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A note that on the generalization of central reduced rings

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Abstract

The 2-central reduced ring, a generalization of central reduced rings, has been introduced, and checked about its relation and properties. Additionally, we have investigated von Neumann regularity of rings with simple singular right R-modules are flat.

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1. Introduction

To begin with, let us see the needed definitions, and assume that every ring in this article is an associative ring with identity without further statements. Let $P(R)$ and $N(R)$ denote the prime radical in R, the set of all nilpotent elements in R, respectively. For any $a \in R$, $l(a)(r(a))$ denotes the left(right) annihilator of a. A ring is reduced if it has no nonzero nilpotent elements, similarly a ring R is called central reduced if every nilpotent element of R is central[16]. A weaker condition than central reduced is defined by Kose in [13]. A ring R is called quasi-reduced if for any $a, b \in R$, $ab = 0$ implies that $(aR) \cap (Rb)$ is contained in the center of R. A ring R is (von Neumann) regular if for every $a \in R$, there exists some $b \in R$ such that $a = aba$. Similarly, a ring R is strongly regular if for every $a \in R$, there exists some $b \in R$ such that $a = a^2b$. It is known that a ring R

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is strongly regular if and only if R is a reduced regular ring. A ring R is called a left (right) SF-ring [14] if simple left (right) R-modules are flat. It is well-known fact that the regular rings are left(right) SF-rings. Ramamurthy [14] initiated the study of left (right) SF-rings and of the question whether a left (right) SF-ring is necessarily regular. R is a left (right) SSF-ring [12], if simple singular left (right) R-modules are flat. That the rings with simple singular left (right) R-modules are flat were studied by authors in [1, 9, 12].

In this article, we introduce 2-central reduced rings as generalization of central reduced rings. We showed that some results of central reduced rings can be extended to 2-central reduced rings for this general settings. Additionally, we prove that 2-central reduced rings are abelian, and there exists an abelian ring which is not 2-central reduced. Therefore the class of 2-central reduced rings lie strictly between the classes of reduced rings and abelian rings. Additionally, for a 2-central reduced ring R, we have proved that R is strongly regular if and only if R is left(right) SF-ring if and only if R is left(right) SSF-ring.

2. The Properties of 2-central reduced rings

To begin with, let us see the central reduced ring-related definitions. Throughout this article, let Z be the ring of integers, and for a positive integer n, Z_n and $Z^{2\times 2}$ denote the ring of integer modulo n and the ring of matrices over Z, respectively.

Definition 2.1. A ring R is said to be central reduced if every nilpotent of R is central [16].

All commutative and reduced rings are central reduced, and the following example shows that there exists central reduced ring which is not reduced ring.

Example 2.2. Let $R = Z[x]/x^2$. Then R is commutative ring and so is central reduced ring. If $a = x + x^2 \in R$, then $a^2 = 0$. Therefore R is not reduced ring.

Definition 2.3. A ring R is called quasi-reduced if for any $a, b \in R$, $ab = 0$ implies $(aR) \cap (Rb)$ is contained in the center of R [13].

Proposition 2.4. If R is central reduced, then R is quasi-reduced [13].

The following example shows that there exists a quasi-reduced ring which is not central reduced ring.

Example 2.5. Consider the subring R (without unit) of the ring 2×2 matrices over Z_2 of the form $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ [11, 13]. The ring R is non-commutative. Let $\begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} c & d \\ c & d \end{pmatrix} \in R$ with $\begin{pmatrix} a & b \\ a & b \end{pmatrix} \begin{pmatrix} c & d \\ c & d \end{pmatrix} = 0$. Then $(a+b)c = 0$ and $(a+b)d = 0$. Two cases arise here. $a+b = 0$ or $a+b \neq 0$.

If $a+b=0$, then $a=b=0$ or $a=b=1$. These cases imply

$$\begin{pmatrix} a & b \\ a & b \end{pmatrix} R \cap R \begin{pmatrix} c & d \\ c & d \end{pmatrix} = 0.$$

Next, assume that $a+b \neq 0$. Then $c=d=0$. Hence $\begin{pmatrix} a & b \\ a & b \end{pmatrix} R \cap R \begin{pmatrix} c & d \\ c & d \end{pmatrix} = 0$. Thus R is a quasi-reduced ring. On the other hand, if $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is non-zero nilpotent element of R, then A is not central in R. Thus R is not central reduced.

Proposition 2.6. If R is a quasi-reduced ring, then for all $a \in R$, $a^2 = 0$ implies a is central [13].

Now we introduce the class of rings, called 2-central reduced rings which is a generalization of central reduced rings. We investigate the properties of reduced rings to hold for the central case. Clearly, reduced rings are central reduced, and central reduced rings are quasi-reduced.

Definition 2.7. A ring R is said to be 2-central reduced if for all $a \in R$, $a^2 = 0$ implies a is central.

We begin with the properties of 2-central reduced rings. Recall that a ring R is semiprime if $aRa = 0$ implies $a = 0$ for $a \in R$. A ring R is called right(left) principally quasi-Baer if the right(left) annihilator of a principally right(left) ideal of R is generated by an idempotent [5]. Finally, a ring R is called right(left) principally projective if the right(left) annihilator of an element of R is generated by an idempotent [6]. The aim of our next propostion is to find the conditions where 2-central reduced ring is reduced.

The following proposition extends known results of [16].

Proposition 2.8. If R is a reduced ring, then R is 2-central reduced. The converse holds if R satisfies any of the following conditions.

(1) R is a semiprime ring.

(2) R is a right(left) principally projective ring.

(3) R is a right(left) principally quasi-Baer ring.

Proof.

(1) Let $a^2 = 0$ for all $a \in R$. Since R is a 2-central reduced ring, a is central. So for all $r \in R$, $ar = ra$. Therefore $ara = a^2r = 0$, so $aRa = 0$. Since R is semi-prime ring, $a = 0$. Hence R is reduced ring.

(2) Let $a^2 = 0$ for all $a \in R$. Assume that R is a right(left) principally projective ring, then there exists an idempotent $e \in R$ such that $r(a) = eR$. Thus $a = ea$. Since

R is 2-central reduced ring, $a = ea = ae = 0$ and so, R is reduced ring. A similar proof may be given for left principally projective ring.

(3) Same with proof (2). ■

Corollary 2.9. If R is a 2-central-reduced ring, then the following conditions are equivalent.

(1) R is a right principally projective ring.

(2) R is a left principally projective ring.

(3) R is a right principally quasi-Baer ring.

(4) R is a left principally quasi-Baer ring.

Note that a ring R is called directly finite if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. In this direction, we prove the following theorem.

Proposition 2.10. If R is a 2-central reduced ring, then R is a directly finite.

Proof. Let R be a 2-central reduced ring and let $a, b \in R$ with $ab = 1$. Then $a(ba-1) = 0$ and $((ba-1)a)^2 = (ba-1)a \cdot (ba-1)a = 0$. Since R is a 2-central reduced ring, $(ba-1)a$ is central. Thus $a[(ba-1)a] = [(ba-1)a]a$, and so, $(ba-1)a^2 = 0$. Since $ab = 1$,

$$[(ba-1)a^2]b^2 = [(ba-1)a](ab)b = [(ba-1)](ab) = 0$$

and $ba - 1 = 0$. Thus we have $ba = 1$. Thus, R is directly finite.

Let R be a ring, $P(R)$ the prime radical and $N(R)$ the set of all nilpotent element of a ring R. Since $P(R)$ is the intersection of all prime ideals of R, it is a nil ideal, therefore $P(R) \subseteq N(R)$. The ring R is called 2-primal if $P(R) = N(R)$[7]. Recall that a ring R is called central semi-commutative[3], if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$. In the next theorem, we prove that all 2-central reduced rings are 2-primal and central semi-commutative. ■

Proposition 2.11. Let R be a 2-central reduced ring. Then we have the following.

(1) R is a 2-primal ring.

(2) R is a central semi-commutative.

(3) For any $x \in l(a)$ and $r \in R$, $xr \in l(x^2) \cap r(x^2)$.

Proof.

(1) To complete the proof it is enough to show that $N(R) \subseteq P(R)$. Let $a \in N(R)$. First assume that $a^2 = 0$, then a is central element since R is 2-central reduced. So for any $s \in R$, $(as)a = sa^2 = 0$. Hence for any prime ideal P, for any $s \in R$, $asa = 0$ implies $a \in P$. Thus $a \in P(R)$. Also assume that $a^3 = 0$, then $(a^2)^2 = aa^3 = 0$, since R is 2-central reduced, a^2 is central element. So for any $r \in R$, $a^r a = ra^3 = 0$ and $(ara)^2 = (ara)(ara) = ar(a^2ra) = 0$ implies ara is central element. Also $ara^2 = a^3r = 0$. Thus for any $s \in R$, $as(ara) = (ara)as = ara^2s = 0$ implies $a \in P$. Therefore $a \in P(R)$. By the induction on the index of nilpotency, we may conclude that $P(R)$ consists of all nilpotent elements of R. Hence R is 2-primal.

(2) Let $a, b \in R$ with $ab = 0$. Then $(ba)^2 = (ba)(ba) = b(ab)a = 0$. Since R is a 2-central reduced ring, ba is central element. Also for any $r \in R$, $(arb)^2 = (arb)(arb) = ar(ba)rb = 0$ for ba is central element. Since R is a 2-central reduced, arb is central element. Hence R is central semi-commutative.

(3) For any $x \in l(a)$, since R is a 2-central reduced ring, xa and ax are central elements. Also for any $r \in R$, $xa = 0$ implies $a^2(xr) = a(ax)r = (ax)ar = a(xa)r = 0$. Thus $xr \in r(a^2)$. And $xa = 0$ implies $(xra)^2 = (xra)(xra) = xr(ax)ra = (xr)r(ax)a = xr^2a(xa) = 0$. Since R is 2-central reduced ring, xra is central element. Therefore, $(xr)a^2 = (xra)a = a(xra) = (ax)ra = r(ax)a = ra(xa) = 0$ implies $xr \in l(a^2)$. ■

Corollary 2.12. Let R be a central reduced ring. Then we have the following.

(1) R is a 2-primal ring.

(2) R is a central semi-commutative.

Note that a ring R is said to be abelian if every idempotent is central.

Lemma 2.13. If R is 2-central reduced ring, then R is an abelian ring.

Proof. Let e be an idempotent element in R. Then we have $(er - ere)^2 = 0$ for all $r \in R$. Since R is 2-central reduced ring, $er - ere$ is a central. Commuting $er - ere$ by e we have $er = ere$. Similarly, $(re - ere)^2 = 0$ implies $re = ere$. Hence R is an abelian ring.

Therefore the class of 2-central reduced rings lie strictly between the classes of reduced rings and abelian rings. The following example shows that the converse of lemma 2.13 may not be true in general. ■

Example 2.14. Consider the ring $R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z,\ a - d \equiv b \equiv c \equiv 0(mod\ 2)\}$. Since the only idempotent of R is zero and identity matrices, we have R is abelian [3]. If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \in R$, then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but A is not a central. Therefore R is not 2-central reduced.

Recall that a ring R is von Neumann regular if for each $a \in R$, there exists $b \in R$ with $aba = a$, while R is strongly regular if for each there exists $b \in R$ with $a^2b = a$. It is well known that R is strongly regular if and only if R is a reduced regular ring or abelian regular ring. Hence by lemma 2.13, the following theorem gives a new characterization of strongly regular rings in terms of 2-central reduced rings.

Theorem 2.15. For a ring R, the following conditions are all equivalent.

(1) R is a strongly regular ring.

(2) R is a regular and reduced ring.

(3) R is a regular and central reduced ring.

(4) R is a regular and 2-central reduced ring.

(5) R is a regular and abelian ring.

We now give our main definition.

Definition 2.16. A ring R is a right SF-ring (SSF-ring) if every simple (simple singular) right R-module is flat.

Goodearl [8] showed regular rings are always right(left) SF, and according to Rege [15], reduced right(left) SF rings are strongly regular. Hence, we have the following Lemmas.

Lemma 2.17. Let R be a ring, and let I be a right ideal of R. Then R/I is a flat right ideal of R if and only if for each $a \in I$ there exists some $b \in I$ such that $a = ba$ [15].

Lemma 2.18. Let I be an ideal of R. If R is a ring whose simple singular left R-module is flat, then R/I is a ring whose every simple singular left R/I-module is flat [9].

Proposition 2.19. The following conditions are equivalent [9].

(1) R is a strongly regular ring.

(2) R is a reduced and left(right) SSF-ring.

Recall that R is a left(right) weakly regular if $I = I^2$ for every left (right) ideal I of R, and R is weakly regular if it is both left and right weakly regular. A regular ring is clearly weakly regular, but the converse is not established.

Proposition 2.20. The following statements are equivalent [1].

(1) R is a weakly regular ring.

(2) R is a a semiprime 2-primal and left(right) SSF-ring.

The following theorem extends known results of [1, 9, 16].

Theorem 2.21. The following statements are equivalent.

(1) R is a strongly regular ring.

(2) R is a 2-central reduced and right(left) SF-ring.

(3) R is a 2-central reduced and right(left) SSF-ring.

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds. Let us show that (3) $\rightarrow$ (1). It is sufficient to show that R is reduced ring by applying Proposition 2.19.

Assume that $a^2 = 0$. If $a \neq 0$, then $r(a) \neq R$, and so, there exists a maximal right ideal M containing $r(a)$. First observe that M is an essential right ideal of R. If not, then M is a direct summand of R. Thus, we can write $M = r(e)$ for some $0 \neq e = e^2 \in R$. This gives $a \in M = r(e)$, $ea = 0$ and R is an abelian by theorem 2.13. Thus, $0 = ea = ae$, and so, $e \in r(a) \subseteq M = r(e)$. Hence $e = 0$, and it is a contradiction. Therefore M must be an essential right ideal of R. Thus $R = M$ is a simple singular right R-module.

Since R is right SSF-ring, by lemma 2.17 and 2.18, there exist $c \in M$ such that $a = ca$. Thus $(1 - c)a = 0$ and since R is a 2-central reduced ring, $a^2 = 0$ implies a is a central element. $a(1 - c) = 0$, and so, $1 - c \in r(a) \subseteq M$. Hence, $1 \in M$, and it is a contradiction. Therefore $a = 0$ and so, R is a reduced ring. ■

Corollary 2.22. The following statements are equivalent.

(1) R is a strongly regular ring.

(2) All R-modules are flat and R is central reduced.

(3) All cyclic R-modules are flat and R is central reduced.

References

[1] Abdul-Jabbar I. M and Mahmood R. D., On rings whose simple singular R-modules are flat, *Raf. J. Comp. & Math.*, **7** (2010), 51–57.

[2] N. Agayev, A. Harmanci and S. Halicioglu, Extened Armendariz Rings, *Alge. Grou, Geom.*, **26** (2009), 343–354.

[3] N. Agayev, T. Ozen and A. Harmanci, On a class of Semicommutative Rings, *Kyung. Math. J.* **51** (2011), 283–291.

[4] J. S. Alin and E.P. Armendariz, A class of rings having all singular simple modules injective, *Math. Scand.* **23** (1968), 233–240.

[5] G. F. Birkenmeier, J. Y. Kim and J. K. Park, Principally quasi-Baer rings, *Comm. Alge.* **29** (2001), 639–660.

[6] G. F. Birkenmeier, J. Y. Kim and J. K. Park, On extensions of Baer and quasi-Baer rings, *J. Pure Appl. Alge.* **159** (2001), 25–42.

[7] Y. Hirano, Some Studies of Strongly-Regular Rings, *Math. J. Okaya. Univ.,* **20** (1978), 141–149.

[8] K. R. Goodearl, *Ring Theory: Non-singular Rings and Modules* Marcel-Dekker, New York, 1974.

[9] X. Guangshi, On strongly regular rings and SF-rings, *Far East J. Math. Sci.*, **32** (2007), 693–699.

[10] S. U. Hwang, C. H. Jeon and K. S. Park, A Generalization of Insertion of Factors Property, *Bull. Kor. Math. Soc.* **44** (2007), 87–94.

[11] T. Y. Lam, *Exercises in Classical Ring Theory* Springer-Verlag, New York, 1995.

[12] R. D. Mahmood, Onweakly regular rings and SSF-rings, *Raf. J. Comp. & Math.* **3**, (2006), 55–59.

[13] H. Kose, B. Ungor, S. Halicioglu and A. Harmanci, Quasi-reduced rings, *Acta Univ. Apul.* **34** (2013), 57–68.

[14] V. S. Ramamurthy, On the injectivity and flatness of certain cyclic modules, *Proc. Amer. Math. Soc.* **48** (1995), 21–25.

[15] M. B. Rege, On von Neumann regular rings and SF-rings, *Math. Japo.* **31** (1986), 927–936.

[16] B. Ungor and S. Halicioglu, H. Kose and A. Harmanci, Rings in which every nilpotent is central, *Alge, Grou. Geom.* **30** (2013), 1–18.