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On An Contact Angle Problem of Rotating Fluid With Free Surface

Hirotada Honda

Abstract

In this paper, we consider the fluid on the rotating plate in 3 dimensional space, which is applicable to thin film coating process, and so on. By considering the contact line of the fluid and the plate, we model the system as a contact angle problem in 3 dimensional space. We model the system mathematically using Stokes equation, and verify the solvability near the edge in weighted Sobolev spaces.

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Introduction

Mathematical modeling of thin film coating process is one of the most important features in the fluid dynamics. It will contribute to thin film coating control process. In the coating process of thin film, the free surface of the fluid and the contact angle problem should be considered. There are several studies of coating processes in the literature, but they did not deal with them at the same time. In this paper, we model the system as a 3 dimensional stationary contact angle problem with free surface. We assume that the diameter of the fluid is small and constant. In the following, we introduce the mathematical formulation on the problem in the next section, consider the linear problem in section 3, nonlinear problem in section4, and conclude this paper in the final section.

Mathematical Formulation

Consider the system depicted in Fig.1.

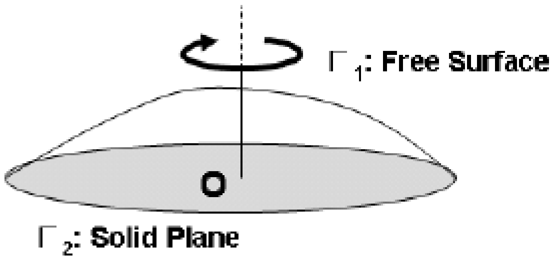

Figure 1: Fluid around Contact Angle

For simplicity, we assume that the contact line of the fluid is a ball on the plane, and the distance between the contact line of the fluid and the rotating center is a positive constant d. To model the system, we introduce the coordinate system shown in Fig.2. In this system, a point in the region is denoted by $(d - r\cos\Psi, \phi, \Psi)$, where $0 < r < d, 0 < \phi < 2\pi, 0 < \Psi < \frac{\pi}{2}$.

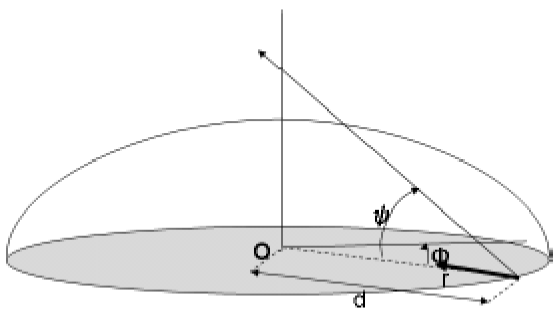

Figure 2: Coordinatge System

Linear problem for stationary problem for the fluid shown in Fig.1 can be described using Stokes equation:

$$\nu \Delta \boldsymbol{v} + \nabla p = \boldsymbol{f}_1, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad x \in \Omega, \tag{1.2}$$

$$\boldsymbol{T}(\boldsymbol{v})\boldsymbol{n} - \sigma H \boldsymbol{n} - p_e \boldsymbol{n} = \boldsymbol{f}_3 \quad x \in \Gamma_1, \tag{1.3}$$

$$\boldsymbol{v} = \boldsymbol{f}_4 \quad x \in \Gamma_2, \tag{1.4}$$

By Ω we denote the region satisfied by the fluid, Γ_1 the free surface of the fluid, Γ_2 the bottom boundary. $\boldsymbol{f}_1$, $\boldsymbol{f}_3$, $\boldsymbol{f}_4$ stand for external forces which are known functions. $\nu > 0$ is viscosity constant, $\boldsymbol{v}$ velocity of the fluid, $\boldsymbol{T}(\boldsymbol{v})$ the stress tensor, σ the surface tension, H twice the mean curvature of the surface, p_e the pressure on the free surface, $\boldsymbol{n}$ is the outer unit normal to the free suraface. For convenience, we assume that $supp(\boldsymbol{v}) \in \{|(r, \phi, \Psi \in \Omega_\theta | 0 < r < \epsilon\}$ for a positive constant ϵ. Operators Δ and ∇ in the coordinate system introduced above is denoted as follows:

$$\triangle f = \frac{1}{(d-rcos\Psi)r}\left\{\frac{\partial}{\partial r}\left(r(d-rcos\Psi)\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial \phi}\left(\frac{\partial r}{\partial d-rcos\Psi}\frac{\partial f}{\partial \phi}\right)+\frac{\partial}{\partial \Psi}\left(\frac{d-rcos\Psi}{r}\frac{\partial f}{\partial \Psi}\right)\right\}, \quad (1.5)$$

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial r} \\ \frac{1}{d-rcos\Psi}\frac{\partial g}{\partial \phi} \\ \frac{1}{r}\frac{\partial g}{\partial \Psi} \end{bmatrix}. \quad (1.6)$$

Next, we can see that the line component of the coordinate system is denoted as $ds^2 = h_r^2 dr^2 + h_\phi^2 d\phi^2 + h_\Psi d\Psi^2$, and we use the following notations:

$$h_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = 1,$$

$$h_\phi^2 = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2 = (d-rcos\Psi)^2,$$

$$h_\Psi^2 = \left(\frac{\partial x}{\partial \Psi}\right)^2 + \left(\frac{\partial y}{\partial \Psi}\right)^2 + \left(\frac{\partial z}{\partial \Psi}\right)^2 = r^2, \quad (1.7)$$

From relations above, we get

$$g_{rr} = 1, \quad g_{\phi\phi} = (d-rcos\phi)^2, \quad g_{\Psi\Psi} = r^2, \quad (1.8)$$

$$g^{rr} = 1, \quad g^{\phi\phi} = \frac{1}{(d-rcos\phi)^2}, \quad g^{\Psi\Psi} = \frac{1}{r^2}. \quad (1.9)$$

Now we introduce new coordinates (b_r, b_ϕ, b_Ψ) defined by $b_r = \sqrt{g^{rr}}a_r$, $b_\phi = \sqrt{g^{\phi\phi}}a_\phi$, $b_\Psi = \sqrt{g^{\Psi\Psi}}a_\Psi$, and using $\sigma_{ij}a^i \otimes a^j = \tilde{\sigma}_{ij}b^i \otimes b^j$, and stand for the (i,j) component of the stress tensor $\boldsymbol{T}(\boldsymbol{v})$ in the old and coordinate system by σ_{ij} and $\tilde{\sigma}_{ij}$, respectively. Then we get the relations

$$\sigma_{rr} = \tilde{\sigma}_{rr}, \quad \sigma_{r\phi} = (d-rcos\phi)\tilde{\sigma}_{r\phi}, \quad \sigma_{r\Psi} = r\tilde{\sigma}_{r\Psi}, \quad \sigma_{\phi\phi} = (d-rcos\phi)^2\tilde{\sigma}_{\phi\phi}, \quad (1.10)$$

$$\sigma_{\phi\Psi} = r(d-rcos\phi)\tilde{\sigma}_{\phi\Psi}, \quad \sigma_{\Psi\Psi} = r^2\tilde{\sigma}_{\Psi\Psi}. \quad (1.11)$$

On the other hand we can calculate Christoffel notations by

$$\Gamma_{kj}^i = \frac{1}{2}g^{li}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^l}\right):$$

$$\Gamma_{rr}^r = \Gamma_{r\phi}^r = \Gamma_{r\Psi}^r = 0, \quad \Gamma_{\phi r}^\phi = -\frac{cos\Psi}{d-rcos\Psi}, \quad \Gamma_{\phi\phi}^r = 0, \quad \Gamma_{\phi\Psi}^\phi = \frac{rsin\Psi}{d-rcos\Psi},$$

$$\Gamma_{\Psi r}^\Psi = \frac{1}{r}, \quad \Gamma_{\Psi\phi}^\Psi = 0, \quad \Gamma_{\Psi\Psi}^\Psi = 0,$$

$$\Gamma_{rr}^\phi = \Gamma_{rr}^\Psi = 0, \Gamma_{\phi\phi}^r = (d-rcos\Psi)cos\Psi, \quad \Gamma_{\phi\phi}^\Psi = -\frac{(d-rcos\Psi)sin\Psi}{r}, \quad \Gamma_{\Psi\Psi}^r = -r, \quad \Gamma_{\Psi\Psi}^\phi = 0.$$

Then we get

$$\sigma_{rr} = 2\nu\frac{\partial v_r}{\partial r}, \quad \sigma_{r\phi} = \nu\left[\frac{\partial v_r}{\partial \phi} + \frac{\partial}{\partial r}\left((d-rcos\Psi)v^\phi\right) + v^\phi cos\Psi\right],$$

$$\sigma_{r\Psi} = \nu\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r\frac{\partial v_\Psi}{\partial r}\right], \quad \sigma_{\phi\phi} = 2\nu(d - rcos\Psi)\frac{\partial v_\phi}{\partial \phi},$$

$$\sigma_{\phi\Psi} = \nu\left[(d - rcos\Psi)\frac{\partial v_\phi}{\partial \Psi} + r\frac{\partial v_\Psi}{\partial \phi}\right], \quad \sigma_{\Psi\Psi} = 2\nu r\frac{\partial v_\Psi}{\partial \Psi},$$

and since the (i, j) component of the deformation tensor σ_{ij} is denoted as:

$$\sigma_{ij} = \nu\left[g_{ik}\left(\frac{\partial x^k}{\partial x^j} + \Gamma^k_{lj}\dot{x}^l\right) + g_{jk}\left(\frac{\partial x^k}{\partial x^i} + \Gamma^k_{li}\dot{x}^l\right)\right], \tag{1.12}$$

we get

$$\tilde{\sigma}_{rr} = 2\nu\frac{\partial v_r}{\partial r}, \quad \tilde{\sigma}_{r\phi} = \nu\left[\frac{1}{d - rcos\Psi}\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} + \frac{2cos\Psi}{d - rcos\Psi}v_\phi\right], \quad \tilde{\sigma}_{r\Psi} = \frac{\nu}{r}\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r\frac{\partial v_\Psi}{\partial r}\right],$$

$$\tilde{\sigma}_{\Psi\Psi} = \nu\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r\frac{\partial v_\Psi}{\partial r}\right], \quad \tilde{\sigma}_{\phi\phi} = \frac{2\nu}{(d - rcos\Psi)}\frac{\partial v_\phi}{\partial \phi},$$

$$\tilde{\sigma}_{\phi\Psi} = \nu\left[\frac{1}{r}\frac{\partial v_\phi}{\partial \Psi} + \frac{1}{d - rcos\Psi}\frac{\partial v_\Psi}{\partial \phi}\right], \quad \tilde{\sigma}_{\Psi\Psi} = \frac{2\nu}{r}\frac{\partial v_\Psi}{\partial \Psi}.$$

From the above considerations, original equations, continuity equations and boundary conditions are described as follows:

$$-\frac{\nu}{(d - rcos\Psi)r}\left\{\frac{\partial}{\partial r}\left(r(d - rcos\Psi)\frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{d - rcos\Psi}\frac{\partial v}{\partial \phi}\right) + \frac{\partial}{\partial \Psi}\left(\frac{d - rcos\Psi}{r}\frac{\partial v}{\partial \Psi}\right)\right\} + \nabla p = \boldsymbol{f}_1, \tag{1.13}$$

$$\frac{1}{(d - rcos\Psi)r}\left\{\frac{\partial}{\partial r}(r(d - rcos\Psi)v_r) + \frac{\partial}{\partial \phi}(rv_\phi) + \frac{\partial}{\partial \Psi}((d - rcos\Psi)v_\Psi)\right\} = f_2 \quad x \in \Omega, \tag{1.14}$$

$$\nu\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r\frac{\partial v_\Psi}{\partial r}\right] = f_{31}, \tag{1.15}$$

$$\nu\left[(d - rcos\Psi)\frac{\partial v_\phi}{\partial \Psi} + r\frac{\partial v_\Psi}{\partial \phi}\right] = f_{32}, \tag{1.16}$$

$$\nu\left[2r\frac{\partial v_\Psi}{\partial \Psi} - p_e\right] = f_{33}, \quad x \in \Gamma_1, \tag{1.17}$$

$$\boldsymbol{v} = \boldsymbol{f}_4 \quad x \in \Gamma_2, \tag{1.18}$$

where

$$\nabla p = \begin{bmatrix} \frac{\partial p}{\partial r} \\ \frac{1}{d - rcos\Psi}\frac{\partial p}{\partial \phi} \\ \frac{1}{r}\frac{\partial p}{\partial \Psi} \end{bmatrix}. \tag{1.19}$$

(1.13)-(1.19) are our mathematical model of the fluid on the plate in the thin film coating process, and is the original problem we deal with in this paper.

Mellin Transform And Function Spaces

In this section, we introduce some function spaces to solve the model problems. Before that, we introduce the Mellin transform with respect to r. From here after, we denote the Mellin transform of a function $f(r)$ by $\tilde{f}(\lambda)(r > 0)$ given as follows:

$$\tilde{f}(\lambda) = \int_0^\infty f(r) r^{\lambda-1} dr, \quad \lambda = \lambda_0 + i\lambda_1 \in \boldsymbol{C}. \tag{2.1}$$

The inverse transformation is given by

$$f(r) = \frac{1}{2\pi i} \int_{Re\lambda=\lambda_0} r^{-\lambda} \tilde{f}(\lambda) d\lambda. \tag{2.2}$$

We know that the following equalities holds[1]:

$$\frac{1}{2\pi} \int_{Re\lambda=\lambda_0} |\tilde{f}(\lambda)|^2 d\lambda = \int_0^\infty r^{2\lambda_0-1} |f(r)|^2 dr, \tag{2.3}$$

$$\widetilde{r^k f(r)} = \tilde{f}(\lambda + k), \tag{2.4}$$

$$\widetilde{r f'(r)} = -\lambda \tilde{f}(\lambda), \tag{2.5}$$

where $f'(r)$ means the derivative of f with r. Next we introduce function spaces.

Let Ω be a domain of the fluid depicted in Fig.1, and we introduce the coordinate system described in the previous section. We also denote by Γ the bottom boundary with $\Psi = \frac{\pi}{2}$. Then, for ardinary $m \in \boldsymbol{N}_+$ and $\nu > 0$, we define weighted Sobolev norms for the functions defined on them:

$$\| u \|^2_{H^m_\mu(\Omega)} = \sum_{0 \le l+\alpha+j \le m} \int_\Omega | D_r^l D_\phi^\alpha D_\Psi^j u(x)|^2 r^{2(\mu-(m-1)+l)} dr d\phi d\Psi,$$

$$\| u \|^2_{H_\mu^{m-\frac{1}{2}}(\Gamma)}$$
$$= \sum_{k=1}^{m} \sum_{l=0}^{m-k} \int_{R_+} \int_0^{2\pi} \int_0^{2\pi} \frac{|D_r^l D_\phi^{k-1} f(r,\phi) - D_r^l D_\phi^{k-1} f(r,\phi')|^2}{|\phi - \phi'|^2} r^{2(\mu-m+l+1)} d\phi d\phi' dr$$
$$+ \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \int_{R_+} \int_0^{2\pi} | D_r^l D_\phi^k f(r,\phi)|^2 r^{2(\mu-m+l+1)} d\phi dr$$
$$+ \sum_{k=0}^{m-1} \int_{R_+} \int_0^{2\pi} d\,\phi r^{2(\mu-k)+1} dr \int_0^r \frac{|D_r^{m-k-1} D_\phi^k f(r+\rho) - D_r^{m-k-1} D_\phi^k f(r)|^2}{\rho^2} d\rho.$$

For domains after Mellin transform, let $\lambda \in \boldsymbol{C}$, $\xi \in \boldsymbol{R}$, $0 < \Psi < \frac{\pi}{2}$, and we define norms of functions on (λ, ξ, Ψ),

$$| \| v | \|^2_{\mu,m} = \int_{\boldsymbol{R}} d\,\xi \int_0^{2\pi} d\Psi \int_{Re\lambda=\mu-m+\frac{3}{2}} \sum_{0\le j+\alpha\le m} (1+|\lambda|^2)^{m-j-\alpha} |\xi^\alpha|^2 |\frac{d^j u}{d\Psi^j}|^2 d\lambda,$$

$$| \| v | \|^2_{\mu,m-\frac{1}{2}} = \sum_{k=1}^{m} \int_{Re\lambda=\mu-m+\frac{3}{2}} (1+|\lambda|^2)^{m-k} |\xi^2|^{k-\frac{1}{2}} |v|^2 d\lambda d\xi$$

$$+ \sum_{k=0}^{m-1} \int_{Re\lambda=\mu-m+\frac{3}{2}} (1+|\lambda|^2)^{m-k-\frac{1}{2}} |\xi^2|^k |v|^2 d\lambda d\xi.$$

Similarly, for $\lambda \in \boldsymbol{C}$, $\xi_i \in \boldsymbol{R}$ $(i=1,2)$, we define norms of functions on (λ, ϕ, Ψ),

$$|| \| v || \|^2_{\mu,m} = \int_{\boldsymbol{R}^2} d\,\xi_1 d\xi_2 \int_0^{2\pi} d\Psi \int_{Re\lambda=\mu-m+\frac{3}{2}} \sum_{0\le j\le m} (1+|\lambda|^2)^{m-j-\alpha} |\xi_1^\alpha|^2 |\xi_2^j|^2 d\lambda.$$

Note that these norms have weights with the distance from the contact line of the fluid. We define function spaces for vector functions in the same manner.
For functions defined on Ω, consider its Mellin transform with respect to r and Fourier transform to ϕ, denoted by $\tilde{\tilde{u}}$. Then, $\| u \|_{H^m_\mu(\Omega)}$ is equivalent to $| \| v | \|_{\mu,m}$, $|| \| v || \|_{\mu,m}$, and $\| u \|_{H_\mu^{m-\frac{1}{2}}(\Gamma)}$ to $| \| v | \|_{\mu,m-\frac{1}{2}}$, respectively.

Proof. We can easily prove the lemma above using Parseval type equalities.
Actually, since $\int_{Re\lambda=\mu-k} |\tilde{h}(\lambda)|^2 (1+|\lambda|^2)^k d\lambda_2$ is equivalent to $\sum_{j=0}^k \int_0^\infty \left|\frac{\partial^j h}{\partial r^j}(r)\right|^2 r^{2(\mu-k+j)-1} dr$ [1],

$$\sum_{j+\alpha=0}^m \int_{Re\lambda=\mu-m+\frac{3}{2}} (1+|\lambda|^2)^{m-(j+\alpha)} \| D_\phi^\alpha D_\Psi^j \tilde{u} \|^2_{L_{2,\phi,\Psi}} d\lambda \qquad (2.6)$$

is equivalent to

$$\sum_{j+\alpha=0}^m \sum_{l=0}^{m-(j+\alpha)} \int_0^\infty \| D_r^l D_\phi^\alpha D_\Psi^j u \|^2_{L_{2,\phi,\Psi}} r^{2(\mu-(m-1)+l)} dr, \qquad (2.7)$$

where $\|\cdot\|_{L_{2,\phi,\Psi}}$ denotes the L_2 norm in the space $\{0<\phi<\frac{\pi}{2}, 0<\Psi<\pi\}$ with respect to ϕ and Ψ.
This completes the first assertion of the proof. The other assertions can be proved similarly. Now, the following is the main theorem of this paper: □

For arbitrary $m \ge 0$, $\boldsymbol{f}_1 \in H^m_\mu(\Omega)$, $\boldsymbol{f}_3 \in H_\mu^{m+\frac{3}{2}}(\Gamma_1)$, $\boldsymbol{f}_4 \in H_\mu^{m+\frac{3}{2}}(\Gamma_2)$, the problem (1.13)-(1.19) has a solution $\boldsymbol{v} \in H^{2+m}_\mu(\Omega)$ satisfying

$$\| \boldsymbol{v} \|_{H_\mu^{2+m}(\Omega)} \le C[\| \boldsymbol{f}_1 \|_{H^m_\mu(\Omega)} + \| \boldsymbol{R} \|_{H_\mu^{m+2}(\Omega)} + \| \boldsymbol{f}_3 \|_{H_\mu^{\frac{3}{2}+m}(\Gamma_1)} + \| \boldsymbol{f}_4 \|_{H_\mu^{\frac{3}{2}+m}(\Gamma_2)}].$$

Approximate Problem

In this section, we consider an approximate problem of (1.14)-(1.20). Replacing r in (1.14) by a constant r_0 that satisfies $0 < r_0 < r$, Expanding (1.14), and denoting the term $(d - r cos\Psi) r \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r}\right) - r cos\Psi r \frac{\partial v}{\partial r} + \frac{r^2}{d - r cos\Psi} \frac{\partial^2 v}{\partial \phi^2} + (d - r cos\Psi) \frac{\partial^2 v}{\partial \Psi^2}$ by $L[\boldsymbol{v}]$ and $(d - r_0 cos\Psi_0) r \frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r}\right) - r_0 cos\Psi_0 r \frac{\partial v}{\partial r} + \frac{r_0^2}{d - r_0 cos\Psi_0} \frac{\partial^2 v}{\partial \phi^2} + (d - r_0 cos\Psi_0) \frac{\partial^2 v}{\partial \Psi^2}$ by $L_0[\boldsymbol{v}]$,

and

$(d - 2r_0 cos\Psi_0) v_r + (d - r_0 cos\Psi_0) r \frac{\partial v_r}{\partial r} + r \frac{\partial v_\phi}{\partial \phi} + r sin\Psi v_\Psi + (d - r_0 cos\Psi_0) \frac{\partial v_\Psi}{\partial \Psi}$

by $L_{30}[\boldsymbol{v}]$,

$(d - 2r cos\Psi) v_r + (d - r cos\Psi) r \frac{\partial v_r}{\partial r} + r \frac{\partial v_\phi}{\partial \phi} + r sin\Psi v_\Psi + (d - r cos\Psi) \frac{\partial v_\Psi}{\partial \Psi}$ by $L_3[\boldsymbol{v}]$,

$$-\frac{\nu}{(d - r_0 cos\Psi_0)}\left\{(d - r_0 cos\Psi_0) r \frac{\partial}{\partial r}\left(r \frac{\partial \boldsymbol{v}}{\partial r}\right) - r_0 cos\Psi_0 r \frac{\partial \boldsymbol{v}}{\partial r} + \frac{r_0^2}{d - r_0 cos\Psi_0} \frac{\partial^2 \boldsymbol{v}}{\partial \phi^2} + (d - r_0 cos\Psi_0) \frac{\partial^2 \boldsymbol{v}}{\partial \Psi^2}\right\} + \begin{bmatrix} r^2 \frac{\partial p}{\partial r} \\ \frac{r_0^r}{d - r_0 cos\Psi_0} \frac{\partial p}{\partial \phi} \\ r \frac{\partial p}{\partial \Psi} \end{bmatrix}$$

$$v = r^2 \boldsymbol{f}_1 - \left[\frac{\nu}{(d - r_0 cos\Psi_0)} - \frac{\nu}{(d - r cos\Psi)}\right] L[\boldsymbol{v}] + \frac{\nu}{(d - r_0 cos\Psi_0)}[L[\boldsymbol{v}] - L_0[\boldsymbol{v}]] + \begin{bmatrix} r^2 \frac{\partial p}{\partial r} (\\ \left\{\frac{r r_0}{d - r_0 cos\Psi_0} - \frac{r^2}{d - r cos\Psi}\right\} \frac{\partial p}{\partial \phi} \\ r \frac{\partial p}{\partial \Psi} \end{bmatrix}, \tag{2.8}$$

$$((d - 2r_0 cos\Psi_0) v_r + (d - r_0 cos\Psi_0) r \frac{\partial v_r}{\partial r} + r_0 \frac{\partial v_\phi}{\partial \phi} + (d - r_0 cos\Psi_0) \frac{\partial v_\Psi}{\partial \Psi} = r f_2 - \frac{r_0 sin\Psi_0}{(d - r_0 cos\Psi_0)} v_\Psi + \left\{\frac{1}{d - r cos\Psi} - \frac{1}{d - r_0 cos\Psi_0}\right\} L_3[\boldsymbol{v}] + \frac{1}{(d - r_0 cos\Psi_0)}[L_{30}[\boldsymbol{v}] - L_3[\boldsymbol{v}]] \tag{2.9}$$

$$\nu\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r \frac{\partial v_\Psi}{\partial r}\right] = f_{31}, \tag{2.10}$$

$$\nu\left[(d - r_0 cos\Psi_0) \frac{\partial v_\phi}{\partial \Psi} + r_0 \frac{\partial v_\Psi}{\partial \phi}\right] = f_{32} + \nu\left[(a_0(r_0, \Psi_0) - a(r, \Psi)) + (r_0 - r) \frac{\partial v_\Psi}{\partial \phi}\right], \tag{2.11}$$

$$2\nu r_0 \frac{\partial v_\Psi}{\partial \Psi} = f_{33} + \nu\left[2(r_0 - r) \frac{\partial v_\Psi}{\partial \Psi} + p_e\right], \quad x \in \Gamma_1, \tag{2.12}$$

$$\boldsymbol{v} = \boldsymbol{f}_4 \quad x \in \Gamma_2, \tag{2.13}$$

where $c'_0 = d - 2r_0 cos\Psi_0$ and $c'_1 = d - r_0 cos\Psi_0$.

In (2.3), we replace the second term of LHS by $+r_0 cos\Psi_0 r\frac{\partial v}{\partial r}$ for a technical reason. So we consider

$$-\frac{\nu}{(d - r_0 cos\Psi_0)}\left\{(d - r_0 cos\Psi_0) r\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{v}}{\partial r}\right) + r_0 cos\Psi_0 r\frac{\partial \boldsymbol{v}}{\partial r} + \frac{r_0^2}{d - r_0 cos\Psi_0}\frac{\partial^2 \boldsymbol{v}}{\partial \phi^2} + (d - r_0 cos\Psi_0)\frac{\partial^2 \boldsymbol{v}}{\partial \Psi^2}\right\} + \begin{bmatrix} r^2\frac{\partial p}{\partial r} \\ \frac{r_0^r}{d - r_0 cos\Psi_0}\frac{\partial p}{\partial \phi} \\ r\frac{\partial p}{\partial \Psi}\end{bmatrix}$$

$$= -2\frac{\nu}{(d - r_0 cos\Psi_0)} r_0 cos\Psi_0 r\frac{\partial \boldsymbol{v}}{\partial r} + r^2\boldsymbol{f}_1 - \left[\frac{\nu}{(d - r_0 cos\Psi_0)} - \frac{\nu}{(d - r cos\Psi)}\right]L[\boldsymbol{v}] + \frac{\nu}{(d - r_0 cos\Psi_0)}[L[\boldsymbol{v}] - L_0[\boldsymbol{v}]] + \begin{bmatrix} r^2\frac{\partial p}{\partial r} \\ \left\{\frac{r r_0}{d - r_0 cos\Psi_0} - \frac{r^2}{d - r cos\Psi}\right\}\frac{\partial p}{\partial \phi} \\ r\frac{\partial p}{\partial \Psi}\end{bmatrix}, \tag{2.14}$$

Replacing RHS of (2.8) and (2.9) by $\boldsymbol{F}_1$ and F_2, (2.10)-(2.13) by $f'_{31}, f'_{32}, f'_{33}, f'_4$ respectively, we consider the nonhomogeneous problem:

$$-\frac{\nu}{(d - r_0 cos\Psi_0)}\left\{(d - r_0 cos\Psi_0) r\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{v}}{\partial r}\right) + r_0 cos\Psi_0 r\frac{\partial \boldsymbol{v}}{\partial r} + \frac{r_0^2}{d - r_0 cos\Psi_0}\frac{\partial^2 \boldsymbol{v}}{\partial \phi^2} + (d - r_0 cos\Psi_0)\frac{\partial^2 \boldsymbol{v}}{\partial \Psi^2}\right\} + \begin{bmatrix} r^2\frac{\partial p}{\partial r} \\ \frac{r_0^r}{d - r_0 cos\Psi_0}\frac{\partial p}{\partial \phi} \\ r\frac{\partial p}{\partial \Psi}\end{bmatrix}$$

$$= r^2\boldsymbol{F}_1, \tag{2.15}$$

$$(c'_0 - c'_1\lambda)v_r + r_0\frac{\partial v_\phi}{\partial \phi} + (d - r_0 cos\Psi_0)\frac{\partial v_\Psi}{\partial \Psi} = rF_2, \tag{2.16}$$

$$\nu\left[\left(\frac{\partial v_r}{\partial \Psi} - v_\Psi\right) + r\frac{\partial v_\Psi}{\partial r}\right] = f_{31}, \tag{2.17}$$

$$\nu\left[(d - r_0 cos\Psi_0)\frac{\partial v_\phi}{\partial \Psi} + r_0\frac{\partial v_\Psi}{\partial \phi}\right] = f_{32} + \nu\left[(a_0(r_0, \Psi_0) - a(r, \Psi)) + (r_0 - r)\frac{\partial v_\Psi}{\partial \phi}\right], \tag{2.18}$$

$$2\nu r_0 \frac{\partial v_\Psi}{\partial \Psi} = f_{33} + \nu\left[2(r_0 - r)\frac{\partial v_\Psi}{\partial \Psi} + p_e\right], \quad x \in \Gamma_1, \tag{2.19}$$

$$\boldsymbol{v} = \boldsymbol{f}_4 \quad x \in \Gamma_2, \tag{2.20}$$

Applying the Mellin transform to r and Fourier transform to ϕ, we get

$$-\frac{\nu}{(d - r_0 cos\Psi_0)}\left\{(d - r_0 cos\Psi_0)\lambda^2 - r_0 cos\Psi_0\lambda)\tilde{\tilde{\boldsymbol{v}}} + \frac{r_0^2}{d - r_0 cos\Psi_0}(i\xi)^2\tilde{\tilde{\boldsymbol{v}}} + (d - r_0 cos\Psi_0)\frac{\partial^2 \boldsymbol{v}}{\partial \Psi^2}\right\}$$

$$+\begin{bmatrix} -(\lambda+1)\tilde{\tilde{q}} \\ \frac{r_0}{d - r_0 cos\Psi_0}(i\xi)\tilde{\tilde{q}} \\ \frac{\partial \tilde{\tilde{q}}}{\partial \Psi} \end{bmatrix} = \widetilde{r^2 \boldsymbol{f}_1}, \tag{2.21}$$

$$(c'_0 - c'_1\lambda)\tilde{\tilde{v}}_r + (i\xi)r_0\widetilde{\tilde{v}_\phi} + (d - r_0 cos\Psi_0)\frac{\partial \widetilde{\tilde{v}_\Psi}}{\partial \Psi} = \widetilde{r f_2}, \tag{2.22}$$

$$\nu\left[\frac{\partial \widetilde{\tilde{v}_r}}{\partial \Psi} - (1+\lambda)\widetilde{\tilde{v}_\Psi}\right] = \tilde{f}'_{31}, \tag{2.23}$$

$$\nu\left[(d - r_0 cos\Psi_0)\frac{\partial \widetilde{\tilde{v}_\phi}}{\partial \Psi} + r_0\frac{\partial \widetilde{\tilde{v}_\Psi}}{\partial \phi}\right] = \tilde{f}'_{32}, \tag{2.24}$$

$$2\nu r_0 \frac{\partial \widetilde{\tilde{v}_\Psi}}{\partial \Psi} = \tilde{f}'_{33}, \tag{2.25}$$

$$\tilde{\boldsymbol{v}} = \tilde{f}'_4, \quad x \in \Gamma_2, \tag{2.26}$$

The following theorem guarantees the solovability of the approximate problem (2.15)-(2.20).

For arbitrary $m \geq 0$, $\boldsymbol{f}_1 \in H_\mu^m(\Omega)$, $\boldsymbol{f}_3 \in H_\mu^{m+\frac{3}{2}}(\Gamma_1)$, $\boldsymbol{f}_4 \in H_\mu^{m+\frac{3}{2}}(\Gamma_2)$, the approximate problem (2.15)-(2.20) has a unique solution $\boldsymbol{v} \in H_\mu^{2+m}(\Omega)$ satisfying

$$\| \boldsymbol{v} \|_{H_\mu^{2+m}(\Omega)} \leq C[\| \boldsymbol{f}_1 \|_{H_\mu^m(\Omega)} + \| \boldsymbol{R} \|_{H_\mu^{m+2}(\Omega)} + \| \boldsymbol{f}_3 \|_{H_\mu^{\frac{3}{2}+m}(\Gamma_1)} + \| \boldsymbol{f}_4 \|_{H_\mu^{\frac{3}{2}+m}(\Gamma_2)}].$$

Half Space Problem

We consider homogeneous problems of (2.15)-(2.20) in the half space $\theta \in \boldsymbol{R}_+$:
We seek a solution such that $\tilde{\tilde{\boldsymbol{v}}} \to 0$ with the form

$$\tilde{\tilde{\boldsymbol{v}}} = e^{-r_1\theta}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + e^{-r_2\theta}\begin{pmatrix} h'_1 \\ h'_2 \\ h'_3 \end{pmatrix}, \tag{2.27}$$

$$q = h_4 e^{-r_2}\theta. \tag{2.28}$$

From the equation (2.15), we get

$$\left[(d-r_0\cos\Psi_0)\lambda^2-(r_0\cos\Psi_0)\lambda-\frac{r_0^2\xi^2}{d-r_0\cos\Psi_0}+(d-r_0\cos\Psi_0)r_1^2\right]\boldsymbol{h}=0, \quad (2.29)$$

$$\left[(d-r_0\cos\Psi_0)\lambda^2-(r_0\cos\Psi_0)\lambda-\frac{r_0^2\xi^2}{d-r_0\cos\Psi_0}+(d-r_0\cos\Psi_0)r_2^2\right]\boldsymbol{h}'+\begin{bmatrix}-(\lambda+1)\\ \frac{r_0}{d-r_0\cos\Psi_0}(i\xi)\\ -r_2\end{bmatrix}h_4=0,$$

(2.30)

$$(c'_0-c'_1\lambda)h_1+i\xi r_0h_2-r_1h_3=0, \quad (2.31)$$

$$(c'_0-c'_1\lambda)h'_1+i\xi r_0h'_2-r_2h'_3=0, \quad (2.32)$$

where $\boldsymbol{h}=\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}, \boldsymbol{h}'=\begin{pmatrix}h'_1\\h'_2\\h'_3\end{pmatrix}$. and from which

$$r_1^2=\frac{r_0\xi^2}{(d-r_0\cos\Psi_0)^2}-\left\{\lambda^2-\frac{(r_0\cos\Psi_0)}{d-r_0\cos\Psi_0}\lambda\right\}, \quad (2.33)$$

$$r_2^2=\frac{r_0^2\xi^2}{(d-r_0\cos\Psi_0)^2}+(1-\lambda^2), \quad (2.34)$$

(2.35)

holds.

On the other hand, from the interface conditions,

$$(-r_1h_1-r_2h'_1)-(1+\lambda)(h_3+h'_3)=f'_{31}, \quad (2.36)$$

$$(d-r_0\cos\Psi_0)(-r_1h_2-r_2h'_2)+r_0(i\xi)(h_3+h'_3)=f'_{32}, \quad (2.37)$$

$$2r_0(-r_1h_3-r_2h'_3)=f'_{33}. \quad (2.38)$$

From (2.21)-(2.24), we get

$$h'_2=\frac{Q}{P}, \quad (2.39)$$

in which

$$P=r_2\left[-\frac{r_1}{c_0-c_1\lambda}\left\{(i\xi)\frac{r_0}{r_1}\left(2-\frac{r_2}{r_1}\right)+r_2\frac{d-r_0\cos\Psi_0}{r_0(i\xi)}\right\}+\frac{(\lambda+1)(d-r_0\cos\Psi_0)}{r_0(i\xi)}+(1+\lambda)\left(1-\frac{r_2}{r_1}\right)\frac{(d-r_0\cos\Psi_0)}{r_0(i\xi)}\right],$$

(2.40)

$$Q=f'_{31}+\frac{r_1}{c_0-c_1\lambda}\left[\frac{i\xi}{r_1(d-r_0\cos\Psi_0)}\left\{\frac{(i\xi)f'_{33}}{2r_1}+f'_{32}\right\}-\frac{f'_{33}}{2r_0}\right]-(1+\lambda)\frac{f'_{33}}{2r_0r_1}, \quad (2.41)$$

denoting $(\xi^2+(1+|\lambda|^2)^{\frac{1}{2}}$ by r, we can see that $|P|>c|r|$ with a positive constant c under the condition $2d(1+d)^2<1$ and $\lambda_0(\lambda_0-\frac{r_0\cos\Psi_0}{d-r_0\cos\Psi_0})>2(1+d)^2$.

Proof. Note that $Re(r_2^2)>0$ holds, and $arg(r_i)\in(-\frac{\pi}{4},\frac{\pi}{4})$ $\quad(i=1,2)$.

$$P=\frac{r_2}{i\xi c_0(1-\lambda)}\left[\xi^2 r_0\left(2-\frac{r_2}{r_1}\right)-r_1r_2\frac{(d-r_0cos\Psi_0)^2}{r_0}+(1-\lambda^2)\left(2-\frac{r_2}{r_1}\right)\frac{(d-r_0cos\Psi_0)^2}{r_0}\right], \tag{2.42}$$

Noting that $\left|\frac{r_0^2\xi^2}{c_0^2}+\left(\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right)\right|\geq\left|\frac{r_0^2\xi^2}{c_0^2}\right|$,

$\left|\frac{r_0^2\xi^2}{c_0^2}+\left(\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right)\right|\geq\left|\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right|$, We can see that

$$\left|\frac{r_2}{r_1}\right|^2=\frac{\left|\frac{r_0^2\xi^2}{c_0^2}+(1-\lambda^2)\right|}{\left|\frac{r_0^2\xi^2}{c_0^2}+\left(\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right)\right|}<1+\frac{|1-\lambda^2|}{\left|\lambda\left(\lambda-\frac{r_0cos\Psi}{c_0}\right)\right|}, \tag{2.43}$$

and

$$\frac{|1+\lambda|^2}{|\lambda|^2}\to 1(|\lambda_1|\to\infty), \tag{2.44}$$

$$\frac{|1+\lambda|^2}{|\lambda|^2}\to \frac{(1+\lambda_0)^2}{\lambda_0^2}(|\lambda_1|\to 0), \tag{2.45}$$

$$\frac{|1-\lambda|^2}{\left|\lambda-\frac{r_0cos\Psi}{c_0}\right|^2}\to 1(|\lambda_1|\to\infty), \tag{2.46}$$

$$\frac{|1-\lambda|^2}{\left|\lambda-\frac{r_0cos\Psi}{c_0}\right|^2}\to \frac{(1-\lambda_0)^2}{\lambda_0-\frac{r_0cos\Psi}{c_0}}(|\lambda_1|\to 0), \tag{2.47}$$

The righth and side of the 4th equation is larger than 1 if λ_0 is small enough, but (2.45) is larger than (2.47), and we get $\left|\frac{r_2}{r_1}\right|^2<1+c_0+\frac{1}{\lambda_0}$. Based on the assumption that c_0 is large enough, we get $\left(1+c_0+\frac{1}{\lambda_0}\right)^{\frac{1}{2}}>2$, and so $Max\left(2-\left|\frac{r_2}{r_1}\right|\right)=2+\left(1+c_0+\frac{1}{\lambda_0}\right)^{\frac{1}{2}}$.

Now, er remind that

$$|r_1r_2|^2\geq\sqrt{\left(\frac{r_0^2\xi^2}{c_0^2}\right)^2+\left|\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right|^2}\sqrt{\left(\frac{r_0^2\xi^2}{c_0^2}\right)^2+|1-\lambda^2|^2}$$

$$\geq\frac{1}{2}\left\{\left(\frac{r_0^2\xi^2}{c_0^2}\right)^2+\left|\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right|\right\}\left\{\left(\frac{r_0^2\xi^2}{c_0^2}\right)^2+|1-\lambda^2|\right\}. \tag{2.48}$$

Based on the assumption that $\frac{1}{2c_0^2}\leq 1$, we can estimate

$$|1-\lambda^2|^2\left|2-\frac{r_2}{r_1}\right|^2\leq\frac{1}{2}|1-\lambda^2|\left|\lambda^2-\frac{r_0cos\Psi}{c_0}\lambda\right|, \tag{2.49}$$

and

$$\xi^4r_0^2\left|2-\frac{r_2}{r_1}\right|^2\leq\frac{r_0^2\xi^4c_0^2}{2}. \tag{2.50}$$

This is the enough condition to show that $|P|>|r|$.

Now, denoting $e_i(\theta) = e^{r_i\theta} (i = 1,2)$, we utilize the following lemma: Concerning $e_i(\theta)$, following inequalities hold:

$$\int_0^\infty |\frac{d^j e_i(\theta)}{d\theta^j}|^2 \le |r_i|^{2j-1} \quad (i = 1,2, j \ge 0). \tag{2.51}$$

Based on the fact that $arg(r_i) \in (-\frac{\pi}{4}, \frac{\pi}{4})$, the proof is similar to that of [3], so we omit it here.

Using the lemma above, we get the following estimate:

$$\begin{aligned} | \parallel \hat{\tilde{v}} | \parallel^2_{\mu,m+2} \le & \sum_{0\le j+\alpha\le m} \int_R d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} (1+|\lambda|^2)^{m+2-(j+\alpha)} |\xi^2|^\alpha (|\xi| \\ & + \sqrt{1+|\lambda|^2})^{2j-1} |\hat{\tilde{f}}'_3|^2 \\ \le & \sum_{j=0,\alpha\ge 1} \int d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} (1+|\lambda|^2)^{m+2-\alpha} |\xi^2|^\alpha (|\xi| + \sqrt{1+|\lambda|^2})^{-1} |\hat{\tilde{f}}'_3|^2 \\ & + \int d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} (1+|\lambda|^2)^{m+2} (|\xi| + \sqrt{1+|\lambda|^2})^{-1} |\hat{\tilde{f}}'_3|^2 \\ & + \sum_{j\ge 1,\alpha=0} \int d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} (1+|\lambda|^2)^{m+2-j} (|\xi| + \sqrt{1+|\lambda|^2})^{2j-1} |\hat{\tilde{f}}'_3|^2 \\ & + \sum_{j\ge 1, 0\le j+\alpha\le m} \sum_{0\le\beta\le 2j-1} \int_R d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta \,(1 \\ & + |\lambda|^2)^{m+2-(j+\alpha)+\frac{\beta}{2}} |\xi^2|^{(j+\alpha)-\frac{1}{2}-\frac{\beta}{2}} |\hat{\tilde{f}}'_3|^2, \end{aligned} \tag{2.52}$$

$$\begin{aligned} & \sum_{j\ge 1, 0\le j+\alpha\le m} \sum_{0\le\beta\le 2j-1} \int_R d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta \,(1+|\lambda|^2)^{m+2-(j+\alpha)+\frac{\beta}{2}} |\xi^2|^{(j+\alpha)-\frac{1}{2}-\frac{\beta}{2}} |\hat{\tilde{f}}'_3|^2 \\ = & \sum_{0\le j+\alpha\le m} \sum_{\delta=0}^{j-1} \int_R d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta \,(1+|\lambda|^2)^{m+2-(j+\alpha)+\delta} |\xi^2|^{(j+\alpha)-\frac{1}{2}-\delta} |\hat{\tilde{f}}'_3|^2 d\lambda \\ & + \sum_{0\le j+\alpha\le m} \sum_{\delta=1}^{j} \int_R d\,\xi \int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta \,(1+|\lambda|^2)^{m+2-(j+\alpha)+\delta-\frac{1}{2}} |\xi^2|^{(j+\alpha)-\delta} |\hat{\tilde{f}}'_3|^2 d\lambda \end{aligned} \tag{2.53}$$

Concerning the first term of RHS of (2.44), denoting $\chi := [(j+\alpha) - \frac{1}{2} - \delta] = (j+\alpha) - 1 - \delta$, $\int |\xi^2|^{(j+\alpha)-\frac{1}{2}-\delta} |f'_3|^2 d\xi$ is equivalent to

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|D_\phi^\chi f'_3(\phi)-D_\phi^\chi f'_3(\phi')|^2}{|\phi-\phi'|^{1+2\{(j+\alpha)-\frac{1}{2}-\delta\}}}d\phi d\phi' = \int_0^{2\pi}\int_0^{2\pi}\frac{|D_\phi^\chi f'_3(\phi)-D_\phi^\chi f'_3(\phi')|^2}{|\phi-\phi'|^2}d\phi d\phi'. \quad (2.54)$$

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|D_\phi^\chi f'_3(\phi)-D_\phi^\chi f'_3(\phi')|^2}{|\phi-\phi'|^{1+2\{(j+\alpha)-\frac{1}{2}-\delta\}}}d\phi d\phi' = \int_0^{2\pi}\int_0^{2\pi}\frac{|D_\phi^\chi f'_3(\phi)-D_\phi^\chi f'_3(\phi')|^2}{|\phi-\phi'|^2}d\phi d\phi'. \quad (2.55)$$

Taking $\mu = \mu' + (j+\alpha) - \delta - \frac{3}{2}$,

$$\sum_{0\le j+\alpha\le m}\sum_{\delta=0}^{j-1}\int_R d\,\xi\int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta\,(1+|\lambda|^2)^{m+2-(j+\alpha)+\delta}|\xi^2|^{(j+\alpha)-\frac{1}{2}-\delta}|f'_3|^2 d\lambda$$

$$= \sum_{0\le j+\alpha\le m}\sum_{\delta=0}^{j-1}\sum_{l=0}^{m+2-(j+\alpha)+\delta}\int_0^\infty r^{2(\mu-m+l-1)}\,|D_r^l\int_0^{2\pi}\int_0^{2\pi}\frac{|D_\phi^\chi f'_3(\phi)-D_\phi^\chi f'_3(\phi')|^2}{|\phi-\phi'|^2}d\phi d\phi'|^2 dr.$$

(2.56)

Concerning the second term of RHS in (2.44), noting that $m-(j+\alpha)+\delta-\frac{1}{2}=[m-(j+\alpha)+\delta-1]+\frac{1}{2}$, let's take $\mu-m+\frac{3}{2}=\mu''-\{m-(j+\alpha)+\delta-1\}$, and

$$\sum_{0\le j+\alpha\le m+2}\sum_{\delta=1}^{j}\int_R d\,\xi\int_{Re\lambda=\mu-m-\frac{1}{2}} c_\beta\,(1+|\lambda|^2)^{m+2-(j+\alpha)+\delta-\frac{1}{2}}|\xi^2|^{(j+\alpha)-\delta}|f'_3|^2 d\lambda$$

$$=\sum_{0\le j+\alpha\le m+2}\sum_{\delta=1}^{j}\int_{Re\lambda=\mu''-\{m+2-(j+\alpha)+\delta-1\}} c_\beta\,(1+|\lambda|^2)^{\{m+2-(j+\alpha)+\delta-1\}+\frac{1}{2}}\,\|D_\phi^{(j+\alpha)-\delta}f'_3\|^2_{L_{2,\phi}(0,\pi)}\,d\lambda.$$

(2.57)

Let denote $m+2-(j+\alpha)+\delta-1$ by χ', then using the fact that $\int_{Re\lambda=\mu-k}(1+|\lambda|^2)^{k+\frac{1}{2}}|\tilde{h}|^2 d\lambda$ is equivalent to

$$\sum_{l=0}^{k}\int_0^\infty |D_r^l h|^2 r^{2(\mu-k+l)-1}dr + \int_0^\infty r^{2\mu}\,dr\int_0^r\frac{|D_r^l h(r+\rho)-D_r^l h(r)|^2}{\rho^2}d\rho,$$

we show (2.41) is equivalent to

$$\sum_{0\le j+\alpha\le m+2}\sum_{\delta=1}^{j}\sum_{i=0}^{\chi'}\int_0^\infty |D_r^i D_\phi^{(j+\alpha)-\delta}f_3|^2 r^{2(\mu-(j+\alpha)+\delta+\frac{1}{2})-2\chi'+2i-1}dr$$

$$+\sum_{0\le j+\alpha\le m+2}\sum_{\delta=1}^{j}\int_0^\infty r^{2(\mu-(j+\alpha)+\delta+\frac{1}{2})}\int_0^r\frac{|D_r^{\chi'}D_\phi^{(j+\alpha)-\delta}f_3(r+\rho)-D_r^{\chi'}D_\phi^{(j+\alpha)-\delta}f_3(r)|^2}{\rho^2}d\rho.$$

Now we get

$$|\,\|v|\,\|_{\mu,m+2}\le C|\,\|f'_3|\,\|_{\mu,m+\frac{3}{2}(R^2)}, \quad (2.58)$$

with some positive constant C.
Similar conclusion holds for the half space problem at Γ_2.

Non Homogeneous Problem

Next, we consider the non-homogeneous problem after Mellin and Fourier transform in the half space. we define $\boldsymbol{w}$ as the solutioin of the following problem:

$$-\nu\left[(-\lambda+\lambda^2)-\frac{\xi_1^2}{sin\theta}-\xi_2^2\right]\widehat{\boldsymbol{w}}=\widehat{\widetilde{\boldsymbol{F}}}_1(\lambda+2),$$

from which we get

$$\widetilde{\widehat{\boldsymbol{w}}}=\frac{\widehat{\widetilde{\boldsymbol{F}}}_1(\lambda+2)}{\nu\left(\frac{\xi_1^2}{sin\theta}+\xi_2^2\right)-(\lambda^2-\lambda)}. \tag{2.59}$$

By (2.50), the following estimate holds:

$$\sum_{j=0}^{m+2}(1+|\lambda|^2)^{m+2-j}(\xi_1^2+\xi_2^2)^j|\widetilde{\widehat{\boldsymbol{w}}}|\leq\sum_{j=0}^{m}(1+|\lambda|^2)^{m-j}(\xi_1^2+\xi_2^2)^j|\widehat{\widetilde{\boldsymbol{F}}}_1(\lambda+2)|. \tag{2.60}$$

Integrating (2.51) over the line $Re\lambda=\mu-m-\frac{1}{2}$,

$$||||\boldsymbol{w}||||_{\mu,m+2}\leq c||||\boldsymbol{F}_1||||_{\mu,m}.$$

Next we consider the problem $\nabla\cdot\boldsymbol{w}'=l_2$.
To consider the above problem, we set $\boldsymbol{w}'=\nabla\Phi$.
Then, the problem to be considered can be denoted as follows:

$$\nabla\cdot\boldsymbol{w}'=\triangle\Phi=l_2, \tag{2.61}$$

$$\Phi|_{\Psi=0}=0. \tag{2.62}$$

Using Cardelon-Zygmund theorem and L_p estimates for elliptic equations [8], we get

$$\sum_{|\alpha|=k}|D^{\alpha}_{r,\phi,\Psi}\boldsymbol{w}'|_{L_2(R^2)}\leq C\sum_{|\alpha|=k}|D^{\alpha}_{r,\phi,\Psi}l_2|_{L_2(R^2)}. \tag{2.63}$$

Multiplying $r^{2(\mu-(m-1)+l)}$ to each term of (2.54), we get the estimate: $\|\boldsymbol{w}'\|_{H_\mu^{m+2}(\Omega)}\leq\|l_2\|_{H_\mu^{m+2}(\Omega)}$.

Now, reminding that $l_2=R-R'$, where $R'=\begin{bmatrix}0\\ \int_0^\phi L_3\,\boldsymbol{v}d\phi\\ 0\end{bmatrix}$,

we get the following estimate:

$$\|\boldsymbol{w}'\|_{H_\mu^{m+2}(\Omega)}\leq\|\boldsymbol{R}\|_{H_\mu^{m+2}(\Omega)}+\|\boldsymbol{u}\|_{H_\mu^{m+2}(\Omega)}+\|\boldsymbol{w}\|_{H_\mu^{m+2}(\Omega)}. \tag{2.64}$$

Now, denoting $r=p+\nabla\cdot\boldsymbol{w}'$, $\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{w}+\boldsymbol{w}'$ we can see (2.21)-(2.26) hold for $\boldsymbol{v}$ and $\boldsymbol{p}$ replaced by q in the half space.

Solution around the Angle

Now we consider the following problem in the transformed domain:

$$-\frac{\nu}{(d-r_0 cos\Psi_0)}\left\{(d-r_0 cos\Psi_0)\lambda^2+r_0 cos\Psi_0\lambda)\tilde{\tilde{v}}+\frac{r_0^2}{d-r_0 cos\Psi_0}(i\xi)^2\tilde{\tilde{v}}+(d-r_0 cos\Psi_0)\frac{\partial^2 \boldsymbol{v}}{\partial\Psi^2}\right\}$$

$$+\begin{bmatrix}-(\lambda+1)\tilde{\tilde{q}}\\ \frac{r_0}{d-r_0 cos\Psi_0}(i\xi)\tilde{\tilde{q}}\\ \frac{\partial\tilde{\tilde{q}}}{\partial\Psi}\end{bmatrix}=\widetilde{r^2 \boldsymbol{f}_1}, \quad (2.65)$$

$$\frac{1}{d-r_0 cos\Psi_0}\{(c'_0-c'_1\lambda)\widetilde{\tilde{v}_r}+(i\xi)r_0\widetilde{\tilde{v}_\phi}\}+\frac{\partial\widetilde{\tilde{v}_\Psi}}{\partial\Psi}=\widetilde{rf_2}, \quad (2.66)$$

$$\nu\left[\frac{\partial\widetilde{\tilde{v}_r}}{\partial\Psi}-(1+\lambda)\widetilde{\tilde{v}_\Psi}\right]=f'_{31}, \quad (2.67)$$

$$\nu\left[(d-r_0 cos\Psi_0)\frac{\partial\widetilde{\tilde{v}_\phi}}{\partial\Psi}+r_0\frac{\partial\widetilde{\tilde{v}_\Psi}}{\partial\phi}\right]=f'_{32}, \quad (2.68)$$

$$2\nu r_0\frac{\partial\widetilde{\tilde{v}_\Psi}}{\partial\Psi}=f'_{33}, \quad (2.69)$$

$$\boldsymbol{v}=\boldsymbol{f}'_4 \quad x\in\Gamma_2, \quad (2.70)$$

We construct coverings of $\tilde{\Omega}_\theta$ and using the method of regulalizer, and note that

$$\|\boldsymbol{v}\|_{H_\mu^{m+1}(\Omega)}\leq C(\epsilon+C_\epsilon\phi_0)\|\boldsymbol{v}\|_{H_\mu^{m+2}(\Omega)}, \quad (2.71)$$

holds for an arbitrary positive number ϵ, where ϕ_0 is the maximum value of ϕ in each coverings. we can construct the solution to (3.44)-(3.47) that satisfies

$$|\,\|\tilde{\tilde{v}}|\,\|_{\mu,m+2}\leq C[\|\tilde{\tilde{\boldsymbol{f}}}_1\|_{\mu,m}+\|\tilde{\tilde{\boldsymbol{R}}}\|_{\mu,m+2}+\|\tilde{\tilde{\boldsymbol{f}}}_3\|_{\mu,m+\frac{3}{2}}+\|\widetilde{\tilde{\boldsymbol{f}_4}}\|_{\mu,m+\frac{3}{2}}+\|\tilde{\tilde{p}}\|_{\mu,m+1}].$$

This implies

$$\|\boldsymbol{v}\|_{H_\mu^{m+2}(\Omega)}\leq C[\|\boldsymbol{f}_1\|_{H_\mu^{m}(\Omega)}+\|\boldsymbol{R}\|_{H_\mu^{m+2}(\Omega)}+\|\boldsymbol{f}_3\|_{H_\mu^{m+\frac{5}{2}}(\Gamma_1)}+\|\boldsymbol{f}_4\|_{H_\mu^{m+\frac{5}{2}}(\Gamma_2)}+\|p\|_{H_\mu^{m+1}(\Omega)}],$$

from which the assertion of theorem2 follows directly.

Solvability of Original Problem

Now, we can verify the solvability of the original problem (2.10)-(2.15). We estimate terms

$$\frac{1}{r^2}\left[\frac{\nu}{(d-r_0 cos\Psi_0)}-\frac{\nu}{(d-rcos\Psi)}\right]L[\boldsymbol{v}], \qquad \frac{1}{r^2}\frac{\nu}{(d-r_0 cos\Psi_0)}[L[\boldsymbol{v}]-L_0[\boldsymbol{v}]],$$

$$\begin{bmatrix}r^2\frac{\partial p}{\partial r}\\ \left\{\frac{rr_0}{d-r_0 cos\Psi_0}-\frac{r^2}{d-rcos\Psi}\right\}\frac{\partial p}{\partial\phi}\\ r\frac{\partial p}{\partial\Psi}\end{bmatrix}.$$ Reminding $supp(\boldsymbol{v})\in\{|(r,\phi,\Psi\in\Omega_\theta|0<r<\epsilon\}$, it is

easily seen that $\|\cdot\|_{H_\mu^m(\Omega)}$ norm of these terms are estimated from above by $\epsilon \| \boldsymbol{v} \|_{H_\mu^m(\Omega)}$.

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