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A Characterization of Normed Algebras

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Abstract

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1. Introduction

The present paper originates in our attempts to extend Hurwitz's theorem to normed near algebras. There are literally scores of different proofs of Hurwitz's 1, 2, 4, 8 theorem which asserts that a finite dimensional alternative normed algebra is isomorphic to the algebra of Reals or Complex numbers or Quaternions or Octonions. In our various attempts to extend the theorem to near algebras we were invariably handicapped by the lack of the missing distributive law. This motivated us to replace the equality $|xy| = |x||y|$ by another equality which is equivalent to it in a normed near algebra but which gives more structure to a near algebra. The equality in (d$'$) below looked as an attractive alternative to (d). But then we were surprised to discover the equivalence presented in this paper. As can be expected and as shown in the proof Theorem 2.1, essential use of both the distributive laws is needed to prove that a normed algebra is a "modified algebra". Thus (d$'$) turns out to be too strong a substitute for (d) in as much as its availability in a modified algebra makes it into a normed algebra. Our search for a condition that properly lies between (d) and (d$'$) to enable the proof of a Hurwitz type theorem for near algebras continues.

In [2]: P;119 we find the following definition of a Normed (nonassociative) algebra: A finite dimensional vector space A over subfield F of $\mathbb{R}$ is called a (nonassociative)normed algebra if there is defined on A a product ab which satisfies

(a) $a(\alpha b) = (\alpha a)b = \alpha(ab)$ for all α in F and a, b in A;

(b) $a(b + c) = ab + ac$, $(b + c)a = ba + ca$ for all a, b, c in A; There exists a basis $\{e_1, e_2, \ldots, e_n\}$ of A such that if $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$ and $b = b_1e_1 + b_2e_2 + \cdots + b_ne_n$ are vectors in A then

(c) $\langle a, b\rangle = \left(\sum_{i=1}^{n} a_ib_i\right)$ defines an inner product on A and

(d) $\|ab\| = \sqrt{\langle ab, ab\rangle} = \|a\|\,\|b\| = \sqrt{\langle a, a\rangle}\sqrt{\langle b, b\rangle}$.

We denote $\|a\|^2$ by $|a|$.

The purpose of this note is to show that an equivalent definition is obtained if the distributive laws in (b) are dropped and (d) is modified into (d$'$) as follows:
$\langle xy, uz\rangle + \langle xz, uy\rangle = 2\langle x, u\rangle\langle y, z\rangle$ for all x, y, u, z in A.

Until we complete the proof that the modified structure is equivalent to a normed algebra, let us agree to call an algebra that satisfies (a),(c) and (d$'$) as a modified algebra.

2. Normed algebra versus Modified algebra

In this section we find a relation between normed algebra and modified algebra.

Theorem 2.1. An algebra $(A, +, \cdot)$ is a normed algebra if and only if $(A, +, \cdot)$ is a modified algebra.

Proof. Suppose $(A, +, \cdot)$ is a normed algebra.
To show that $(A, +, \cdot)$ is a modified algebra, we need to verify that $\langle xy, uz\rangle + \langle xz, uy\rangle = 2\langle x, u\rangle\langle y, z\rangle$ for all x, y, u, z in A......(i)
First we observe that $|a + b| = \langle a + b, a + b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle$

$=|a| + 2\langle a, b\rangle + |b|$ [By the bilinearity of inner product and its symmetry].
So $\langle a, b\rangle = \frac{1}{2}[|a + b| - |a| - |b|]$.
Second we observe that $|xy + xz| = |x(y + z)| = |x||y + z| = |x| \ [|y| + 2\langle y, z\rangle + |z|]$.
But $|xy + xz| = |xy| + 2\langle xy, xz\rangle + |xz| = |x||y| + 2\langle xy, xz\rangle + |x||z|$.
Equating the two expressions for $|xy + xz|$, we obtain $\langle xy, xz\rangle = |x|\langle y, z\rangle$ and similarly $\langle xz, yz\rangle = |z|\langle x, y\rangle$.
We are now ready to prove (i).

$$\begin{aligned}
\langle xy, wz\rangle + \langle xz, wy\rangle &= \langle xy, wz\rangle + \langle wy, xz\rangle \\
&= \langle (x + w)y, (x + w)z\rangle - \langle xy, xz\rangle - \langle wy, wz\rangle \\
&= |x + w|\langle y, z\rangle - |x|\langle y, z\rangle - |w|\langle y, z\rangle \\
&= [|x + w| - |x| - |w|]\langle y, z\rangle \\
&= 2\langle x, w\rangle\langle y, z\rangle.
\end{aligned}$$

Suppose conversely that $(A, +, \cdot)$ is a modified algebra. To conclude that $(A, +, \cdot)$ is a normed algebra, we have to verify the distributive laws and $|xy| = |x||y|$ for all x, y in A. Let us take $u = x$ and $z = y$ in (c$'$) to obtain $\langle xy, xy\rangle + \langle xy, xy\rangle = 2\langle x, x\rangle\langle y, y\rangle$ so that $|xy| = |x||y|$. We need to do some preliminary work for proving the distributive laws. We may assume that $dimA \geq 1$. On P 115 of [1], we find the following remark: "Our first observation is that there is no loss of generality in assuming that A has an identity element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$". Though the remark referred to above talks about normed algebras, its verification of does not use the distributive laws and hence is valid in a modified algebra as well. Hence there is no loss of generality in assuming that there is an element 1 in A such that $1 \cdot x = x \cdot 1 = x$ for all $x \in A$.

Now for x in A, we define $x^* = 2\langle x, 1\rangle 1 - x$ and prove that $\langle xy, z\rangle = \langle y, x^*z\rangle$ and $\langle xy, z\rangle = \langle x, zy^*\rangle$..........(ii) for x, y, z in A. $\langle xy, z\rangle = 2\langle x, 1\rangle\langle y, z\rangle - \langle xz, y\rangle$ [by taking $w = 1$ in (d$'$)]. Also $\langle y, x^*z\rangle = \langle y, (2\langle x, 1\rangle 1 - x)z\rangle$

$$\begin{aligned}
&= \langle y, 2\langle x, 1\rangle z\rangle - \langle y, xz\rangle \\
&= 2\langle x, 1\rangle\langle y, z\rangle - \langle xz, y\rangle.
\end{aligned}$$

Hence $\langle xy, z\rangle = \langle x, zy^*\rangle$. The other equality of (ii) can be proved similarly.

$$\langle x(y + z), t\rangle = \langle y + z, x^*t\rangle = \langle y, x^*t\rangle + \langle z, x^*t\rangle = \langle xy, t\rangle + \langle xz, t\rangle.$$

Since this is true for all $t \in A$ and since $dimA \geq 1$, it follows that $x(y + z) = xy + xz$. The other distributive law can be proved similarly. ■

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