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Graded Weakly Semiprime Ideals that are not Graded Semiprime

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Let G be a group and let R be a G-graded ring. In this article, we focus on the graded weakly semiprime ideals that are not graded semiprime and we introduce further results concerning such ideals. For example, if I is such ideal, we prove that $I_g \subseteq Nil(R)$ for some $g \in supp(R,G)$ and we prove that there exists $g \in supp(R,G)$ and $a \in R_g - I_g$ such that $4aI_g = 0$. Also, we prove that there exists $g \in supp(R,G)$ and $y \in R_g - I_g$ such that $y^2 = 0$. Assuming that 2 is not a zero divisor of R, we show that there exists $g \in supp(R,G)$ such that $x^2 = 0$ for all $x \in I_g$ and we show that there exists $g \in supp(R,G)$ and $a \in R_g - I_g$ such that $aI_g = 0$. Also, we show that there exists $g \in supp(R,G)$ such that $I_g^2 = 0$.

Graded weakly semiprime ideals, graded semiprime ideals, graded prime ideals, graded weakly prime ideals.

1 Introduction and Preliminaries

Let G be a group with identity e. A ring R is said to be G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and R_e (the identity component of R) is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g. Also we write $h(R) = \bigcup_{g \in G} R_g$ and $supp(R,G) = \{g \in G : R_g \neq 0\}$. For more details, see [7].

Let R be a G-graded ring and I be an ideal of R. Then I is called G-graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., if $x \in I$ and $x = \sum_{g \in G} x_g$, then $x_g \in I$ for all $g \in G$. An ideal of a G-graded ring need not be G-graded. To see this, consider $R = Z[i]$ and $G = Z_2$. Then R is G-graded by $R_0 = Z$ and $R_1 = iZ$. Now, $I = \langle 1 + i \rangle$ is an ideal of R with $1 + i \in I$. If I is G-graded, then $1 \in I$, so $1 = a(1 + i)$ for some $a \in R$, i.e.,

$1 = (x + iy)(1 + i)$ for some $x, y \in Z$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, I is not G-graded.

A proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$. Also, a proper graded ideal I of R is said to be graded weakly prime if whenever $a, b \in h(R)$ such that $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Recently, various generalizations of graded prime (graded weakly prime) ideals are studied in [2,3,4,5,6].

In [1], we define a proper graded ideal I of R to be graded semiprime if whenever $a \in h(R)$ such that $a^2 \in I$, then $a \in I$ and we define a proper graded ideal I of R to be graded weakly semiprime if whenever $a \in h(R)$ such that $0 \neq a^2 \in I$, then $a \in I$. Clearly, every graded semiprime ideal is a graded weakly semiprime ideal. However, the next example shows that the converse need not be true in general.

Example 1.1 *Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbf{Z}_4$. Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$ and $R_1 = R_3 = 0$. Consider $I = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$. Then I is a graded weakly semiprime ideal of R. However, I is not graded semiprime since $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R_2 \subseteq h(R)$ such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$ but $A \notin I$.*

In this article, we focus on the graded weakly semiprime ideals that are not graded semiprime and we introduce further results concerning such ideals. For example, if I is such ideal, we prove that $I_g \subseteq Nil(R)$ for some $g \in supp(R, G)$ and we prove that there exists $g \in supp(R, G)$ and $a \in R_g - I_g$ such that $4aI_g = 0$. Also, we prove that there exists $g \in supp(R, G)$ and $y \in R_g - I_g$ such that $y^2 = 0$. Assuming that 2 is not a zero divisor of R, we show that there exists $g \in supp(R, G)$ such that $x^2 = 0$ for all $x \in I_g$ and we show that there exists $g \in supp(R, G)$ and $a \in R_g - I_g$ such that $aI_g = 0$. Also, we show that there exists $g \in supp(R, G)$ such that $I_g^2 = 0$.

Throughout this article, all rings are commutative with nonzero unity.

2 Graded Weakly Semiprime Ideals that are not Graded Semiprime

In this section, we introduce several results concerning the graded weakly semiprime ideals that are not graded semiprime.

Definition 2.1 *Let I be a graded ideal of a G-graded ring R. Then $g \in G$ is said to be an unbreakable zero element of I if $R_{g^2} = 0$ and $R_g \not\subseteq I$*

Theorem 2.2 *Let I be a graded ideal of a G-graded ring R and g be an unbreakable zero element of I. Then*

1. $g \in supp(R, G)$.
2. there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$.
3. I is not graded semiprime ideal of R.

Proof. Since g is an unbreakable zero element of I, $R_{g^2} = 0$ and $R_g \nsubseteq I$

1. If $g \notin supp(R, G)$, then $R_g = 0$ and then $R_g \nsubseteq I$ a contradiction. So, $g \in supp(R, G)$.
2. Since $R_g \nsubseteq I$, there exists $a \in R_g$ such that $a \notin I$ and then $a^2 \in R_g R_g \subseteq R_{g^2} = 0$ and hence $a^2 = 0$.
3. By part 2, there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$ and then $a \in h(R)$ such that $a^2 \in I$. If I is graded semiprime, then $a \in I$ a contradiction. So, I is not graded semiprime ideal of R.

Theorem 2.3 *Let I be a graded weakly semiprime ideal of a G-graded ring R. If g is an unbreakable element of I, then there exists $a \in R_g$ such that $(a + x)^2 = (a - x)^2 = 0$ for all $x \in I_g = I \cap R_g$.*

Proof. By Theorem 2.2 part 2, there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Let $x \in I_g$. Then $(a + x)^2 = a^2 + 2ax + x^2 = 2ax + x^2 \in I$ (as $x \in I$). Since $a, x \in R_g$, $a + x \in R_g \subseteq h(R)$.If $(a + x)^2 \neq 0$, then since I is graded weakly semiprime, $a + x \in I$ and since $x \in I$, $a \in I$ a contradiction.So, $(a + x)^2 = 0$.Similarly, $(a - x)^2 = 0$.

Theorem 2.4 *Let I be a graded ideal of a G-graded ring R. If I is a graded weakly semiprime ideal of R that is not graded semiprime, then there exists $g \in supp(R, G)$ such that $R_g \nsubseteq I$ and hence there exists $a \in R_g$ such that $a^2 = 0$and $a \notin I$.*

Proof. Since I is not graded semiprime, there exists $a \in h(R)$ such that $a^2 \in I$ but $a \notin I$. Since $a \in h(R)$, $a \in R_g$ for some $g \in G$ and since $a \notin I$, $R_g \nsubseteq I$ and then $R_g \neq 0$, so $g \in supp(R, G)$.Now, $a^2 \in I$.If $a^2 \neq 0$, then since I is graded weakly semiprime, $a \in I$ a contradiction.So, $a^2 = 0$.
For a ring R, $Nil(R)$ denotes the ideal of nilpotent elements of R.

Theorem 2.5 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. Then $I_g \subseteq Nil(R)$ for some $g \in supp(R, G)$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R, G)$ such that $R_g \nsubseteq I$ and hence there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = 0$ and then $a, a+x \in Nil(R)$ and hence $x \in Nil(R)$. So, $I \subseteq Nil(R)$.

Theorem 2.6 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. If $char(R) = 2$ or 2 is not a zero divisor of R, then there exists $g \in supp(R,G)$ such that $x^2 = 0$ for all $x \in I_g$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ such that $R_g \nsubseteq I$ and hence there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = (a-x)^2 = 0$. If $char(R) = 2$, then $x^2 = a^2 + x^2 = (a+x)^2 = 0$. Suppose $char(R) \neq 2$ and 2 is not a zero divisor of R. Then $0 = (a+x)^2 + (a-x)^2 = 2x^2$ and then $x^2 = 0$.

Theorem 2.7 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. If 2 is not a zero divisor of R, then there exists $g \in supp(R,G)$ such that $I_g^2 = 0$.*

Proof. By Theorem 2.6, there exists $g \in supp(R,G)$ such that $x^2 = 0$ for all $x \in I_g$. Let $x, y \in I_g$. Then $x+y \in I_g$ and then $0 = (x+y)^2 = x^2 + 2xy + y^2 = 2xy$ and hence $xy = 0$.Thus, $I_g^2 = 0$.

Theorem 2.8 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. If 2 is not a zero divisor of R, then there exists $g \in supp(R,G)$ and $a \in R_g - I_g$ such that $aI_g = 0$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ such that $R_g \nsubseteq I$ and hence there exists $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Since $a \notin I$, $a \notin I_g$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = 0$ and by the proof of Theorem 2.6, $x^2 = 0$.So, $0 = (a+x)^2 = a^2 + 2ax + x^2 = 2ax$ and then $ax = 0$.Hence, $aI_g = 0$.

Theorem 2.9 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. Then there exists $g \in supp(R,G)$ such that $x^3 = 0$ for all $x \in I_g$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ and $a \in R_g$ such that $a^2 = 0$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = (a-x)^2 = 0$ and then $0 = (a+x)^2 + (a-x)^2 = 2x^2$ and hence $0 = x(a+x)^2 = x(a^2 + 2ax + x^2) = 2ax^2 + x^3 = x^3$.

Theorem 2.10 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. Then there exists $g \in supp(R,G)$ and $a \in R_g - I_g$ such that $4aI_g = 0$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ and $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Since $a \notin I$, $a \notin I_g$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = (a-x)^2 = 0$ and then $0 = (a+x)^2 + (a-x)^2 = 2x^2$ and hence $0 = 2(a+x)^2 = 2(a^2 + 2ax + x^2) = 4ax + 2x^2 = 4ax$.Hence, $4aI_g = 0$.

Theorem 2.11 *Let I be a graded weakly semiprime ideal of a graded ring R that is not graded semiprime. Then there exists $g \in supp(R,G)$ and $y \in R_g - I_g$ such that $y^2 = 0$.*

Proof. Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ and $a \in R_g$ such that $a^2 = 0$ and $a \notin I$. Let $x \in I_g$. Then by the proof of Theorem 2.3, $(a+x)^2 = (a-x)^2 = 0$ and then $0 = (a+x)^2 + (a-x)^2 = 2x^2$ and hence $(a+2x)^2 = a^2 + 4ax + 4x^2 = 4ax + 2(2x^2) = 4ax = 0$ by the proof of Theorem 2.10. Choose $y = a + 2x$, then $y^2 = 0$. If $y = a + 2x \in I_g$, then as $2x \in I_g$, $a \in I_g \subseteq I$ a contradiction. So, $y = a + 2x \in R_g - I_g$ such that $y^2 = 0$.

Theorem 2.12 *Let R_1 and R_2 be two G-graded rings and $R = R_1 \times R_2$. If I is a graded weakly semiprime ideal of R that is not graded semiprime, then $I = I_1 \times I_2$ where I_1 and I_2 are graded weakly semiprime ideals of R_1 and R_2 respectively and I_1 is not graded semiprime ideal of R_1 or I_2 is not graded semiprime ideal of R_2.*

Proof. We know that $I = I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2 respectively and R is G-graded by $R_g = (R_1)_g \times (R_2)_g$. Firstly, we prove that I_1 is graded. Let $x \in I_1$. Then $x \in R_1$ and since R_1 is graded, $x = \sum_{g \in G} x_g$ and then $\sum_{g \in G} (x_g, 0) = \left(\sum_{g \in G} x_g, 0\right) = (x,0) \in I_1 \times I_2 = I$ and since I is graded, $(x_g, 0) \in I$ for all $g \in G$ and then $x_g \in I_1$ for all $g \in G$. Hence, I_1 is a graded ideal of R_1. Similarly, I_2 is a graded ideal of R_2. Now, let $a \in h(R_1)$ such that $0 \neq a^2 \in I_1$ and let $b \in h(R_2)$ such that $0 \neq b^2 \in I_2$. Then $(a,b) \in h(R_1) \times h(R_2) = h(R_1 \times R_2)$ such that $0 \neq (a,b)^2 \in I$ and since I is graded weakly semiprime, $(a,b) \in I$ and then $a \in I_1$ and $b \in I_2$. Hence, I_1 and I_2 are graded weakly semiprime ideals of R_1 and R_2 respectively. On the other hand, Since I is graded weakly semiprime that is not graded semiprime, by Theorem 2.4, there exists $g \in supp(R,G)$ and $(a,b) \in R_g$ such that $(a,b)^2 = (0,0)$ and $(a,b) \notin I$ and then $a^2 = b^2 = 0$. Since $(a,b) \notin I$, either $a \notin I_1$ or $b \notin I_2$. So, either $a \in h(R_1)$ such that $a^2 \in I_1$ with $a \notin I_1$ or $b \in h(R_2)$ such that $b^2 \in I_2$ with $b \notin I_2$ and hence I_1 is not graded semiprime ideal of R_1 or I_2 is not graded semiprime ideal of R_2.

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