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Developing A Desired Power Series Of The Function

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Abstract

Algorithmic generalized form of a power series which yields infinitely many expansions of a function $f(x)$. $f(x)$ Can be broken down into simpler functions and by assigning suitable parameters a new generalized expansion of the function is obtained.

1. Introduction

Power series is a representation of a function in infinite series of simple terms, usually represented through a sigma notation[1],this series is usually centered at some 'a'. A real valued function whose higher order derivative(s) exist can only be represented in the form of Power series [4]. One of the most commonly used forms is that of Taylor series [5], this Taylor series when centered at '0' then this series is called Maclaurin series [3]. Taylor series is one of the finest numeric algorithms for solution of various differential equations, Taylor series is considered as a very powerful tool used in various engineering and science field. Taylor series is a variant of power series, as it is impossible to compute till infinity we usually represent our generalized power series (Taylor series) as sum till big natural number 'N 'followed by the remainder term. In this paper a new algorithmic approach is generalized which aims to prove that infinitely many expansions of the given functions are possible, that is infinitely many series of one given function is possible, while the given function may or may not have higher order derivative(s). The function is split into two or more than two forms these forms of the function possess higher order derivatives, while the given function may not be 'n' times differentiable. Thereby proving every given function which is continuous on the given interval [a,b] can be represented in the form of this generalized Power series. Aim of this paper is to generalize a method by which: (i) The series of given function whose$(n)^{th}$order derivative(s) does not exist can also be obtained by this method. (ii) One who is willing to approximate one function using the other, this method is suitable for that.(iii) Infinitely many series exists for the given function. (iv)If a series is required which propagates slowly so that analysis becomes easier it is possible. I split the function into various different forms in such a way that infinitely many solutions of the given function exist, and hence by this one can generate a desired series of the given said function. One can obtain the expansion even for various discontinuous function(s).

2. Expanding the given function in numerous ways

In this section of the paper we will see a method and establish a generalization (sigma notation) such that infinitely many expansions of one given function are possible.

A Power series representation consists of a real number a_n and an existing variable 'x'. If all the given a_n's are zero then the given expansion will yield 'x'. If all $a_n = 1$ then the series is believed to be converging in nature.

A given function will have remainder as it is impossible to compute till infinity this remainder term is represented as R_n. This remainder term as per Cauchy's form is $R_n = h^n \dfrac{(1-\theta)^{n-1}}{(n-1)!} \cdot f^n(a+\theta h)$

Theorem 2.1 *A real valued function $f(x)$ whose higher order derivative exists, it's $(n-1)^{th}$ order is continuous on $[a, a+h]$ and $(n)^{th}$ order exists for $(a, a+h)$ then there exist unique functions $s(x)$, $p(x)$, $k(x)$ and $l(x)$ such that $f(s(x)) - f(p(x)) = f(x)$ and $f(x) = k(x) - l(x)$ then there exists Θ which lies between 0 and 1, $0 < \Theta < 1$ such that,*

$$f(x) = [k(a) - l(a)] + \frac{(S-P)}{1!} \cdot [k'(a) - l'(a)] + \frac{(S-a)^2 - (P-a)^2}{2!} \cdot [k''(a) - l''(a)] + ...R_n(x)$$

Where $s(x) = S$ and $p(x) = P$ while the $R_n(x)$ as per Lagrange's form of remainder is

$$R_n(x) = \frac{(S-a)^n - (P-a)^n}{n!} \cdot [k^n(a+\Theta(S-P)) - l^n(a+\Theta(S-P))]$$

. The $R_n(x)$ is the error term or remainder of the entire expansion.

Proof Let a function $g(x)$ be defined on $[a, a+h]$ as

$$g(x) = f(x) + \frac{a+h-x}{1!} \cdot f'(x) + \frac{(a+h-x)^2}{2!} \cdot f''(x) + ... + (a+h-x)^k \cdot A \quad \textbf{(2.1)}$$

Such that $f(x)$ and it's higher order derivatives are continuous on the given interval $[a, a+h]$. And differentiable on $(a, a+h)$, Put $x = a+h$ then $g(a+h) = f(a+h)$, now let $x = a$ then,

$$g(a) = f(a) + \frac{h}{1!} \cdot f'(a) + \frac{h^2}{2!} \cdot f''(a) + ... + h^k \cdot A$$

Here 'A' is unknown but it's value is such that $g(a) = g(a+h)$, as per Rolle's Theorem when $g(a) = g(a+h)$ there exists $c \in (a, a+h)$ such that

$g'(c) = g'(a+\theta h) = 0$. Thus differentiating **(2.1)** at some 'c' should equate to zero hence we obtain 'A'.

$$g'(a+\theta h) = \frac{h^{n-1}}{(n-1)!} \cdot (1-\theta)^{n-1} \cdot f^n(a+\theta h) - kA(1-\theta)^{k-1} \cdot h^{k-1} = 0$$

The value of 'A' is $$A = \frac{(h^{n-k})((1-\theta)^{n-k})}{(n-1)! \cdot k} \cdot f^n(a+\theta h)$$

$$f(a+h) = f(a) + \frac{h}{1!} \cdot f'(a) + \frac{h^2}{2!} \cdot f''(a) + ... + \frac{h^{n-1}}{(n-1)!} \cdot f^{(n-1)}(a) + h^n \cdot \frac{(1-\theta)^{(n-k)}}{k(n-1!)} f^{(n)}(a+\theta h)$$

Remainder after n terms as per Lagrange's form is

$$f(a+h) = f(a) + \frac{h}{1!} \cdot f'(a) + \frac{h^2}{2!} \cdot f''(a) + ... \frac{h^n}{n!} \cdot f^n(a+\Theta h)$$

i.e when $k = n$. Substitute, $a + h = x$ gives our function $f(x)$, consider functions $s(x)$ and $p(x)$ [Whose $(n)^{th}$ order derivatives may or may not exist] such that $f(s(x)) - f(p(x)) = f(x)$ (provided $f(x)$ is not a constant). Assuming $s(x) = S$ and $p(x) = P$, both 'S' and 'P' are corresponding functions of 'x' thus, these exist in terms of 'x'.

Thus,

$$f(x) = f(a) + \frac{[S-P]}{1!} \cdot f'(a) + \frac{[(S-a)^2 - (P-a)^2]}{2!} f''(a) + ... \frac{[(S-a)^n - (P-a)^n]}{n!} \cdot f^n(a+\Theta(S-P)) \quad \textbf{(2.2)}$$

Consider functions $k(x)$ and $l(x)$ such that $f(x) = k(x) - l(x)$, These given functions are such that their high order derivatives exists till $(n)^{th}$ order and are also continuous on the same interval as that of the given function. Substituting this result to **(2.2)** we get

$$f(x) = [k(a) - l(a)] + \frac{[S-P]}{1!} \cdot [k'(a) - l'(a)] + \frac{[(S-a)^2 - (P-a)^2]}{2!} \cdot [k''(a) - l''(a)] + ... \frac{[(S-a)^n - (P-a)^n]}{n!} \cdot [k^n(a+\Theta(S-P)) - l^n(a+\Theta(S-P))] \quad \textbf{(2.3)}$$

As per Cauchy's form of remainder after n terms is given by

$$R_n(x) = \frac{[(S-a)^n - (P-a)^n]}{(n-1)!} \cdot [(1-\Theta)^{n-1} \cdot f^n(a+\Theta(S-P))].$$

Case 1 If higher order derivative of function $f(x)$ does not exist and yet power series of the function is desired then one may alter functions $k(x)$ and $l(x)$ such that $k^n(x)$ and $l^n(x)$ exists and then expand as per **(2.3)**.

Here this $f(x)$ is such that it can be split into simpler form numerous number of times i.e

$f(x) = k(x) - l(x) = M(x) - n(x) = P(x) - q(x) = ...$ **(2.4)** And

$$f(x) = f(a(x)) - f(b(x)) = f(c(x)) - f(d(x)) = f(e(x)) - f(j(x)) = \ldots\ldots \quad \textbf{(2.5)}$$

Thus this proves that n number of expansions of one given function is possible in one or different function forms.

Generalization of the above proof gives us

$$f(x) = [k(a) - l(a)] + \sum_{n=1}^{\infty} \frac{[(S-a)^n - (P-a)^n]}{n!} \cdot [k^n(a) - l^n(a)]$$

NsOTE: In our above expansion if $s(x) = S = x$, $p(x) = P = a$ and $l(x) = 0$ then this expansion is our Taylor's expansion.

Conclusion

The above results prove that one can expand the given function numerous number of times in numerous number of ways and by using suitable desired parameters of $k(x)$, $l(x)$ and $s(x)$,$p(x)$ one can obtain desired expansion of the given said function and the important thing to keep in mind is that this function $f(x)$ can be broken down into more simpler functions.

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