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Black Scholes Option Pricing Model – Brownian Motion Approach

Dr S Prabakaran

Associate Professor, Department of Accounting & Finance,
Faculty of Economics and Administrative Sciences,
Pontificia Universidad Javeriana Cali – Cali,
Colombia. Valle Del Cauca, South América
prabakaran@javerianacali.edu.co, Cell No: - +573044587735

Abstracts

Brownian motion has become one of the fundamental building blocks of modern quantitative finance. The mathematical theory of Brownian motion has been applied in contexts ranging far beyond the movement of particles in fluids. Until recently, stock market researchers have confronted the same problem. While they can chart the path of the market on a minute by minute basis it is very hard for them to observe who buys, who sells and how demand and supply affects price fluctuations. There exist many researchers about how the behavior of different investors makes the option price movement in a stock market. The purpose of this paper is to construct the Black Scholes option pricing model in the stock markets by using Brownian motion approach. The main ambition of this study is fourfold:

1) First we begin our approach to construction of Brownian motion from the simple symmetric random walk.
2) Next we introduce the Black – Scholes option pricing model with stock price movement by using of Geometric Brownian motion.
3) Then we extent this Brownian motion approach in the stock market and
4) Finally we construct the model for the generalization based on the deformation of the standard Brownian motion and Black Scholes pricing formula. And this paper will end with conclusion.

KEY WORDS: - Brownian motion, Option pricing model, Random walk, Stock Market

INTRODUCTION

Brownian motion is a simple continuous stochastic process that is widely used in physics and finance for modeling random behavior that evolves over time. Examples of such behavior are the random movements of a molecule of gas or fluctuations in an asset’s price. Brownian motion gets its name from the botanist Robert Brown [1] in (1828) who observed in 1827. While Brown was studying how particles of pollen suspended in water moved erratically on a microscopic scale. The motion was caused by water molecules randomly buffeting the particle of pollen and he observed minute particles in the pollen grains executing the jittery motion. After repeating the experiment with particles of dust, he was able to conclude that the motion was due to pollen being “alive” but the origin of the motion remained unexplained. The first one to give a theory of Brownian motion was Louis Bachelier in 1900 in his PhD thesis “The theory of speculation”. However, it was only in 1905 that Albert Einstein, using a probabilistic model, could sufficiently explain Brownian motion. He observed that if the kinetic energy of fluids was right, the molecules of water moved at random. Thus, a small particle would receive a random number of impacts of random strength and from random directions in any short period of time. This random bombardment by the molecules of the fluid would cause a sufficiently small particle to move exactly just how Brown described it [2].

However, stock markets, the foreign exchange markets, commodity markets and bond markets are all assumed to follow Brownian motion, where assets are changing continually over very small intervals of time and the position, namely the change of state on the assets, is being altered by random amounts. More importantly, the mathematical models used to describe Brownian motion are the fundamental tools on which all financial asset pricing and derivatives pricing models are based.

The purpose of this paper is to construct the Black Schools Option Pricing Model through the Brownian motion approach.

1. <u>CONSTRUCTION OF BROWNIAN MOTION</u>

Brownian motion has become one of the fundamental building blocks of modern quantitative finance. Indeed, the basic continuous – time model for financial assets prices assumes that the log – returns of a given financial assets follow a Brownian motion with drift. A convenient way to understand Brownian motion is as a limit of random walk with smaller steps taking places more and more often.

Throughout, we use the following notation for the real numbers, the non – negative real numbers, but integers, and the non – negative integers respectively:

$$\mathbb{R} \overset{def}{=} -\infty,\infty \qquad (1)$$

$$\mathbb{R}_{+} \overset{def}{=} 0,\infty \qquad (2)$$

$$Z \overset{def}{=} ...,-2,-1,0,1,2... \qquad (3)$$

$$\mathbb{N} \overset{def}{=} 0,1,2,... \tag{4}$$

Our particular importance in our study is the normal distribution, $N(\mu,\sigma^2)$, with mean $-\infty < \mu < \infty$ and the variance $0 < \sigma^2 < \infty$; the density and cdf are given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}, \tag{5}$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy, x \in \mathbb{R}, \tag{6}$$

The normal distribution is also called the Gaussion distribution after the famous German mathematician and physicist Carl Friedrich Gauss.

When $\mu = 0$ and $\sigma^2=1$ we obtain the standard (or unit) normal distribution, $N(0,1)$, and the density and cdf reduce to

$$\theta(x) \overset{def}{=} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \tag{7}$$

$$\Theta(x) \overset{def}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy. \tag{8}$$

As we shall see over and over again in our study of Brownian motion, one of its nice features is that many computations involving it are based on evaluating $\Theta(x)$, and hence are computationally elementary.

If $Z \sim N(0,1)$, then $X=\sigma Z+\mu$ has the $N(\mu,\sigma^2)$ distribution. Conversely, if $X \sim N(\mu,\sigma^2)$, then $Z=(X-\mu)/\sigma$ has the standard normal distribution.

If thus follow that if $X \sim N(\mu,\sigma^2)$, then

$F(x)=P(X \le x)=\Theta((x-\mu)/\sigma)$.

Letting $X \sim N(\mu,\sigma^2)$, the moments generating function of the normal distribution is given by $M(s)=E(e^{sX})=\int_{-\infty}^{\infty} e^{sx} f(x)dx=e^{s\mu+s^2\sigma^2/2}, \quad -\infty < s < \infty.$ (9)

For deriving this above equation (9), first we compute $M(s)$ when $X=Z$ the unit is normal. Denoting this by $M_z(s)$ apply the lower and upper limit and we will compute $M_z(s)=E(e^{sZ})=e^{s^2/2}$

Normally generally if $X \sim N(\mu,\sigma^2)$, then it can be expressed as $X=\sigma Z+\mu$, and thus

$E(e^{sX})=e^{s\mu}E(e^{\sigma sZ})=e^{s\mu}Mz(\sigma s)=e^{s\mu}e^{(\sigma s)^2/2}=e^{s\mu+s^2\sigma^2/2}$, and we proved equation no 9.

Now the central limit theorem (CLT) says, if $\{X_i : i \geq 1\}$ are iid with finite mean $E(X) = \mu$ and finite non – zero variance $\sigma^2 = Var(X)$, then

$$Z_n \stackrel{def}{=} \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right) \Rightarrow N(0,1), n \to \infty, \text{ in distribution;}$$

$\lim_{n\to\infty} P(Z_n \leq x) = \Theta(x), x \in \mathbb{R}$.

If $\mu = 0$ and $\sigma^2 = 1$, then the CLT becomes

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \Rightarrow N(0,1).$$

Moreover, for any constant $c \neq 0$, since $cN(0,1) \sim N(0, c^2)$, we obtain:

If $\mu = 0$ and $\sigma^2 = 1$, then the CLT becomes for any constant $c \neq 0$

$$c\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \Rightarrow N(0, c^2).$$

Now, we are constructing the Brownian motion from the simple symmetric random walk. Recall the simple symmetric random walk, $R_0 = 0$,

$$R_n = \Delta_1 + ... + \Delta_n = \sum_{i=1}^{n} \Delta_i, n \geq 1,$$

Where the Δ_i are iid with $P(\Delta = -1) = P(\Delta = -1) = 0.5$. thus $E(\Delta) = 0$ and $Var(\Delta) = E(\Delta^2) = 1$.

We view time n in minutes, and R_n as the position at time n of a particles, moving in $\mathbb{R}$, which every minute takes a step, of size 1, equally likely to be forwards or backwards. Because $E(\Delta) = 0$ and $Var(\Delta) = 1$, it follow that $E(R_n) = 0$ and $Var(R_n) = n, n \geq 0.$

Choosing a large integer $k > 1$, if we instead make the particle take a step every $1/k$ minutes and make the step size $1/\sqrt{k}$, then by time t the particle will have taken a large number, $n = tk$, of steps and its position will be

$$B_t(t) = \frac{1}{\sqrt{k}}\sum_{i=1}^{tk} \Delta_i.$$

This leads to the particle taking many more iid steps, but each of small magnitude, in any given interval of time. We expect that as $k \to \infty$, these small steps become a continuous and the process $\{B_k(t) : t \geq 0\}$ should converge to a process $\{B(t) : t \geq 0\}$ with continuous sample paths. We call this process Brownian motion after the Scottish botanist Robert Brown. Its properties will be derived next.

Notice that for fixed k, any increment

$$B_t(t) - B_k(s) = \frac{1}{\sqrt{k}}\sum_{i=sk}^{tk} \Delta_i, 0 \leq s < t,$$

has a distribution that only depends on the length, $t-s$, of the time interval $(s,t]$ because it only depends on the number, $k(t-s)$, of iid Δ_i making up its construction. Thus we deduce that the limiting process $as\, k \to \infty$ will posses stationary increments. The distribution of any increments $B(t)-B(s)$ has a distribution that only depends on the length of the time interval $t-s$.

Notice further that given two overlapping time intervals, $(t_1,t_2]$ and $(t_3,t_4]$, $0 \le t_1 < t_2 < t_3 < t_4$, the corresponding increments

$$B_k(t_4) - B_k(t_3) = \frac{1}{\sqrt{k}} \sum_{i=t_3k}^{t_4k} \Delta_i, \tag{11}$$

$$B_k(t_2) - B_k(t_1) = \frac{1}{\sqrt{k}} \sum_{i=t_1k}^{t_2k} \Delta_i, \tag{12}$$

are independents because they are constructed from different Δ_i. Thus we deduce that the limiting process $as\, k \to \infty$ will also posses independent increments. For any non – overlapping time intervals, $(t_1,t_2]$ and $(t_3,t_4]$, the rvs $X_1 = B(t_2) - B(t_1)$ and $X_2 = B(t_4) - B(t_3)$ are independent.

Observing that $E(B_t(t)) = 0$ and $Var(B_k(t)) = tk/k \to t, k \to \infty$ we infer that the limiting process will satisfy $E(B(t)) = 0, Var(B(t)) = t$ just like the random walk $\{R_n\}$ does in discrete – time n $E(R_n) = 0, Var(R_n) = n$.

Finally, a direct application of the CLT yield (via setting $n = tk$, $\mu = 0$, $\sigma^2 = 1$, $c = \sqrt{t}$)

$$B_t(t) = \sqrt{t}\left(\frac{1}{\sqrt{kt}} \sum_{i=1}^{tk} \Delta_i\right) \Rightarrow N(0,t), k \to \infty \text{ in distribution.}$$

And we conclude that for each fixed $t > 0$, $B(t)$ has a normal distribution with mean 0 and variance *t*. similarly, using the stationary and independent increments property, we conclude that $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t-s$, and more generally.

The limiting BM process is a process with continuous sample paths that has both stationary and independent normally distributed (Gaussian) increments: if $t_0 = 0 < t_1 < t_2 < ... < t_n$, *then the rvs.* $B(t_{i-1}), i \in \{1,2,...n\}$, are independent with

$B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$.

If we define $X(t) = \sigma B(t) + \mu t$, then $X(t) \sim N(\mu, \sigma^2 t), \sigma \in \mathbb{R}_+, \mu \in \mathbb{R}$, and we obtain, by such scaling and translation, more generally, a process with stationary and independent increments in which $X(t) - X(s)$ has a normal distribution with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

When $\sigma^2 = 1$ and $\mu = 0$ (as in our construction) the process is called Standard Brownian Motion, and denoted by $B(t) : t \geq 0$. Otherwise, it is called Brownian motion with variance term σ^2 and *drift* μ.

2. BLACK SCHOOLS OPTION PRICING MODEL

Here we assume that the Black – Scholes option pricing model is instantaneous stock price movement and it can differentiate by

$$dS = \mu S dt + \sigma S dz \qquad (13)$$

Here, S is the stock price, μ and σ are constants, t is the time, and z follows a stochastic process called Wiener process, under which $dz = \varepsilon\sqrt{dt}$ where ε is a random draw from the standardized normal distribution. Equation (13) known as geometric Brownian motion, with μ and σ called the drift parameter and the volatility parameters, respectively.

The (1) implies $dInS = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$ (14)

The stochastic process as characterized by equation (14) indicates that InS is normal distributed.

Equivalently, S is lognormally distributed. With S_0 and S_T denoted as the stock prices at time 0 and time T, respectively, equation (14) leads to

$$S_T = S_0 exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\varepsilon\sqrt{T}\right]. \qquad (15)$$

Further, the expected value and the variance of S_T are

$$E(S_T) = S_0 \exp(\mu T) \qquad (16)$$

and $Var(S_T) = S_0^2\left[\exp(2\mu T)\right]\left[\exp(\sigma^2 T) - 1\right] = \left[E(S_T)\right]^2\left[\exp(\sigma^2 T) - 1\right]$, (17)

respectively. Equation (17) can be written as

$$\sqrt{Var(S_T)} = E(S_T)\sqrt{\exp(\sigma^2 T) - 1} \qquad (18)$$

for graphical convenience.

Given equation (16), we have

$$\frac{\partial E(S_T)}{\partial \mu} = TE(S_T) \quad \text{and} \qquad (19)$$

$$\frac{\partial E(S_T)}{\partial T} = \mu E(S_T) \qquad (20)$$

Accordingly, $E(S_T)$ increase with μ for any given $T > 0$. It increase with T if $\mu > 0$, decreases instead if $\mu < 0$, and remains unchanged if $\mu = 0$.

Likewise, given equation (17), we have

$$\frac{\partial Var\left(S_T\right)}{\partial \mu}=2TVar\left(S_T\right), \tag{21}$$

$$\frac{\partial Var\left(S_T\right)}{\partial \sigma}=2\sigma T\left[E\left(S_T\right)\right]^2\exp\left(\sigma^2 T\right), \tag{22}$$

and $$\frac{\partial Var\left(S_T\right)}{\partial T}=S_0^2\left\{\left(2\mu+\sigma^2\right)\exp\left[\left(2\mu+\sigma^2\right)T\right]-2\mu\exp\left(2\mu T\right)\right\}. \tag{23}$$

Thus, $Var\left(S_T\right)$ increases with μ and σ for any given $T>0$. As $\exp\left[\left(2\mu+\sigma^2\right)T\right]>\exp\left(2\mu T\right)>0$ for $T>0$, $Var\left(S_T\right)$ increase with T if $2\mu+\sigma^2\geq 0$. However, if μ is negative and low enough to make $2\mu+\sigma^2$ negative, the graph of $Var\left(S_T\right)$ versus T will not always be upward sloping; if $2\mu+\sigma^2<0$, the graph will be download sloping once T exceeds a threshold value. Specifically, with the threshold value of T being

$$T^*=\frac{1}{\sigma^2}\ln\left(\frac{2\mu}{2\mu+\sigma^2}\right), \tag{24}$$

$T \lesseqgtr T^*$ corresponds to $\partial Var\left(S_T\right)/\partial T \lesseqgtr 0$.

In addition, the Black Scholes formulas for the prices at time 0 of a European call and put option on a non dividend paying stock are

$$c=S_0N\left(d_1\right)-Ke^{-rt}N\left(d_2\right) \tag{25}$$

and $$p=Ke^{-rt}N\left(-d_2\right)-S_0N\left(-d_1\right) \tag{26}$$

where $$d_1=\frac{In\left(S_0/K\right)+\left(r+\sigma^2/2\right)T}{\sigma\sqrt{T}} \text{ and } d_2=d_1-\sigma\sqrt{T} \tag{27}$$

the function $N\left(x\right)$ is the cumulative probability distribution function for a standard normal distribution. In other, words, it is probability that a variables with a standard normal distribution, $\phi\left(0,1\right)$, will be less than *x*. The variables *c* and *p* are the European call and put price, S_0 is the stock price at time zero, *K* is the strike price, *r* is the continuously compound risk – free rate, σ is the price volatility, and *T* is the time to maturity of the option.

Next, we considered a portfolio consisting of a long position in a European call option and a short position in a European put option and assumes that both contracts have the same underlying assets, time to maturity *T* and strike price *K*. The terminal payoff of this portfolio is

$$\max\left(S_T-K,0\right)-\max\left(K-S_T,0\right)=\max\left(S_T-K,0\right)+\min\left(S_T-K,0\right)=S_T-K$$

We observe that the payoff of the portfolio is equal to that of a long forward contract with a strike price of *K*. Furthermore, forward contracts that the current value V_0 of a long position in an existing forward contract with strike price *K* is given by

$$V_0=\left(F_0-K\right)e^{-rT}, \tag{28}$$

Where F_0 is the fair delivery price of a forward contract with the same maturity as the two options. Remember that F_0 is being determined such that the initial value of the contract is zero. We can now argue that by the law of one price, two portfolios that have the same value at a future in time need to have the same current value.

Thus $C_0 - P_0 = \ F_0 - K \ \ e^{-rT}$ or equivalently

$$C_0 + Ke^{-rT} = P_0 F_0 e^{-rT} \tag{29}$$

This result is known as the put – call parity and provides a model – independent relation between the current price of a European call and put option with the sane strike price. In particular, given the price of one of the two options, we can solve for the no – arbitrage price of the other

$$\begin{aligned} C_0 &= P_0 + F_0 e^{-rT} - Ke^{-rT} \\ P_0 &= C_0 - F_0 e^{-rT} + Ke^{-rT} \end{aligned} \tag{30}$$

Note that we did not make any assumptions so far regarding the type of the underlying security. This general form of the put – call parity formula always applies and is the only one that we have to remember. In order to obtain the put – call parity relationship for the diffident underlying types, we just need to substitute the respective forward prices. We get

UNDERLYING TYPE	**FORWARD**	**PUT –CALL PARITY**
No Holding returns	$S_0 e^{rT}$	$C_0 + Ke^{-rT} = P_0 + S_0$
Fixed Discrete Holding Returns	$S_0 - De^{-rtD} \ \ e^{rT}$	$C_0 + Ke^{-rT} = P_0 + S_0 - De^{-rtD}$
Proportional Discrete holding returns	$1-q \ \ S_0 e^{rT}$	$C_0 + Ke^{-rT} = P_0 + \ 1-q \ \ S_0$
Continuous Proportional Holding returns	$S_0 e^{\ r-q \ T}$	$C_0 + Ke^{-rT} = P_0 + S_0 e^{-qT}$

We now consider an interesting special case of the put – call parity formula. As mentioned earlier, the fair delivery price F_0 of a forward contract is determined such that the initial value of the contract is zero. If the strike price K of the two option is equal to F_0, then it follows from the put – call parity that $C_0 = P_0$ and the two option prices must be agreed.

3. BROWNIAN MOTION APPROACH – STOCK MARKET

The mathematical theory of Brownian motion has been applied in contexts ranging far beyond the movement of particles in fluids. Until recently, stock market researchers have confronted the same problem. While they can chart the path of the market on a minute by minute basis it is very hard for them to observe who buys, who sells and how demand and supply affects price movements. There exist many interesting theories about how the behavior of different investors makes the prices move, but there is no empirical evidence to support the critical link between the investor decisions and the price dynamics [3].

However, stock markets, the foreign exchange markets, commodity markets and bond markets are all assumed to follow Brownian motion, where assets are changing continually over very small intervals of time and the position, namely the change of state on the assets, is being altered by random amounts. More importantly, the mathematical models used to describe Brownian motion are the fundamental tools on which all financial asset pricing and derivatives pricing models are based.

These models are of key importance to the work that is being done here on market models and risk analysis.

We consider a non dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{(\mu t+\sigma W_t^P)}$. The logarithm of the stock price $Y_t = In\, S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dW_t^P \tag{31}$$

where μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price and W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E^P(dW_t dW_{t'}) = dtdt'\delta(t-t')$ and on some filtered probability space $(\Omega, (F_t), P)$.

Application of Ito's formula yields the following SDE for the stock price process

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t^P \tag{32}$$

And the above equation has analytically solution

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma dW_t}$$

for an arbitrary initial value S_0. This model will use in the option pricing.

In finance, an option is a contract whereby the contract buyer has a right to exercise a feature of the contract (the option) on or before a future date (the exercise date). The 'writer' (seller) has the obligation to honor the specified feature of the contract. Since the option gives the buyer a right and the seller an obligation, the buyer has received something of value. The amount the buyer pays the seller for the option is called the option premium.

Having rights without obligations has financial value, so option holders must purchase these rights, making them assets. These assets derive their value from some other asset, so they are called derivative assets. Modern option pricing techniques, with roots in stochastic calculus, are often considered among the most mathematically complex of all applied areas of finance [4].

For an exercise price K, and an exercise date *T*, one has the rights to buy stocks with price K and sell it with S_T in the markets if $S_T > K$. If not, one has no obligation to buy. This option called an European call option and we define claim *C* (payoff at time *T*) by

$$C = (S_T - K)^+ = \max(S_T - K, 0) \tag{33}$$

So, if $S_T > K$ then the owner of the option will obtain the payoff C at time T while if $S_T \leq K$ then the owner will not exercise his option and the payoff is 0 [5].

The Geometric Brownian motion is the basis of Black Scholes model and the key assumptions of this model are

1. The stock pays no dividends during the option's life

Most companies pay dividends to their share holders, so this might seem a serious limitation to the model considering the observation that higher dividend yields elicit lower call premiums. A common way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price.

2. European exercise terms are used

European exercise terms dictate that the option can only be exercised on the expiration date. American exercise term allow the option to be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. This limitation is not a major concern because very few calls are ever exercised before the last few days of their life. This is true because when you exercise a call early, you forfeit the remaining time value on the call and collect the intrinsic value. Towards the end of the life of a call, the remaining time value is very small, but the intrinsic value is the same.

3. Markets are efficient

This assumption suggests that people cannot consistently predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process. To understand what a continuous Itô process is, you must first know that a Markov process is "one where the observation in time period t depends only on the preceding observation." An Itô process is simply a Markov process in continuous time. If you were to draw a continuous process you would do so without picking the pen up from the piece of paper.

4. No commissions are charged

Usually market participants do have to pay a commission to buy or sell options. Even floor traders pay some kind of fee, but it is usually very small. The fees that Individual investor's pay is more substantial and can often distort the output of the model.

5. Interest rates remain constant and known

The Black and Scholes model uses the risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate, but the discount rate on U.S. Government Treasury Bills with 30 days left until maturity is usually used to represent it. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

6. Returns are lognormal distributed

This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

According to the Black Scholes Model the current fair price of the claim is

$C = S_0\Phi(T_+) - Ke^{-rT}\Phi(T_-)$

Where $T_+ = \dfrac{\left[\log\left(\dfrac{S_0}{K}\right)+\left(r+\dfrac{1}{2}\sigma^2 T\right)\right]}{\left[\sigma\sqrt{T}\right]}$ (34)

$$T_- = \frac{\left[\log\left(\frac{S_0}{K}\right)+\left(r-\frac{1}{2}\sigma^2 T\right)\right]}{\left[\sigma\sqrt{T}\right]} \quad (35)$$

$\Phi(x)$ is a cumulative standard normal distribution $\mathbf{P}(Z<x)$ for

$$Z \sim N(0,1) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}du} \quad (36)$$

C is the Call premium (amount per share that an option buyer pays to the seller), T is time until expiration, K is the Option exercise price, and r is the Risk free interest rate. From the above equation $S_0\Phi(T_+)$ derives the expected benefits from acquiring the stock outright. This is found by multiplying the initial price by the change in the call premium with respect to a change in the underlying stock price [4] . And $Ke^{-rT}\Phi(T_-)$ gives the present value of paying the exercise price on the expiration day. The fair market value of the call option is then calculated by taking the difference between the two parts.

4. CONSTRUCTION OF OPTION PRICING MODEL

The aforesaid analysis of the Black Scholes formula for option pricing presupposes that the stock price follows the lognormal distribution. However, significant empirical evidence now subsists of the stock returns deviating from the lognormal distribution with “fat tails” and a “sharp peak” which better fit the truncated Levy flights or other power law distributions [6 - 8]. To broad base the Black Scholes model, generalizations by way of “Levy noise” and “jump diffusions” [9] have already been studied. In this paper, we propose a model that incorporates a “weighted Brownian motion” as the stochastic (noise) term, where the weights themselves are a function of the “Brownian motion / noise” i.e.

$$dW_t^P \rightarrow dU_t^P = f\left(U_t^P, t\right)dW_t^P \quad (37)$$

W_t^P is a regular Brownian motion representing Gaussian white noise with zero mean and δ correlation in time i.e. $E^P\left(dW_t dW_{t'}\right) = dtdt'\delta\left(t-t'\right)$ and on some filtered probability space $\left(\Omega, \left(F_t\right), P\right)$. We, further, mandate that the function that the

$f\left(U_t^P,t\right)$ satisfy the Novikov condition and that the process $U_t^P = \int_0^t f\left(U_t^P,s\right) dW_s^P$ is a local P- martingale with a non normal distribution. This requirement is not as restrictive as it may seem at first sight in context of the applications envisaged. We will address this issue again in the sequel.

This generalization contemplates a statistical feedback process. In this context, several studies on stock market data have shown the existence of nonlinear characteristics and chaotic behavior that lend credence to the existence of a statistical feedback mechanism of market players. Explanations for the existence of "fat tails" in stock market data have been offered on this statistical feedback process e.g. "extremal events" cause "disproportionate reactions" among market players. This deformed noise may also capture the "herd behavior" of stock market investors. The model also captures time dependent return processes since f is a function of U_t^P and t so that the drift term varies with time.

We define the European call option. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max\left(S_T - E, 0\right) = \left(S_T - E\right)^+$ where S_T is the stock price on the exercise date and E is the exercise price.

We consider a non dividend paying stock, the price process of which follows the geometric Brownian motion with drift $S_t = e^{\left(\mu t + \sigma W_t^P\right)}$, where μ and σ are constants representing the long term drift and the noisiness (diffusion) respectively in the stock price. The logarithm of the stock price $Y_t = In\, S_t$ follows the stochastic differential equation

$$dY_t = \mu dt + \sigma dU_t^P = \mu dt + \left[\sigma f\left(U_t^P,t\right)\right] dW_t^P \tag{38}$$

Application of Ito´s formula yields the following SDE for stock price process is then given by

$$dS_t = \left(\mu + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2\right) S_t dt + \left[\sigma f\left(U_t^P,t\right)\right] S_t dW_t^P \tag{39}$$

Let $C\left(S_t,t\right)$ denote the instantaneous price of a call option with exercise price E at any time t before maturity when the price per unit of the underlying is S_t. We assume that $C\left(S_t,t\right)$ does not depend on the past price history of the underlying. Applying the Ito formula to $C\left(S_t,t\right)$ yields and the deformed SDE for the contingent claim $C\left(S_t,t\right)$ gets modified to

$$dC_t = \left[\left(\mu + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2\right) S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\left[\sigma f\left(U_t^P,t\right)\right]^2 S_t^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \frac{\partial C}{\partial S}\left[\sigma f\left(U_t^P,t\right)\right] S_t dW_t^P, \tag{40}$$

Following the same procedure as in the standard case, we obtain the following PDE for the contingent claim that is subject to the deformed noise

$$rS_t \frac{\partial C(S_t,t)}{\partial S} + \frac{1}{2}\left[\sigma f(U_t^P,t)\right]^2 S_t^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} + \frac{\partial C(S_t,t)}{\partial t} = rC(S_t,t) \qquad (41)$$

Applying Girsanov's theorem to the price process (39), we perform a change of measure and define a probability measure Q such that the discounted stock price process $Z_t = S_t e^{-rt}$ or equivalently

$$dZ_t = \left(\mu - r + \frac{1}{2}\left[\sigma f(U_t^P,t)\right]^2\right) Z_t dt + \left[\sigma f(U_t^P,t)\right] Z_t dW_t^P \qquad (42)$$

behaves as a martingale with respect to Q. This is performed by eliminating the drift term through the transformation

$$\frac{\left(\mu - r + \frac{1}{2}\left[\sigma f(U_t^P,t)\right]^2\right)}{\sigma f(U_t^P,t)} \to \gamma_t \qquad (43)$$

whence $W_t^Q = W_t^P + \gamma_t t$ is a Brownian motion without drift with respect to the measure Q and $dZ_t = \left[\sigma f(U_t^Q,t)\right] Z_t dW_t^Q$ which is driftless under the measure Q and hence, Z_t is a Q martingale. The two measures P & Q are related through the Radon Nikodym derivative which in the deformed case takes the form

$$\xi(t) = \frac{dQ}{dP} = \exp\left(-\int_0^t \gamma_t dW_t^P - \frac{1}{2}\int_0^t \gamma_t^2 dt\right) \qquad (44)$$

And the expectation operators under the two measures are related as

$$E^Q\left[X_t | F_s\right] = \xi^{-1}(s)\, E^P\left[\xi(t) X_t | F_s\right]$$

Our next step in martingale based pricing is to constitute a Q martingale process that hits the discounted values of the contingent claim i.e. call option. This is formed by taking the condition expected of the discounted terminal payoff from the claim under the Q measure i.e. $E_t = E^Q\left[e^{-rT}(S_T - E)^+ | F_t\right]$ (45)

We now constitute a self – financing strategy that exactly replicates the claim and whose value is know with certainty. For this purpose, we introduce a 'bond' in our model that evolves according to the following price process

$$\frac{dB_t}{B_t} = rdt, B_T = 1, \qquad (46)$$

where r is the relevant risk free interest rate.

Making use of ϕ_t units of the underlying asset and ψ_t units of the bond, where $\phi_t = \frac{\partial C(S_t,t)}{\partial S}$, $B_t\psi_t = C(S_t,t) - \phi_t S_t$, we can now construct a trading strategy that has the following properties

(a) it exactly replicates the price process of the call option i.e.

$$\phi_t S_t + \psi_t B_t = C(S_t,t), \forall t \in [0,T] \qquad (47)$$

(b) it is self financing i.e. $\phi_t dS_t + \psi_t dB_t = dV_t, \forall t \in [0,T]$ (48)

Using eqs. (37), (39), (47) & (48) we have

$$dC = \left(\phi_t \mu S_t + \frac{1}{2}\phi_t \sigma^2 S_t + \psi_t r B_t\right)dt + \phi_t \sigma S_t dW_t^P. \tag{49}$$

Matching the diffusion terms of (39) & (44) and using (42), we get the aforesaid expressions for ϕ_t and ψ_t respectively. The value of this portfolio at any time t be shown to be equal to $V_t = e^{rt}E_t$ with E_t being given (45). It follows that the value of the replicating portfolio and hence of the call option at time t is given by

$$V_t = e^{rt}E_t = e^{-r(T-t)}E^Q\left[(S_T - E)^+ \middle| F_t\right] = e^{-r(T-t)}E^Q\left[(S_T - E)1_{S_T \geq E} \middle| F_t\right] \tag{50}$$

$$= e^{-r(T-t)} \int_{U_T^Q : S(U_T^Q, T) \geq E} \left(S(U_T^Q, T) - E\right) \bar{f}\left(U_T^Q, T \middle| U_T^Q, t\right) dU_T^Q \tag{51}$$

The expectation value of the contingent claim $\max(S_T - E, 0) = (S_T - E)^+$ under the measure Q depends only on the martingale distribution of the stock price process S_t under the measure Q which is obtained by writing it in terms of Q Brownian motion W_t^Q. We have from eq (38) for the deformed stock price process under the measure Q as

$$d\left(\ln S_t\right) = \mu dt + \sigma f\left(U_t^P, t\right) dW_t^P = \left(r - \frac{1}{2}\left[\sigma f\left(U_t^Q, t\right)\right]^2\right)dt + \sigma f\left(U_t^Q, t\right) dW_t^Q \tag{52}$$

which on integration yields

$$S_t = S_0 \exp\left[\int_0^t \sigma f\left(U_t^Q, t\right) dW_t^Q + \int_0^t \left(r - \frac{1}{2}\left[\sigma f\left(U_t^Q, t\right)\right]^2\right)ds\right]. \tag{53}$$

The value of the call option can now be computed by using eq. (51).

CONCLUSION

From the above discussion, we described the Black scholes option pricing model through Brownian motion approach and we stated in the introduction Brownian motion is a simple continuous stochastic process that is widely used in physics and finance for modeling random behavior that evolves over time and it´s extend this approach in the constructed the black schools option pricing model in the stock markets by using Brownian motion approach. And we concluded that, initially we begin our approach to construction of Brownian motion from the simple symmetric random walk and we introduced the black – scholes option pricing model with stock price movement by using of geometric Brownian motion, then we extended this Brownian motion approach in the stock market and finally we construct the model for the generalization based on the deformation of the standard Brownian motion and black scholes pricing formula.

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