^*}{\sigma'_{ij}}^*, b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}\end{aligned}$$

Hence the deterministic equivalent of the fuzzy probabilistic constraint in (4.3) is expressed as:

$$\sum_{j=1}^{n}\ln x_j \leq n\ln b_i - n\ln n - {\mu'_{ij}}^* + k_{\beta_i{}^*}{\sigma'_{ij}}^*, b_i\epsilon\left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\} \tag{4.9}$$

Let a_{ij} follows log-normal distribution with mean and variance $\mu_{a_{ij}}$ and $\sigma^2_{a_{ij}}$ respectively. Then the mean and variance of $\ln a_{ij}$ follows normal distribution with mean and variance

μ_{ij} and σ_{ij}^2 respectively. The relation between the mean and variances of log-normal and normal distributions are given by:

$$\mu_{ij} = \ln \mu_{a_{ij}} - \frac{1}{2} \ln \left(1 + \frac{\sigma_{a_{ij}}^2}{\mu_{a_{ij}}^2}\right) \tag{4.10}$$

$$\sigma_{ij}^2 = \ln \left(1 + \frac{\sigma_{a_{ij}}^2}{\mu_{a_{ij}}^2}\right) \tag{4.11}$$

Now we have the relation between μ_{ij} and μ'_{ij} is:

$$\mu'_{ij} = E\left(\sum_{j=1}^{n} \ln a_{ij}\right) = \sum_{j=1}^{n} \mu_{ij} \tag{4.12}$$

The relation between σ_{ij}^2 and ${\sigma'_{ij}}^2$ is:

$${\sigma'_{ij}}^2 = Var\left(\sum_{j=1}^{n} \ln a_{ij}\right) = \sum_{j=1}^{n} \sigma_{ij}^2 \tag{4.13}$$

So equation (4.9) is equivalent to

$$\begin{aligned}
\sum_{i=1}^{n} \ln x_j &\leq n \ln b_i - n \ln n - \sum_{j=1}^{n} \mu_{ij}^* + k_{\beta_i{}^*} \sqrt{\sum_{j=1}^{n} {\sigma_{ij}^*}^2}, b_i \epsilon \left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\} \\
\Rightarrow \sum_{j=1}^{n} \ln x_j &\leq n \ln b_i - n \ln n - \sum_{j=1}^{n} \left[\ln \mu_{a_{ij}}^* - \frac{1}{2} \ln \left(1 + \frac{{\sigma_{a_{ij}}^*}^2}{{\mu_{a_{ij}}^*}^2}\right)\right] \\
&+ k_{\beta_i{}^*} \sqrt{\sum_{j=1}^{n} \ln \left(1 + \frac{{\sigma_{a_{ij}}^*}^2}{{\mu_{a_{ij}}^*}^2}\right)}, b_i \epsilon \left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\}
\end{aligned}$$

$$\Rightarrow \sum_{j=1}^{n} \ln x_j \leq n \ln \left(\frac{b_i}{n}\right) - \sum_{j=1}^{n} \left[\ln \mu_{a_{ij}}^* - \frac{1}{2} \ln \left(1 + \frac{{\sigma_{a_{ij}}^*}^2}{{\mu_{a_{ij}}^*}^2}\right)\right] \tag{4.14}$$

$$+ k_{\beta_i{}^*} \sqrt{\sum_{j=1}^{n} \ln \left(1 + \frac{{\sigma_{a_{ij}}^*}^2}{{\mu_{a_{ij}}^*}^2}\right)}, b_i \epsilon \left\{b_i^{(1)}, b_i^{(2)}, ...b_i^{(h_i)}\right\} \tag{4.15}$$

Hence the theorem. ■

The deterministic equivalent of the MOFPP problem (3.7)-(3.9) is expressed as:

$$\max : (Z_k) = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1, 2, \dots, K \tag{4.16}$$

Subject to

$$\Rightarrow \sum_{j=1}^{n} \ln x_j \le n \ln\left(\frac{b_i}{n}\right) - \sum_{j=1}^{n}\left[\ln \mu^*_{a_{ij}} - \frac{1}{2}\ln\left(1 + \frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right)\right] + k_{\beta_i^*}\sqrt{\sum_{j=1}^{n} \ln\left(1 + \frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right)} \tag{4.17}$$

$b_i \epsilon \left\{b_i^{(1)}, b_i^{(2)}, \dots\dots b_i^{(h_i)}\right\}, i = 1, 2 \cdots, m$

$$x_j \ge L_j > 0, \quad , j = 1, 2, \dots, n. \tag{4.18}$$

where $\tilde{\mu}_{a_{ij}}$ and $\tilde{\sigma}^2_{a_{ij}}$ are mean and variance of $\tilde{a}_{ij}$, which follows fuzzy log-normal distribution. Exactly one element is to be selected from the set $\left\{b_i^{(1)}, b_i^{(2)}, \dots\dots b_i^{(h_i)}\right\}$ for ith constraint. This problem is a non-linear programming problem (NLPP) with multi choice parameters. This can not be solved as a standard NLPP. In order to solve it,we may use multi-choice programming method.

Theorem 4.2. If $\tilde{a}_{ij}$ and $\tilde{b_i}$, $i = 1, 2, \dots, m$ and $j = 1, 2, 3, \dots, n$ are independent FRVs distributed log-normally then

$$\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b_i}\right) \succeq \tilde{\beta}_i, \tilde{b}_i \epsilon \left\{\tilde{b_i}^{(1)}, \tilde{b_i}^{(2)}, \dots, \tilde{b_i}^{(h_i)}\right\} \tag{4.19}$$

is equivalent to

$$\sum_{i=1}^{n} \ln x_j \le -n\left[ln(\mu^*_{b_i}) - \frac{1}{2}\ln\left(1 + \frac{\sigma^{*2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)\right] \tag{4.20}$$

$$- \; n \ln n - \sum_{j=1}^{n}\left[\ln \mu^*_{a_{ij}} - \frac{1}{2}\ln\left(1 + \frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right)\right] \tag{4.21}$$

$$+ \; k_{\beta_i^*}\sqrt{\sum_{j=1}^{n} \ln\left(1 + \frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right) + n^2 \ln\left(1 + \frac{\sigma^{*2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)}, \mu^*_{b_i}\epsilon$$

$$\times \; \left\{\mu^*_{b_i^{(1)}}, \mu^*_{b_i^{(2)}}, \dots, \mu^*_{b_i^{(h-i)}}\right\}, \sigma^{*2}_{b_i}\epsilon \left\{\sigma^{*2}_{b_i^{(1)}}, \sigma^{*2}_{b_i^{(2)}}, \dots, \sigma^{*2}_{b_i^{(h-i)}}\right\} \tag{4.22}$$

where $\tilde{\mu_{b_i}}$ and $\tilde{\sigma_{b_i}}^2$ are mean and variance of $\tilde{b_i}$ and $\tilde{\mu}_{a_{ij}}$ and $\tilde{\sigma}^2_{a_{ij}}$ are mean and variance of $\tilde{a}_{ij}$, which follows fuzzy log-normal distribution.

Proof. It is assumed that $\tilde{a}_{ij}$ and $\tilde{b}_i$, $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, \ldots, n$ are independent FRVs distributed log-normally. $\ln \tilde{a}_{ij}$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}_{ij}$ and $\tilde{\sigma}_{ij}^2$ respectively [2.8].

$\ln \tilde{b}_i$ is a fuzzy random variable, whose mean and variance are $\tilde{\mu_i}$ and $\tilde{\sigma_i}$ [2.8]. $\tilde{k} = \sum_{j=1}^{n} \ln \tilde{a}_{ij} - n \ln \tilde{b}_i$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}_{ij}^{'}$ and $(\tilde{\sigma}_{ij}^{'})^2$ respectively [2.9]. The α-cuts of $\tilde{k}_i$, $\tilde{\mu}_{ij}^{'}$ and $(\tilde{\sigma}_{ij}^{'})^2$ are $\tilde{k}_i[\alpha] = [k_{i*}, {k_i}^*]$, $\tilde{\mu}_{ij}^{'}[\alpha] = [\tilde{\mu}_{ij_*}^{'}, \tilde{\mu}_{ij}^{'*}]$, $(\tilde{\sigma}_{ij}^{'})^2[\alpha] = [(\tilde{\sigma}_{ij*}^{'})^2, (\tilde{\sigma}_{ij}^{'*})^2]$ respectively. Let us consider the i-th fuzzy probabilistic constraint:

$$\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i\right) \succeq \tilde{\beta}_i, \tilde{b}_i \epsilon \left\{\tilde{b_i}^{(1)}, \tilde{b_i}^{(2)}, \ldots\ldots, \tilde{b_i}^{(h_i)}\right\} \tag{4.23}$$

where $\tilde{\beta}_i$, $(i = 1, 2, 3, \ldots, m)$ are fuzzy numbers and whose α-cuts are $\tilde{\beta}_i[\alpha] = [\beta_{i*}, \beta_i^*]$. The α-cuts of i-th fuzzy probabilistic constraint is:

$$\begin{aligned}
&\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i\right)[\alpha] \\
&= \left\{P\left(\sum_{j=1}^{n} a_{ij} x_j \le b_i\right) |a_{ij} \in \tilde{a}_{ij}[\alpha], b_i \in \tilde{b}_i[\alpha]\right\} && (4.24)\\
&= \left\{P\left(n(\prod_{j=1}^{n} a_{ij}\ x_j\right)^{1/n} \le b_i)|a_{ij} \in \tilde{a}_{ij}[\alpha], b_i \in \tilde{b}_i[\alpha]\right\} && (4.25)
\end{aligned}$$

Taking logarithm on both sides inside the parenthesis of (4.25) yields

$$\begin{aligned}
&\left\{P(\ln n(\prod_{j=1}^{n} a_{ij}\ x_j)^{1/n} \le \ln b_i)|D\right\} \\
&= \left\{P\left(\ln n + \frac{1}{n}\left(\sum_{j=1}^{n} \ln a_{ij} + \sum_{j=1}^{n} \ln x_j\right) \le \ln b_i\right)|D\right\} \\
&= \left\{P\left(n \ln n + \sum_{j=1}^{n} \ln a_{ij} + \sum_{j=1}^{n} \ln x_j \le n \ln b_i\right)|D\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ P\left(\sum_{j=1}^{n} \ln a_{ij} - n \ln b_i \le -n \ln n - \sum_{j=1}^{n} \ln x_j \right) |D \right\} \\
&= \left\{ P\left(k_i \le -n \ln n - \sum_{j=1}^{n} \ln x_j \right) |k_i \in \tilde{k}_i[\alpha] \right\} \\
&= \left\{ P\left(\frac{k_i - \mu'_{ij}}{\sigma'_{ij}} \le \frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - \mu'_{ij}}{\sigma'_{ij}} \right) |T \right\} \qquad (4.26)
\end{aligned}
$$

where

$$
\begin{aligned}
&T = (k_i, \mu'_{ij}, {\sigma'_{ij}}^2 | k_i \in \tilde{k}_i[\alpha], \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], {\sigma'_{ij}}^2 \in \tilde{\sigma}'^2_{ij}[\alpha]) \text{ and} \\
&D = \{a_{ij} \in \tilde{a}_{ij}[\alpha], b_i \in \tilde{b}_i[\alpha]\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\tilde{P}\left(\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i \right)[\alpha] \\
&= \left(\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\frac{-n \ln n - \sum_{i=1}^{n} \ln x_j - \mu'_{ij}}{\sigma'_{ij}}} e^{\frac{-z^2}{2}} dz | T \right), \ where\ z = \left(\frac{k_i - \mu'_{ij}}{\sigma'_{ij}} \right) \qquad (4.27) \\
&= \left\{ \Phi\left(\frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - \mu'_{ij}}{\sigma'_{ij}} \right) |T \right\} \qquad (4.28)
\end{aligned}
$$

where $z = \dfrac{k_i - \mu'_{ij}}{\sigma'_{ij}}$ and Φ is the cumulative distribution function of $N(0, 1)$ distribution, which is an increasing function.

$$
\begin{aligned}
\min \ : \ & \left\{ \Phi\left(\frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - \mu'_{ij}}{\sigma'_{ij}} \right) |T \right\} \\
= \ & \left\{ \Phi\left(\frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - {\mu'_{ij}}^*}{{\sigma'_{ij}}^*} \right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
\max \ : \ & \left\{ \Phi\left(\frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - \mu'_{ij}}{\sigma'_{ij}} \right) |T \right\} \\
= \ & \left\{ \Phi\left(\frac{-n \ln n - \sum_{j=1}^{n} \ln x_j - \mu'_{ij_*}}{\sigma'_{ij_*}} \right) \right\}
\end{aligned}
$$

$$\tilde{P}\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j \leq \tilde{b_i}\right)[\alpha]$$
$$= \left[\Phi\left(\frac{-n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^{*}}{{\sigma'_{ij}}^{*}}\right), \Phi\left(\frac{-n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}_{*}}{{\sigma'_{ij}}_{*}}\right)\right]$$

Using fuzzy inequality, the α-cuts of the fuzzy probabilistic constraint can be expressed as:

$$\begin{aligned}
\tilde{P}\left(\sum_{j=1}^{n}\tilde{a}_{ij}x_j \leq \tilde{b_i}\right)[\alpha] &\succeq \tilde{\beta}_i[\alpha], \tilde{b_i}\epsilon\left\{\tilde{b_i}^{(1)}, \tilde{b_i}^{(2)}, ..., \tilde{b_i}^{(h_i)}\right\}\\
\Rightarrow \Phi\left(\frac{-n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^{*}}{{\sigma'_{ij}}^{*}}\right) &\geq \beta_i{}^{*}, \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], \sigma'_{ij} \in \tilde{\sigma}'_{ij}[\alpha]\\
\Rightarrow \Phi\left(\frac{-n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^{*}}{{\sigma'_{ij}}^{*}}\right) &\geq \Phi(-k_{\beta_i{}^{*}}), \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], \sigma'_{ij} \in \tilde{\sigma}'_{ij}[\alpha]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \left(\frac{-n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^{*}}{{\sigma'_{ij}}^{*}}\right) &\geq (-k_{\beta_i{}^{*}}), \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], \sigma'_{ij} \in \tilde{\sigma}'_{ij}[\alpha]\\
\Rightarrow -n\ln n - \sum_{j=1}^{n}\ln x_j - {\mu'_{ij}}^{*} &\geq -k_{\beta_i{}^{*}}{\sigma'_{ij}}^{*}, \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], \sigma'_{ij} \in \tilde{\sigma}'_{ij}[\alpha]
\end{aligned}$$

Hence the deterministic equivalent of the fuzzy probabilistic constraint in (4.19) is expressed as:

$$\sum_{j=1}^{n}\ln x_j \leq -n\ln n - {\mu'_{ij}}^{*} + k_{\beta_i{}^{*}}{\sigma'_{ij}}^{*}, \quad \mu'_{ij} \in \tilde{\mu}'_{ij}[\alpha], \quad \sigma'_{ij} \in \tilde{\sigma}'_{ij}[\alpha] \tag{4.29}$$

Now we have the relation between μ_i, μ_{ij} and μ'_{ij} is:

$$\mu'_{ij} = \sum_{j=1}^{n}\mu_{ij} - n\mu_i, \quad \mu_i\epsilon\{\mu_i^{(1)}, \mu_i^{(2)}, ..., \mu_i^{(h_i)}\} \tag{4.30}$$

The relation between σ_i^2, σ_{ij}^2 and ${\sigma'_{ij}}^2$ is:

$${\sigma'_{ij}}^2 = \sum_{j=1}^{n}\sigma_{ij}^2 + n^2\sigma_i^2, \quad \sigma_i^2\epsilon\{\sigma_i^{(1)}, \sigma_i^{(2)}, ..., \sigma_i^{(h_i)}\} \tag{4.31}$$

So equation (4.29) is equivalent to

$$\sum_{i=1}^{n} \ln x_j \leq -n\left[ln(\mu^*_{b_i}) - \frac{1}{2}\ln\left(1+\frac{\sigma^{*\,2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)\right] \tag{4.32}$$

$$- \; n\ln n - \sum_{j=1}^{n}\left[\ln \mu^*_{a_{ij}} - \frac{1}{2}\ln\left(1+\frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right)\right] \tag{4.33}$$

$$+ \; k_{\beta_i{}^*}\sqrt{\sum_{j=1}^{n}\ln\left(1+\frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right) + n^2\ln\left(1+\frac{\sigma^{*\,2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)}, \mu^*_{b_i}\epsilon$$
$$\times \; \left\{\mu^*_{b_i{}^{(1)}}, \mu^*_{b_i{}^{(2)}}, \ldots, \mu^*_{b_i{}^{(h-i)}}\right\}, \sigma^{*2}_{b_i}\epsilon\left\{\sigma^{*2}_{b_i{}^{(1)}}, \sigma^{*2}_{b_i{}^{(2)}}, \ldots, \sigma^{*2}_{b_i{}^{(h-i)}}\right\} \tag{4.34}$$

where $\tilde{\mu_{b_i}}$ and $\tilde{\sigma_{b_i}}^2$ are mean and variance of $\tilde{b_i}$ and $\tilde{\mu}_{a_{ij}}$ and $\tilde{\sigma}^2_{a_{ij}}$ are mean and variance of $\tilde{a}_{ij}$, which follows fuzzy log-normal distribution. Hence the theorem. ■

The deterministic equivalent of the MOFPP problem (3.7)-(3.9) is expressed as:

$$\max : (Z_k) = \sum_{j=1}^{n} c^k_j x_j, \quad k = 1, 2, \ldots, K \tag{4.35}$$

Subject to

$$\sum_{i=1}^{n} \ln x_j \leq -n\left[ln(\mu^*_{b_i}) - \frac{1}{2}\ln\left(1+\frac{\sigma^{*\,2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)\right] \tag{4.36}$$

$$- \; n\ln n - \sum_{j=1}^{n}\left[\ln \mu^*_{a_{ij}} - \frac{1}{2}\ln\left(1+\frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right)\right] \tag{4.37}$$

$$+ \; k_{\beta_i{}^*}\sqrt{\sum_{j=1}^{n}\ln\left(1+\frac{\sigma^{*\,2}_{a_{ij}}}{\mu^{*\,2}_{a_{ij}}}\right) + n^2\ln\left(1+\frac{\sigma^{*\,2}_{b_i}}{\mu^{*\,2}_{b_i}}\right)}, \mu^*_{b_i}\epsilon$$
$$\times \; \left\{\mu^*_{b_i{}^{(1)}}, \mu^*_{b_i{}^{(2)}}, \ldots, \mu^*_{b_i{}^{(h-i)}}\right\}, \sigma^{*2}_{b_i}\epsilon\left\{\sigma^{*2}_{b_i{}^{(1)}}, \sigma^{*2}_{b_i{}^{(2)}}, \ldots, \sigma^{*2}_{b_i{}^{(h-i)}}\right\} \tag{4.38}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{4.39}$$

where $\tilde{a_{ij}}$ and $\tilde{b_i}$ are FRVs distributed log-normally and $c^k_j \epsilon R$, for all i, j and k. $\tilde{\mu_{b_i}}$ and $\tilde{\sigma_{b_i}}^2$ are mean and variance of b_i and $\tilde{\mu}_{a_{ij}}$ and $\tilde{\sigma}^2_{a_{ij}}$ are mean and variance of $\tilde{a}_{ij}$, which follows fuzzy log-normal distribution. Exactly one element is to be selected from the set $\left\{\mu^*_{b_i{}^{(1)}}, \mu^*_{b_i{}^{(2)}}, ..., \mu^*_{b_i{}^{(h-i)}}\right\}$ *and* $\left\{\sigma^{*2}_{b_i{}^{(1)}}, \sigma^{*2}_{b_i{}^{(2)}}, \ldots, \sigma^{*2}_{b_i{}^{(h-i)}}\right\}$ for i-th constraint.

Theorem 4.3. If $\tilde{b_i}$, $i = 1, 2, \ldots, m$ are independent FRVs distributed log-normally then

$$\tilde{P}\left(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b_i}\right) \succeq \tilde{\beta_i}, \quad i = 1, 2, \ldots, m \tag{4.40}$$

is equivalent to

$$\ln\left(\sum_{j=1}^{n} a_{ij}x_j\right) \le \left[\ln \mu_{b_{i*}} - \frac{1}{2}\ln\left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)\right] + k_{\beta_i{}^*}\sqrt{\ln\left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)} \tag{4.41}$$

where $\tilde{\mu}_{b_i}$ and $\tilde{\sigma}^2_{b_i}$ are mean and variance of $\tilde{b_i}$, which follows fuzzy log-normal distribution.

Proof. It is assumed that $\tilde{b_i}$, $i = 1, 2, 3, \ldots, m$ are independent FRVs distributed log-normally. $\ln \tilde{b_i}$ is a fuzzy normal random variable, whose mean and variance are $\tilde{\mu}_i$ and $\tilde{\sigma}_i^2$ respectively [2.8]. The α-cuts of $\tilde{b_i}$, $\tilde{\mu}_i$ and $(\tilde{\sigma}_i)^2$ are $\tilde{b_i}[\alpha] = [b_{i*}, {b_i}^*]$, $\tilde{\mu}_i[\alpha] = [\tilde{\mu}_{i*}, \tilde{\mu}_i^*]$, $(\tilde{\sigma}_i)^2[\alpha] = [(\tilde{\sigma}_{i*})^2, (\tilde{\sigma}_{ij}^*)^2]$ respectively. Now let us consider the i-th constraint:

$$\tilde{P}\left(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b_i}\right) \succeq \tilde{\beta_i} \tag{4.42}$$

Where $\tilde{\beta_i}$, $(i = 1, 2, 3, \ldots, m)$ are fuzzy number and whose α-cuts are $\tilde{\beta_i}[\alpha] = [\beta_*, \beta^*]$. The α-cut of i-th fuzzy probabilistic constraint is:

$$\tilde{P}\left(\sum_{j=1}^{n} a_{ij}x_j \le \tilde{b_i}\right)[\alpha]$$

$$, \quad = \quad \tilde{P}(y_i \le \tilde{b_i})[\alpha], where y_i = \sum_{j=1}^{n} a_{ij}x_j \tag{4.43}$$

$$= \left\{P(y_i \le b_i)|b_i \in \tilde{b_i}[\alpha]\right\} \tag{4.44}$$

$$= \left\{P(\ln y_i \le \ln b_i)|b_i \in \tilde{b_i}[\alpha]\right\} \tag{4.45}$$

$$= \left\{1 - P(\ln b_i \le \ln y_i)|b_i \in \tilde{b_i}[\alpha]\right\} \tag{4.46}$$

$$= \left\{1 - P\left(\frac{\ln b_i - \mu_i}{\sigma_i} \le \frac{\ln y_i - \mu_i}{\sigma_i}\right)|b_i \in \tilde{b_i}[\alpha], \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha]\right\} \tag{4.47}$$

$$= \left\{1 - \left(\frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{\frac{\ln y_i - \mu_i}{\sigma_i}} e^{\frac{-z^2}{2}} dz|K\right),\ where\ z = \left(\frac{\ln b_i - \mu_i}{\sigma_i}\right)\right\} \tag{4.48}$$

$$= \left\{1 - \phi\left(\frac{\ln y_i - \mu_i}{\sigma_i}\right)|\mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha]\right\} \tag{4.49}$$

where $K = b_i \in \tilde{b}_i[\alpha], \mu_i \in \tilde{\mu}_i[\alpha], \sigma_i \in \tilde{\sigma}_i[\alpha]$ and ϕ is the cumulative distribution function of $N(0,1)$ distribution. Now let

$$\begin{aligned} \min \quad : \quad & \left\{1 - \Phi\left(\frac{\ln y_i - \mu_i}{\sigma_i}\right)\right\} \\ = \quad & \left\{1 - \Phi\left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right)\right\} \end{aligned}$$

$$\begin{aligned} \max \quad : \quad & \left\{1 - \Phi\left(\frac{\ln y_i - \mu_i}{\sigma_i}\right)\right\} \\ = \quad & \left\{1 - \Phi\left(\frac{\ln y_i - {\mu_i}^*}{{\sigma_i}^*}\right)\right\} \end{aligned}$$

So,

$$\begin{aligned} & \tilde{P}\left(\sum_{j=1}^{n} a_{ij}x_j \leq \tilde{b_i}\right)[\alpha] \\ , \quad = \quad & \left[1 - \Phi\left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right), 1 - \Phi\left(\frac{\ln y_i - {\mu_i}^*}{{\sigma_i}^*}\right)\right] \end{aligned} \tag{4.50}$$

Using fuzzy inequality, the $\alpha - cut$ of the fuzzy constraint (4.42) is expressed as:

$$\begin{aligned} \tilde{P}(\sum_{j=1}^{n} a_{ij}x_j \leq \tilde{b}_i)[\alpha] \quad & \succeq \quad \tilde{\beta}_i[\alpha] \\ \Rightarrow 1 - \Phi\left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right) & \geq \beta_i^* \\ \Rightarrow \Phi\left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right) & \leq 1 - \beta_i^* \\ \Rightarrow \Phi\left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right) & \leq \Phi(k_{\beta_i^*}) \\ \Rightarrow \left(\frac{\ln y_i - \mu_{i*}}{\sigma_{i*}}\right) & \leq k_{\beta_i^*} \\ \Rightarrow \ln y_i - \mu_{i*} & \leq k_{\beta_i^*}\sigma_{i*} \\ \Rightarrow \ln y_i & \leq k_{\beta_i^*}\sigma_{i*} + \mu_{i*} \end{aligned}$$

Hence the deterministic equivalent of the fuzzy probabilistic constraint (4.42) is expressed as:

$$\ln\left(\sum_{j=1}^{n} a_{ij}x_j\right) \leq k_{\beta_i^*}\sigma_{i*} + \mu_{i*} \tag{4.51}$$

Let b_i follows log-normal distribution with mean and variance μ_{b_i} and $\sigma_{b_i}^2$ respectively. Then the mean and variance of $\ln b_i$ follows normal distribution with mean and variance μ_i and σ_i^2. The relation between the mean and the variance of log-normal and normal distribution are given by:

$$\mu_i = \ln \mu_{b_i} - \frac{1}{2} \ln \left(1 + \frac{\sigma_{b_i}^2}{\mu_{b_i}^2}\right) \tag{4.52}$$

$$\sigma_i^2 = \ln \left(1 + \frac{\sigma_{b_i}^2}{\mu_{b_i}^2}\right) \tag{4.53}$$

So the equation (4.51) is equivalent to:

$$\ln \left(\sum_{j=1}^{n} a_{ij} x_j \right) \leq \left[\ln \mu_{b_{i*}} - \frac{1}{2} \ln \left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)\right] + k_{\beta_i{}^*} \sqrt{\ln \left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)} \tag{4.54}$$

Hence the theorem. ■

The deterministic equivalent of the MCMOFPP problem:

$$\max : (z_k) = \sum_{j=1}^{n} c_j^{(k)} x_j, \;\; c_j^{(k)} \in \left\{ c_j^{k_1}, c_j^{k_2}, \ldots, c_j^{k_R} \right\}, k = 1, 2, \ldots, K. \tag{4.55}$$

Subject to

$$\tilde{P} \left(\sum_{j=1}^{n} a_{ij} x_j \leq \tilde{b_i} \right) \geq \tilde{\beta_i}, \;\; i = 1, 2, \ldots, m. \tag{4.56}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{4.57}$$

is expressed as:

$$\max : (z_k) = \sum_{j=1}^{n} c_j^{(k)} x_j, \;\; c_j^{(k)} \in \left\{ c_j^{k_1}, c_j^{k_2}, \ldots, c_j^{k_R} \right\}, k = 1, 2, \ldots, K. \tag{4.58}$$

Subject to

$$\ln \left(\sum_{j=1}^{n} a_{ij} x_j \right) \leq \left[\ln \mu_{b_{i*}} - \frac{1}{2} \ln \left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)\right] + k_{\beta_i{}^*} \sqrt{\ln \left(1 + \frac{{\sigma_{b_{i*}}}^2}{{\mu_{b_{i*}}}^2}\right)} \tag{4.59}$$

$$x_j \geq L_j > 0, \quad , j = 1, 2, \ldots, n. \tag{4.60}$$

where $\beta_i, i = 1, 2, \ldots, m$ are fuzzy numbers and $a_{ij}, c_j^k \in R$, for all i, j, k and $\tilde{\mu}_{b_i}$ and $\tilde{\sigma}_{b_i}^2$ are mean and variance of $\tilde{b_i}$ which follows log-normal distribution.

4.2. Step-2: Dealing with multi-choice parameter

In this step, we deal with multi-choice parameter by least square approximation technique. We fit 'n' number of data to a polynomial of degree 'n-1'. A polynomial that passes through a given set of points is called an interpolating polynomial. There are 'n' points in a plane $(i, y_i), i = 0, 1, 2, \dots, n-1$ with distinct 'i' then there exists a unique polynomial in x whose degree is n- 1.

Let the polynomial be:

$$P_{R_k-1}(x) = a_0 + a_1 x + a_2(x)^2 + a_3(x)^3 + a_4(x)^4 + \cdots + a_{n-2}(x)^{n-2} + a_{n-1}(x)^{n-1}$$

Now, according to concept of least square approximation

$$\begin{aligned} & P(x) = y(x) \\ \implies & P(x) = y(x) = a_0 + a_1 x + a_2(x)^2 + a_3(x)^3 + a_4(x)^4 + \cdots \\ & \quad + a_{n-2}(x)^{n-2} + a_{n-1}(x)^{n-1} \end{aligned}$$

After obtaining the values of $a_0, a_1, \dots, a_{n-1}$ the required polynomial can be derived. Now the data.

i	0	1	2	3	$\cdots$	$n-2$	$n-1$
y_i	y_0	y_1	y_2	y_3	$\cdots$	y_{n-2}	y_{n-1}

must fit to the polynomial. Now,

$$P(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + a_4(0)^4 + \cdots + a_{n-2}(0)^{n-2} + a_{n-1}(0)^{n-1} = y_0$$

$$P(1) = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 + a_4(1)^4 + \cdots + a_{n-2}(1)^{n-2} + a_{n-1}(1)^{n-1} = y_1$$

$$P(2) = a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 + a_4(2)^4 + \cdots + a_{n-2}(2)^{n-2} + a_{n-1}(2)^{n-1} = y_2$$

$$\vdots$$

$$\begin{aligned} P(n-1) = {} & a_0 + a_1(n-1) + a_2(n-1)^2 + a_3(n-1)^3 + a_4(n-1)^4 + \cdots \\ & + a_{n-2}(n-1)^{n-2} + a_{n-1}(n-1)^{n-1} = y_{n-1} \end{aligned}$$

The system of equations can be written as $AX = B$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1^2 & 1^3 & 1^4 & \cdots & 1^{n-2} & 1^{n-1} \\ 1 & 2 & 2^2 & 2^3 & 2^4 & \cdots & 2^{n-2} & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & n-1 & (n-1)^2 & (n-1)^3 & (n-1)^4 & \cdots & (n-1)^{n-2} & (n-1)^{n-1} \end{bmatrix}$$

$$X = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-2} \end{bmatrix}$$

The value of $a_1, a_2, \ldots a_{n-1}$ can be found out by solving $AX = B$. If A is non singular then the system can be solved.

Using the least square approximation technique, a polynomial with integer restrictions is incorporated for multi-choice parameters in Model-I, Model-II, and Model-III to obtain a mixed integer multi-objective non-linear mathematical models.

4.3. Step-3: Fuzzy Programming Method

Steps of the fuzzy programming technique are given below:

Step-1: Select the first objective function (i.e. $Z_k(x), k = 1$) and solve it as a single objective MP problem subject to the constraints. Let $x^{(1)}$ be the ideal solution. Then select the second objective function and find the ideal solution as $x^{(2)}$, continue the process K times for K different objective functions. Let $x^{(1)}, x^{(2)}, \ldots, x^{(K)}$ be the ideal solutions for the respective objective functions.

Step-2: Evaluate all these objective functions at all these ideal solutions and formulate a pay-off matrix (Table 3) of size K by K as follows.

Table 1: Pay-Off Matrix

	$Z_1(x)$	$Z_2(x)$	$\ldots$	$Z_K(x)$
$x^{(1)}$	Z_{11}	Z_{12}	$\ldots$	Z_{1K}
$x^{(2)}$	Z_{21}	Z_{22}	$\ldots$	Z_{2K}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x^{(K)}$	Z_{K1}	Z_{K2}	$\ldots$	Z_{KK}

Step-3: From pay-off matrix (Table 3) determine the bounds for k-th objective function $Z_k(x), r = 1, 2, \ldots, K$. If an objective function is of maximization type find the best upper bound U_k^* and worst lower bound L_k^-. If an objective function is of minimization type find the best lower bound L_k^* and worst upper bound U_k^-, $k = 1, 2, \dots, K$.

Step-4: Associate a membership function $\mu_{Z_k}(x)$ to the k-th objective function $Z_k(x)$ as:

$$\mu_{Z_k}(x) = \begin{cases} 1, & if\ Z_k(x) \geq U_k^* \\ \dfrac{Z_k(x) - L_k^-}{U_k^* - L_k^-}, & if\ L_k^- < Z_k(x) < U_k^*, \quad k = 1, 2, 3, \dots, K \\ 0, & if\ Z_k(x) \leq L_k^- \end{cases} \tag{4.61}$$

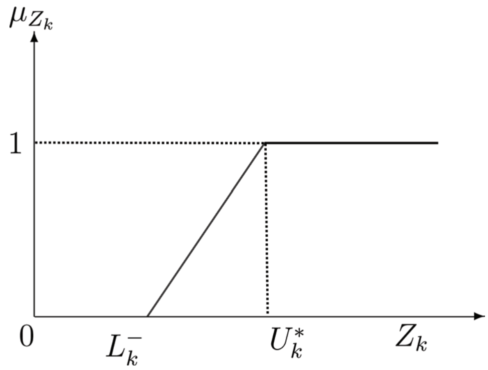

Figure 1: Membership Function of a Vector Maximization Problem

$$\mu_{Z_k}(x) = \begin{cases} 1, & if\ Z_k(x) \leq L_k^* \\ \dfrac{U_k^- - Z_k(x)}{U_k^- - L_k^*}, & if\ L_k^* < Z_k(x) < U_k^-, \quad k = 1, 2, 3, \dots, K \\ 0, & if\ Z_k(x) \geq U_k^- \end{cases} \tag{4.62}$$

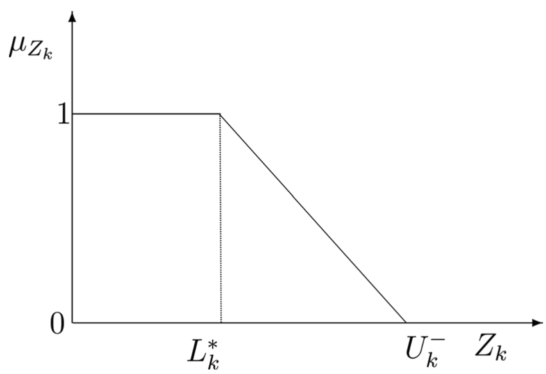

Figure 2: Membership Function of a Vector Minimization Problem

Step-5: **(a)** Use max-min operator with an augmented variable λ and formulate a single objective crisp MP problem as:

$$\max : \lambda \tag{4.63}$$

subject to

$$\lambda \leq \mu_{Z_k}(x),\ \ k = 1, 2, 3, \ldots, K \tag{4.64}$$

$$x \in S \tag{4.65}$$

where S is the feasible region of the MOMP model.

(b) Similarly, if we use min-max operator with an augmented variable λ, a single objective crisp MP problem can be formulated as:

$$\min : \lambda \tag{4.66}$$

subject to

$$\lambda \geq \mu_{Z_k}(x),\ \ k = 1, 2, 3, \ldots, K \tag{4.67}$$

$$x \in S \tag{4.68}$$

where S is the feasible region of the MOMP model.

Step-6: Solve the crisp model by using an appropriate mathematical programming method to find an optimal compromise solution x^*. Then evaluate all the objective functions at the optimal compromise solution x^*.

5. Numerical Example

$$\max : Z_1 = 3x_1 + 2x_2 \tag{5.1}$$

$$\max : Z_2 = 5x_1 + x_2 \tag{5.2}$$

subject to,

$$\tilde{p}(x_1 + 3x_2 \leq b_1) \succeq \tilde{0.3} \tag{5.3}$$

$$\tilde{p}(4x_1 + 2x_2 \leq b_2) \succeq \tilde{0.4} \tag{5.4}$$

$$x \geq 0 \tag{5.5}$$

$$y \geq 0 \tag{5.6}$$

Where $\tilde{a}_{ij}$ are independent log-normally distributed FRV with mean $\tilde{\mu}_{1ij} and \tilde{\mu}_{2ij}$, variance $\tilde{\sigma}^2_{1ij} and \tilde{\sigma}^2_{2ij}$ as triangular fuzzy numbers and $\tilde{0.3}$ and $\tilde{0.4}$ are also an fuzzy numbers. Exactly one element is to be selected from each of the set, $b_1 = \{2, 3, 5, 7, 8\}, b_2 = \{4, 6, 7, 9\}$. $\tilde{\mu}_{1ij} =< 5, 6, 7 >, \tilde{\mu}_{2ij} =< 6, 7, 9 >, \tilde{\sigma}_{1ij} =< 3, 4, 5 >, \tilde{\sigma}_{2ij} =< 4, 6, 7 >, \tilde{0.3} =< 0.2, 0.3, 0.4 >, \tilde{0.4} =< 0.3, 0.4, 0.5 >$.

Solution. $\alpha - cuts$ of $\tilde{\mu}_{1ij}, \tilde{\mu}_{2ij}, \tilde{\sigma}_{1ij}, \tilde{\sigma}_{2ij}$ and $\tilde{0.3}, \tilde{0.4}$ an be calculated. The $\alpha - cuts$ are

$$\begin{aligned}
\tilde{\mu}_{1ij}[\alpha] &= [5+\alpha, 7-\alpha] && (5.7)\\
\tilde{\mu}_{2ij}[\alpha] &= [6+\alpha, 9-2\alpha] && (5.8)\\
\tilde{\sigma}_{1ij}[\alpha] &= [3+\alpha, 5-\alpha] && (5.9)\\
\tilde{\sigma}_{2ij}[\alpha] &= [4+2\alpha, 7-\alpha] && (5.10)\\
\tilde{0.3}[\alpha] &= [0.2+0.1\alpha, 0.4-0.1\alpha] && (5.11)\\
\tilde{0.4}[\alpha] &= [0.3+0.1\alpha, 0.5-0.1\alpha] && (5.12)
\end{aligned}$$

Using theorem (2.1) the deterministic equivalent model is

$$\begin{aligned}
\max: Z_1 &= 3x_1 + 2x_2 && (5.13)\\
\max: Z_2 &= 5x_1 + x_2 && (5.14)
\end{aligned}$$

subject to,

$$\ln(x_1+3x_2) \leq 2\ln\left(\frac{b_1}{2}\right) - \left[\ln(7-\alpha) - \frac{1}{2}\ln\left(1+\frac{(5-\alpha)^2}{(7-\alpha)^2}\right)\right] + k(0.4-0.1\alpha)\sqrt{\ln\left(1+\frac{(5-\alpha)^2}{(7-\alpha)^2}\right)} \quad (5.15)$$

$$\ln(4x_1+2x_2) \leq 2\ln\left(\frac{b_2}{2}\right) - \left[\ln(9-2\alpha) - \frac{1}{2}\ln\left(1+\frac{(7-\alpha)^2}{(9-2\alpha)^2}\right)\right] + k(0.5-0.1\alpha)\sqrt{\ln\left(1+\frac{(7-\alpha)^2}{(9-2\alpha)^2}\right)} \quad (5.16)$$

$x_1, x_2 > 0$.

For $\alpha = 0.2$ the above deterministic equivalent becomes:

$$\begin{aligned}
\max: Z_1 &= 3x + 2y && (5.17)\\
\max: Z_2 &= 5x + y && (5.18)
\end{aligned}$$

subject to,

$$\ln(x_1+3x_2) \leq 2\ln\left(\frac{b_1}{2}\right) - 1.397558 \quad (5.19)$$

$$\ln(4x_1+2x_2) \leq 2\ln\left(\frac{b_2}{2}\right) - 1.46087 \quad (5.20)$$

$x_1, x_2 > 0$. Where,

$$b_1 = -0.16667u^3 + u^2 + 0.16667u + 2 \quad (5.21)$$
$$b_2 = 0.3333v^3 - 1.5v^2 + 3.16667v + 4 \quad (5.22)$$

$u \leq 3, v \leq 2$. The above deterministic multi-objective multi-choice fuzzy non-linear programming problem is solved using fuzzy programming method. Applying the steps of the fuzzy programming method two ideal solutions are obtained. $X^1 = (x_1, x_2)^T = (0.6183296, 1.112224)^T$ and $X^2 = (x_1, x_2)^T = (1.174440, 0.0000)^T$. Where the values of objective functions are, $Z_1 = 4.079436 and Z_2 = 5.872201$. Using the two ideal solution a pay-off matrix is formulated as

Table 2: Pay-Off Matrix for Numerical Example

	Z_1	Z_2
X^1	4.079436	4.203872
X^2	3.52332	5.872201

and from the pay-off matrix the bounds of the objective functions are obtained. Finally a crisp model is formulated as:

$$max : \lambda$$

subject to,

$$3x_1 + 2x_2 - 0.556116\lambda \geq 3.52332 \quad (5.23)$$
$$5x_1 + x_2 - 1.668329\lambda \geq 4.203872 \quad (5.24)$$
$$\ln(x_1 + 3x_2) \leq 2\ln\left(\frac{b_1}{2}\right) - 1.397558 \quad (5.25)$$
$$\ln(4x_1 + 2x_2) \leq 2\ln\left(\frac{b_2}{2}\right) - 1.46087 \quad (5.26)$$
$$u \leq 4 \quad (5.27)$$
$$v \leq 3 \quad (5.28)$$
$$x_1,\ x_2 \geq 0 \quad (5.29)$$

The crisp problem is solved using the LINGO soft-ware and an optimal compromise solution is obtained for $\alpha = 0.2$ as $X^* = (0.8963861, 0.5561106)$, $\lambda = 0.5000028$. The objective functionals for first and second objective functions are 3.80138 and 5.038041 respectively.Solutions obtained for ten different values of α are given in the following table-3.

Table 3: Solution for different value of α

Sl.n.	α	x_1	x_2	λ	Z_1	Z_2
2	0.2	0.8963861	0.5561106	0.5000028	3.80138	5.038041
3	0.3	0.9216348	0.5518419	0.4999979	3.868588	5.160016
4	0.4	0.9475293	0.5489263	0.5000043	3.940441	5.286573
5	0.5	0.9749306	0.5458901	0.4999992	4.016572	5.420543
6	0.6	1.0039690	0.5427168	0.5000000	4.097341	5.562562
7	0.7	1.0348100	0.5393883	0.5000105	4.183207	5.713438
8	0.8	1.0676090	0.5358815	0.5000047	4.27459	5.873927
9	0.9	1.103434	0.5324531	0.4999972	4.375208	6.049623
10	1.0	1.140801	0.5284983	0.4999993	4.4794	6.232503

6. Conclusions

A solution procedure to solve a multi-choice multi-objective fuzzy probabilistic programming problem is considered in this paper. Using log-normal distribution we first transformed the probabilistic constraint into its equivalent crisp model. We handled the multi-choice parameter using least square approximation technique. Then the transformed equivalent multi-objective programming problem tackled by using fuzzy approximation technique. The final model was solved by using existing methodology or software. The result obtained from the numerical example is solved by taking $\alpha = 0.1, 0.2, 0.3, \ldots, 1.0$. The proposed mathematical model can also be solved using hybrid optimization technique.

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