Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4555–4561

http://www.ripublication.com/gjpam.htm

Stability Analysis of Impulsive Functional Differential Equations

Kanwalpreet Kaur
IKG Punjab Technical University, Jalandhar
Punjab (INDIA) - 144601.
E-mail: kanwal.2015@rediffmail.com

S. K. Srivastava
Beant College of Engineering and Technology,
Punjab (INDIA).
E-mail: sks64-bcet@yahoo.co.in

Abstract

This paper studies the stability of impulsive functional differential system where the time delay related to the state variables on the impulses are studied. By employing the Razumikhin technique and Lyapunov functionals some criteria of exponential stability is derived, which can be applied to impulsive functional differential equations.This result extends some existing results in the literature.

AMS subject classification: 34K20, 93D05.
Keywords: Impulsive functional differential equations, exponential stability, Razumikhin technique, Lyapunov function.

1. Introduction

The theory of impulsive delay differential equations is applied in many practical problems like in the field of engineering, communication network, medical, agricultural sciences, chemistry and many more [2, 4, 6]. In last few decades the theory of stability analysis has been extensively explored by the researchers. In these investigations many qualitative properties have been developed in study of systems with the impulse effect [3, 7], also the concept of time delay has attained major attention in recent years. In [1, 5] the stability of impulsive differential equations with delay is considered and some results are obtained. In these obtained results researchers mainly assume the relation between state variables on impulses to present state variables. But very little work is done in which state variables at the time of impulses are related to the time delay. So in this paper we get some results on exponential stability for impulsive functional differential equations in which state variables at the time of impulses related to the time delay is considered.

This paper is organized as follows. In section 2, we introduce basic notations and definitions. In sections 3, we get the several criteria about the exponential stability of the systems of impulsive functional differential equations. Concluding remarks are given in section 4.

2. Preliminaries

Consider Impulsive functional differential equations

$$\begin{aligned} x' &= f(t, x_t), && t \in [t_{k-1}, t_k), \\ \Delta x(t_k) &= I_k(x(t_k^-)) + J_k(x(t_k^- - \tau)), && t = t_k, k \in N \\ x_{t_0} &= \eta \end{aligned} \tag{1}$$

we assume the function

$$f : R_+ \times PC([-\tau, 0], R^n) \to R^n;$$

$$\eta \in PC([-\tau, 0], R^n); I_k, J_k : R_+ \times PC([-\tau, 0], R^n) \to R^n;$$

$[t_k]_{k=1}^{\infty}$ satisfies $0 = t_0 < t_1 < \cdots < t_k < \cdots, lim_{k\to\infty} t_k = \infty . \Delta x(t_k) = x(t_k) - x(t_k^-)$, where $x_t, x_{t^-} \in PC([-\tau, 0], R^n)$. For the given constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $||.||$ defined by $||.|| = sup_{-\tau<s<0}||\psi(s)||$. Denote $x(t) = x(t, t_0, \eta)$,the unique solution of equation (1), where $x_{t_0} = \eta$, we further suppose that $f(t, 0) = 0$, $I_k(0) = 0$ and $J_k(0) = 0, t \in R_+$ and $k \in N$,so that system (1) admits the zero solution.

Definition 2.1. For a function $V : R_+ \times R^n \longrightarrow R_+$,the upper right-hand derivative of the function V with respect to system(1) is defined by

$$D_+V(t, \vartheta(0)) = \lim_{\kappa \longrightarrow 0^+} sup \frac{1}{\kappa}[V(t + \kappa, \vartheta(0) + \kappa f(t, \phi)) - V(t, \vartheta(0))]$$

for $(t, \phi) \in R_+ \times PC([-\tau, 0], R^n)$.

Definition 2.2. The function $V : R_+ \times R^n \to R_+$ belongs to class v_0 if it holds following conditions:

(i) the function V is continuous on each of the sets $[t_{k-1}, t_k) \times R^n$ and for each $x, r \in R^n, t \in [t_{k-1}, t_k), k \in N, lim_{(t,r)\to(t_k^-,x)} V(t, r) = V(t_k^-, x)$ exists.

(ii) V is locally Lipschitzian with respect to $x \in R^n$ and $\forall t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.3. The function $V : R_+ \times PC([-\tau, 0], R^n) \rightarrow R_+$ belongs to class $v_0(.)$ if it holds following conditions:

(i) the function V is continuous on $[t_{k-1}, t_k) \times PC([-\tau, 0], R^n$ and for all $\psi, \phi \in PC([-\tau, 0], R^n)$ and $k \in N$, $lim_{(t,\psi)\rightarrow(t_k^-,\phi)} V(t, \psi) = V(t_k^-, \phi)$ exists.

(ii) $V(t, \psi)$ is locally Lipschitzian in each compact set in $PC([-\tau, 0], R^n)$ and $\forall t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.4. A functional $V(t, \psi) : R_+ \times PC([-\tau, 0], R^n) \rightarrow R_+$ belongs to class $v_0(.)$ if $V(t, \psi)$ for any $x \in PC([-\tau, \infty), R^n)$=$\{x : [t_0 - \tau, \infty) \rightarrow R^n$ is piecewise continuous$\}$,$V(t, x_t)$ is continuous for $t \geq t_0$.

Definition 2.5. The zero solution of the system (1) is said to be exponentially stable if $\exists$ some constants $w > 0$ and for every $\epsilon > 0$,there exist $\lambda = \lambda(\epsilon) > 0$ such that for any initial value $x_{t_0} = \eta$

$$||x(t, t_0, \eta)|| < \epsilon e^{-w(t-t_0)}, \forall t \geq t_0,$$

whenever $||\eta||_\tau < \lambda$ and $t_0 \in R_+$.

3. Main Results

Theorem 3.1. Suppose that the hypotheses (H1)-(H3) are satisfied, also there exist $V_1 \in v_0$, $V_2 \in v_0^*(.)$, $q_1, q_2 > 0$ where $q_1 \leq q_2$ and the constants $\gamma, \sigma, w, w_1, w_2, w_3 > 0$ such that

(i)

$$w_1||x||^{q_1} \leq V(t, x) \leq w_2||x||^{q_2}, 0 \leq V_2(t, \phi) \leq w_3||\phi||_\tau^{q_2},$$

for any $t \in R_+, x \in R^n, \phi \in PC([-\tau, 0], R^n)$.

(ii) for each $k \in N$ and

$$\begin{aligned} x \in R^n, V_1(t_k, x(t_k^-) + I_k(x(t_k^-)) + J_k(x(t_k^- - \tau))) \\ \leq (1 + \beta_k)(V_1(t_k^-, x(t_k^-)) + V_1(t_k^- + s, x(t_k^- + s))) \end{aligned}$$

for $s \in [-\tau, 0]$, where $\beta_k \geq 0, k \in N$.

(iii) for

$$V(t, \phi) = V_1(t, \phi(0)) + V_2(t, \phi),$$

$$V^{'}(t, \phi) \leq wV(t, \phi),$$

$$\forall t \in [t_{k-1}, t_k), \phi \in PC([-\tau, 0], R^n), k \in N.$$

(iv) for any $k \in N, \tau \leq t_k - t_{k-1} \leq \sigma$ and

$$\beta_k + \frac{w_3}{w_1} e^{(\frac{q_2}{q_1}-1)\sigma wk} \leq e^{-(\gamma+w)\sigma} - 1.$$

Then the zero solution of impulsive differential equation (1) is exponentially stable.

Proof. Consider $x(t) = x(t, t_0, \eta)$ be the solution of the impulsive differential system (1) with $||\phi||_\tau < \lambda$.Let $v_1(t) = V_1(t, x(t))$ and $v_2(t) = V_2(t, x_t)$, $v(t) = v_1(t) + v_2(t)$.For any given $\epsilon \in (0, 1]$, select $\lambda = \lambda(\epsilon) > 0$ such that

$$w_2\lambda^{q_1} + w_3\lambda^{q_2} < w_1\epsilon^{q_1}e^{-(\gamma+w)\sigma}$$

Form the condition (iii), we have

$$v(t) \leq v(t_{k-1})e^{w(t-t_{k-1})}, t \in [t_{k-1}, t_k), k \in N. \tag{2}$$

we shall prove that

$$\begin{aligned} v(t) &< w_1\epsilon^{q_1}e^{-(\gamma+w)\sigma k}e^{w(t-t_0)} \\ ||x|| &< \epsilon e^{-\frac{\gamma}{q_1}(t-t_0)}, t \in [t_{k-1}, t_k), k \in N \end{aligned} \tag{3}$$

For $k = 1$, from conditions (i)–(iv) and (2), we get

$$\begin{aligned} v(t) &\leq v(t_0)e^{w(t-t_0)} \\ &\leq w_2\lambda^{q_1} + w_3\lambda^{q_2}e^{w(t-t_0)} \\ &< w_1\epsilon^{q_1}e^{-(\gamma+w)\sigma}e^{w(t-t_0)} \end{aligned}$$

Thus

$$\begin{aligned} ||x(t)||^{q_1} &\leq \frac{1}{w_1}v(t) \\ &\leq \epsilon^{q_1}e^{-(\gamma+w)\sigma}e^{w(t_1-t_0)} \\ &\leq \epsilon^{q_1}e^{-(\gamma+w)(t_1-t_0)}e^{w(t_1-t_0)} \\ &\leq \epsilon^{q_1}e^{-\gamma(t-t_0)}, t \in [t_0, t_1) \end{aligned} \tag{4}$$

Therefore

$$||x(t)|| \leq \epsilon e^{-\frac{\gamma}{q_1}(t-t_0)}, t \in [t_0, t_1).$$

Let us assume that the equation (3), holds for $k = j$, i.e.

$$\begin{aligned} v(t) &< w_1\epsilon^{q_1}e^{-(\gamma+w)\sigma j}e^{w(t-t_0)}, and \\ ||x(t)|| &< \epsilon e^{-\frac{\gamma}{q_1}(t-t_0)}, t \in [t_{j-1}, t_j), j \geq 2 \end{aligned} \tag{5}$$

Now we shall prove that the result (3) holds for $k = j + 1$. For $t \in [t_{j-1}, t_j)$, from condition (i) and equation (5), we have

$$\begin{aligned} ||x(t)||^{q_1} &\leq \frac{1}{w_1}v_1(t) \\ &\leq \frac{1}{w_1}v(t) \\ &< \epsilon^{q_1}e^{-(\gamma+w)\sigma j}e^{w(t_j-t_0)} \end{aligned}$$

Therefore

$$\begin{aligned}||x(t_j^-)||_\tau &= sup_{-\tau \le s < 0}||x(t_j + s)|| \\ &< \epsilon e^{-\frac{(\gamma+w)}{q_1} j\sigma} e^{\frac{w}{q_1}(t_j - t_0)}\end{aligned}$$

By using continuity of $v_2(t)$ at each t_j and the condition (ii), we have

$$\begin{aligned}v_1(t_j) &\le (1+\beta_j)(v_1(t_j^-) + v_1(t_j^- + s)) \\ &< (1+\beta_j) w_1 \epsilon^{q_1} e^{-(\gamma+w) j\sigma} e^{w(t_j - t_0)}\end{aligned}$$

and

$$\begin{aligned}v_2(t_j) = v_2(t_j^-) &\le w_3 ||x_{t_j^-}||_\tau^{q_2} \\ &< w_3 \epsilon^{q_2} e^{-\frac{q_2}{q_1}(\gamma+w) j\sigma} e^{\frac{q_2}{q_1} w(t_j - t_0)}\end{aligned}$$

Therefore by using $q_1 \le q_2$, we get

$$\begin{aligned}v(t_j) &= v_1(t_j) + v_2(t_j) \\ &< (1+\beta_j) w_1 \epsilon^{q_1} e^{-(\gamma+w) j\sigma} e^{w(t_j - t_0)} + w_3 \epsilon^{q_2} e^{-\frac{q_2}{q_1}(\gamma+w) j\sigma} e^{\frac{q_2}{q_1} w(t_j - t_0)} \\ &\le w_1 \epsilon^{q_1} e^{-(\gamma+w) j\sigma} ((1+\beta_j) e^{w(t_j - t_0)} + \frac{w_3}{w_1} e^{\frac{q_2}{q_1} w(t_j - t_0)} \\ &\le w_1 \epsilon^{q_1} e^{-(\gamma+w) j\sigma} e^{w(t_j - t_0)} (1 + \beta_j + \frac{w_3}{w_1} e^{(\frac{q_2}{q_1} - 1) w j\sigma)}\end{aligned} \tag{6}$$

By using condition (iv), we get

$$\begin{aligned}v(t_j) &= w_1 \epsilon^{q_1} e^{-(\gamma+w) j\sigma} e^{w(t_j - t_0)} e^{-(\gamma+w)\sigma} \\ &\le w_1 \epsilon^{q_1} e^{-(\gamma+w)(j+1)\sigma} e^{w(t_j - t_0)}\end{aligned}$$

By using equation (2) and (6), for $t \in [t_j, t_{j+1})$, we have

$$\begin{aligned}v(t) &\le v(t_j) e^{w(t - t_j)} \\ &< w_1 \epsilon^{q_1} e^{-(\gamma+w)(j+1)\sigma} e^{w(t - t_0)}\end{aligned}$$

Thus

$$\begin{aligned}||x(t)||^{q_1} &< \epsilon^{q_1} e^{-(\gamma+w)(j+1)\sigma} e^{w(t - t_0)} \\ &\le \epsilon^{q_1} e^{-(\gamma+w)(j+1)\sigma} e^{w(t_{j+1} - t_0)} \\ &\le \epsilon^{q_1} e^{-(\gamma+w)(j+1)\sigma} e^{w(j+1)\sigma} \\ &\le \epsilon^{q_1} e^{-(j+1)\gamma\sigma} \\ &\le \epsilon^{q_1} e^{-\gamma(t_{j+1} - t_j + t_j + \cdots + t_1 - t_0)} \\ &\le \epsilon^{q_1} e^{-\gamma(t_{j+1} - t_0)} \\ &\le \epsilon^{q_1} e^{-\gamma(t - t_0)}, \; t \in [t_j, t_{j+1})\end{aligned}$$

which implies that result (3) holds for $k = j + 1$. Thus by induction we conclude that result (3) holds $\forall k \in N$. Hence,

$$||x(t)|| < \epsilon e^{-\frac{\gamma}{q_1}(t-t_0)}, t \geq t_0,$$

The proof is complete. Thus the zero solution of impulsive differential equation (1) is exponentially stable. ■

Corollary 3.2. Suppose that the hypotheses (H1)-(H3) are satisfied, also there exist $V_1 \in \upsilon_0$, $V_2 \in \upsilon_0^*(.)$ and the constants $\gamma, \sigma, q, w, w_1, w_2, w_3 > 0$ such that

(i)
$$w_1||x||^q \leq V(t,x) \leq w_2||x||^q, 0 \leq V_2(t,\phi) \leq w_3||\phi||_\tau^q,$$
for any $t \in R_+, x \in R^n, \phi \in PC([-\tau,0], R^n)$.

(ii) for each $k \in N$ and $x \in R^n$,
$$\begin{aligned} &V_1(t_k, x(t_k^-) + I_k(x(t_k^-)) + J_k(x(t_k^- - \tau))) \\ &\quad \leq (1+\beta_k)(V_1(t_k^-, x(t_k^-)) + V_1(t_k^- + s, x(t_k^- + s))) \end{aligned}$$
for $s \in [-\tau, 0]$, where $\beta_k \geq 0, k \in N$.

(iii) for $V(t,\phi) = V_1(t,\phi(0)) + V_2(t,\phi)$, $V^{'}(t,\phi) \leq wV(t,\phi)$, $\forall t \in [t_{k-1}, t_k), \phi \in PC([-\tau,0], R^n), k \in N$.

(iv) for any $k \in N$, $\tau \leq t_k - t_{k-1} \leq \sigma$ and $\beta_k + \dfrac{w_3}{w_1} \leq e^{-(\gamma+w)\sigma} - 1$.

Then the zero solution of impulsive differential equation (1) is exponentially stable.

Proof. Consider $q_1 = q_2 = q$ in theorem 3.1. ■

4. Conclusion

In this paper, we extended the concept of exponential stability criteria to the systems of impulsive functional differential equations. With the use of Lyapunov function along with Razumikhin technique, we have obtained some results for exponential stability of the system in which state variables on impulses are related to time delay.

References

[1] Ballinger, G., Liu, X.Z., 1999, "Existence and uniqueness results for impulsive delay differential equations", Dyn. Cont. Disc. Impuls., Syst. 5, 579–591.

[2] Lakshmikantham, V., Bainov, D.D., Simeonov, P.S., "Theory of Impulsive Differential Equations", Vol. 6 of Series in Modern Applied Mathematics, World Scientific, Teaneck, NJ, USA (1989).

[3] Liu, X.Z., Ballinger, G., 2001, “Uniform asymptotic stability of impulsive delay differential equations”, Comp. Math. App., 41, 903–915.
[4] Liu, X.Z., Wang, Q., 2007, “The method of lyapunov functionals and exponential stability of impulsive systems with time delay”, Nonlinear Analysis, 66(7), 1465–1484.
[5] Shen, J.H., 1999, “Razumikhin techniques in impulsive functional differential equations”, Nonlinear Analysis, 36, 119–130.
[6] Wang, Q., Liu, X.Z., 2007, “Impulsive stabilization of delay differential systems via the Lyapunov-Razumikhin method”, Appl. Math. Letter, 20(8), 839–845.
[7] Zhang, Y., Sun, J.T., 2005, “Eventual pratical stability of impulsive differential equations with time delay in terms of two measurements”, J. Comput. Appl. Math., 176(1), 223–229.